Functional Analysis

Prof. P. D. Srivastava

Department of Mathematics

Indian Institute of Technology, Kharagpur

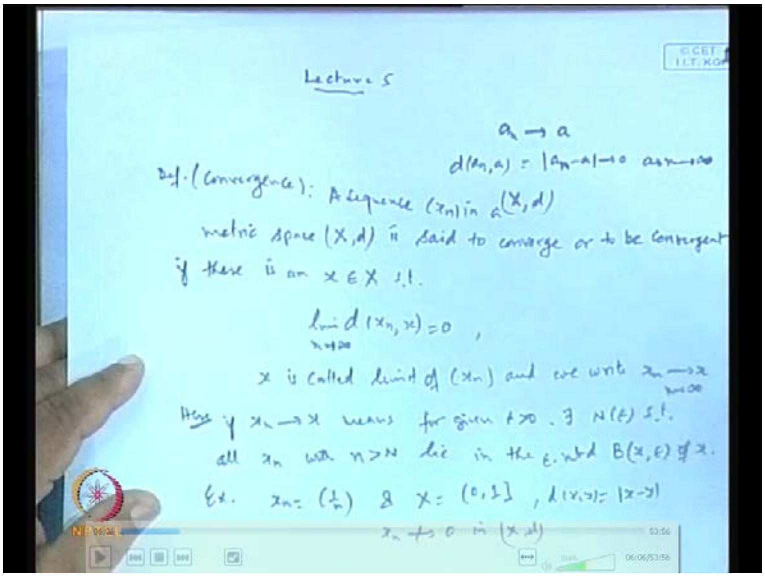
Module No. # 01

Lecture No. # 05

Convergence, Cauchy Sequence, Completeness

So, today we will discuss concept of Cauchy sequence and completeness; the convergence concept we already discussed. However, we will repeat what we have done in last time. We know the sequence of the real or complex number, it plays a vital role in calculation and the metric define on it, that is the usual metrics enables us to define the concept of the convergence.

(Refer Slide Time: 00:50)



When we say a sequence of the real or a complex number n goes to a, it means the difference between a n minus a; this tends to 0 as n tends to infinity. So, this basically define is nothing but the distance between a n and a and this concept on the real line, we wanted to enhance over an arbitrary metric space X, d where, X is the set of point and d is the notion of the distance defined on it.

So, we say a sequence X n, a sequence convergence. First, a sequence X n in a metric space X d is said to converge or to be convergent, if there is an X, belonging to capital X such that, the distance between x n X, when n is sufficiently large is 0 tending to 0 or limit of this is 0, there, X is called the limit of of the sequence x n and we write of course, write it as X n goes to X under the limit, when n tends to infinity on the limit of this 1.

Now, here, if we look, this what is d of x n x. It is basically a real number, it is basically a real number a sequence of the real number. So, the concept of the sequence of that real number, the same concept is applied here. We are converting the sequence a points in x n in x which goes to x in the form of the real number and then applying the same definition of the convergence of a real or complex numbers, real numbers.

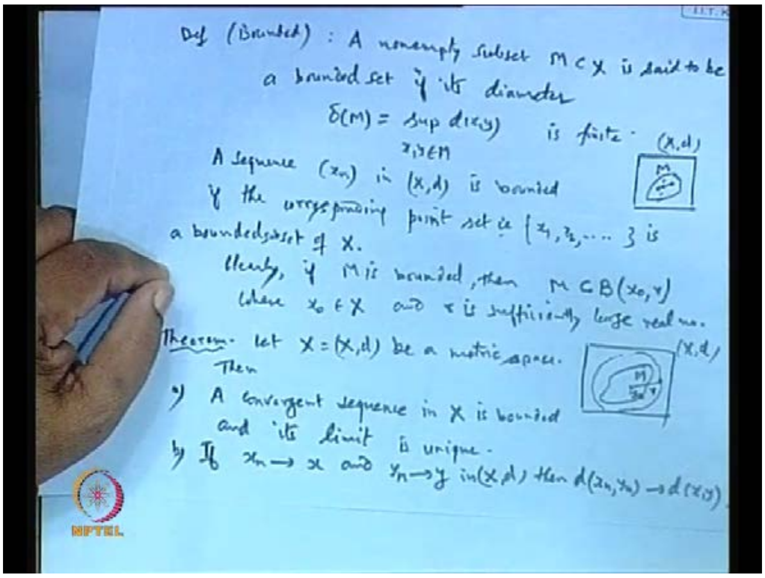
Now, here, when we say the x n converges to x, this means, here we say when x n converges to x means for given epsilon greater than 0, there exist an integer n which depends on epsilon such that, all x n with n greater than capital N lie in the neighborhood, in the epsilon neighborhood of x, that in in epsilon neighborhood B x epsilon of x, that is the meaning of this. So, in terms of the neighborhood, we can say sequent x n convergent. ok

Now, the point x must be a point in x, that if x does not belong to capital x, then we cannot say the limit converges. For example, if we take a sequence x n say, 1 by n and our capital X is a semi close, say, interval 0 to 1 and the metric d, if I take as x minus y, then, though the sequence x n goes to 0 when tends to infinity. In the sense of real over a real line, but the 0 is a point which is not available in this plus x. So, we say the x n does not converge to 0 in the metric space x d because, the limiting point is not available here ok.

So, the important part, area that limiting point, must be a point of the space x. That is why the (()) convergence of a sequence is not an intrinsic property, it basically depends on the metric on the space which you are choosing. Here, if I take this closed interval, then the same sequence will behave as a convergent sequence in this replacement.

So, this much we have already discussed earlier. Now based on this or using this concept of the convergence, we cannot define the concept of the boundedness and the relation between the convergence and the bounded sequence.

(Refer Slide Time: 06:15)



So, boundedness, bounded a non empty subset M of x is said to be bounded, to be a bounded set, if it is diameter delta M, which is equal to supremum d of x y x y are the points of M is finite. So, a set M, which is a subset of a metric space x d is said to be bounded if the diameter of the set is finite. That is, if we pick up any 2.0 here, find out the distances and like this if you continue, then supremum of this, if it comes out to be finite, then we say the set m is bounded.

And similarly, we say a sequence x n in a metric space x d is bounded, is said to be bounded if the corresponding point point set that is x 1 comma x 2 and so on is a bounded set, is a bounded set a bounded subset or set subset of x just like.

So, basically what we if. So, clearly, if a set m is bounded, then m can be content inside a open ball centered at x 0 with a suitable radius r where, x 0 is a point in capital X and any point in capital X and r is sufficiently large large real number, if we have the set x d a metric space and this is our set M.

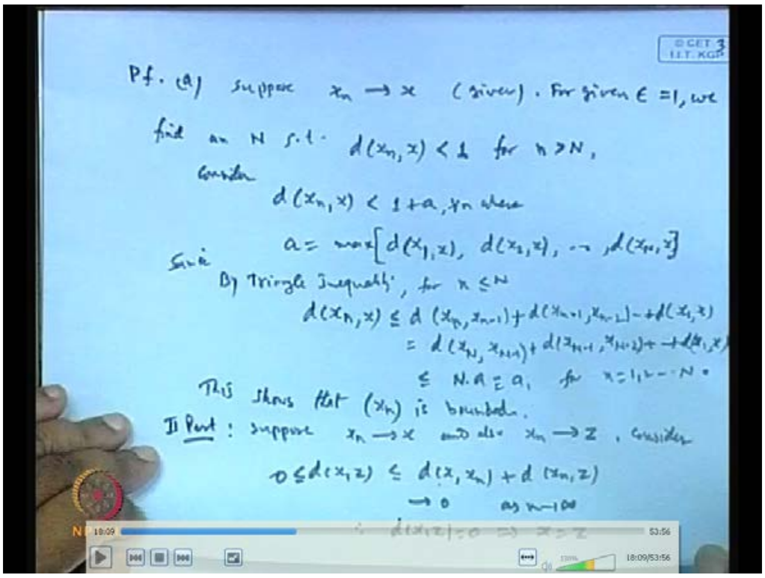
If the set is bounded, it means if I pickup any arbitrary point x 0 in capital X need not be then corresponding to this point, we can always find an open ball centered at this point, with a suitable radius, say r, such that all the points of the set M or the points of this ball, when we say, the set m is a bounded set or bounded subset of x. So, there is a relation between the convergence and bounded sequence.

So, we go for some relation in the form theorem or let it be, let x which is a metric space x d be a metric space, then the following thing of course, a convergence sequence in x is bounded and its limit is unique, this is the first session we made it.

Second point which is true if sequence x n converges to x and y n converges to y in the metric space x d, then the corresponding distances x n y n, this will go to d of x y this. So, basically, what this result says, every convergence sequence is a bounded sequence. We are not talking about the converse part; converse basic is basically not true always. In general, the sequence need not be a convergent always ok.

Second part says, if there are 2 sequences which goes to the limit x and y, then the difference if I take the x n y n and find out the distances, then that distance will also go to the distance between x and y. So, that is what is. So, let us see the proof of this.

(Refer Slide Time: 12:31)



Part a what is given a a convergent sequence is given, we wanted to show the convergent sequence is bounded. So, suppose sequence x n converges to x, this is given so, by definition of the convergent. So, for given epsilon, say suppose 1, we can find we can find an n an integer n such that, distance between x n and x can be made less than epsilon, that is, 1 after certain integer capital N, so, all n, greater then capital N.

So, let us consider d of x n comma x. We claim that this is less than 1 plus a, where a stands for maximum of d x 1 comma x d x 2 comma x d x n comma x. So, if this is true, then obviously, a point set x 1 x 2 x n will be a bounded set because, d of f x n x is less than a. So, supremum was taken its finite, therefore, the convergence sequence will be a bounded sequence ok.

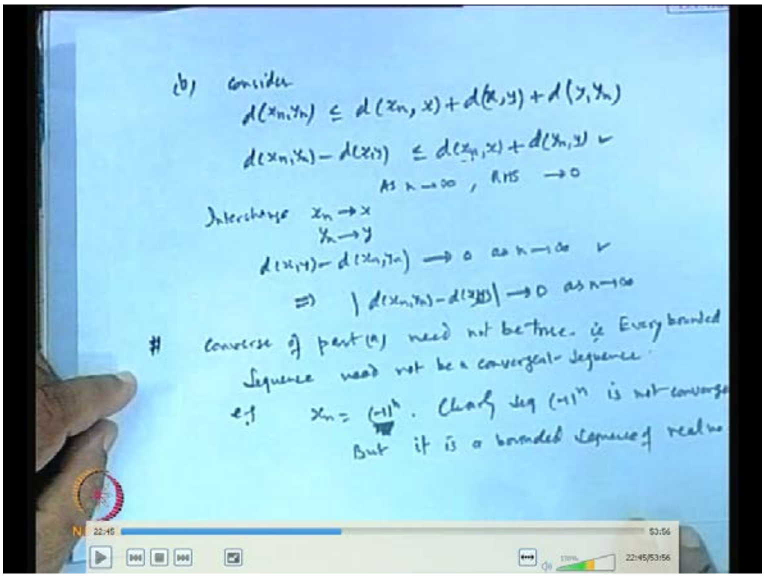
Now, how does it follow this? By the triangle inequality, we can write d of x n x, when n is say, n is less than or equal to n, we can write this thing as this will be less than equal to d of x n comma x n minus 1 plus d of x n minus 1 comma x n minus 2 and lastly, it goes to d of x 1 comma x like this. So, here when n is there, so, we can write this as capital N also. So, we can say capital N, that is, it goes a d of x comma x comma x x n minus 1 and d of x n minus 1 x n minus 2 and like this, continues till d of x 1 x.

Now, the maximum value of this, say a, then we can write, this is less than equal to some n times a. So, let it be another constant, say, a. So, we can say d of x n x n is less than a 1 for all n 1 2 up to n minus 1 and for n greater then this 1 n its already less than 1. So, basically, when you take the both these simultaneously, we can write the d of x n x is less than 1 plus a for all n. This this is true for all n, for all n for every n. So, it is. So, we get this, shows that sequence x n is bounded ok.

Now, second part of this is, the limit is unique. So, for the second part, suppose x n has 2 limit, suppose x n converges to x and also x n converges to z, we wanted x and z are identical, equal. So, now, consider d of x comma z since the distance cannot be negative, it will be always be greater than equal to 0 apply the triangle inequality. So, we get d of x comma x n plus d of x n comma z.

Now, as n tends to infinity, x n converges to x x as well as x n converges to z; it is our assumption. So, the right hand side will go to 0, therefore, this d of x z is less then equal to 0, but basically it is always greater than 0. So, this implies d of x z is 0 that implies x equal to 0 by definition of the metric. So, limit will always be unique, if a sequence converges, it will converge to the same limit. So, this will

(Refer Slide Time: 18:18)



The second part of it proof that we wanted to show that, if x n converges to x y n converges to y, then d of x n y n will go to d of x y. So, consider d of x n y; start with this and then apply that triangle inequality d of x n comma x plus d of x comma y plus d of y comma y n ok.

Now, transfer this d of x y this side. So, we get d of x n y n minus d of x y is less than equal to d of x n comma x plus d of y n comma y. Because, it is symmetric in nature now, as n tends to infinity, the right hand side tends to 0; right hand side will go to 0 because, this goes to 0, this goes to, so, this will go to 0.

Now, interchange the role of x n x n and x that is x n is interchange with x y n interchange with y, then here you are getting d of x y minus d of x n y n and this can be further written because, if I interchange x n x, there will be no change. So, this will also go to 0 as n tends to infinity. So, what we get it from here d of x n y n minus d x y tends to 0, while in this case d of x y minus d of x n y n goes to 0. This implies the modulus of d x n y n minus d of x y this modulus will go to 0, as n tends to infinity and that is what we wanted to prove clear.

Now, so far, we have considered the convergence of the sequence and the corresponding sequence said to be bounded set and the relation between the convergent bounded the converse of this converse of part a need not we need not be true or be true, that is every bounded sequence need not be convergent need not be a convergent sequence.

For example, if I take a sequence x n to be minus 1 to the power n, by say n now, obviously, the sequence when or just we take minus 1 to the power n, where this is convergent, we can say minus 1 to the power n. Take the sequence x n to be minus 1 to the power n.

Now, this sequence is not convergent sequence, minus 1 to the power n is not convergent as n tends to n because, when n is say, even, it goes to plus 1, when it is odd it goes to minus 1. So, it is not a convergent, but it is a bounded sequence of real numbers. So, every bonded sequence need not be convergent that we.

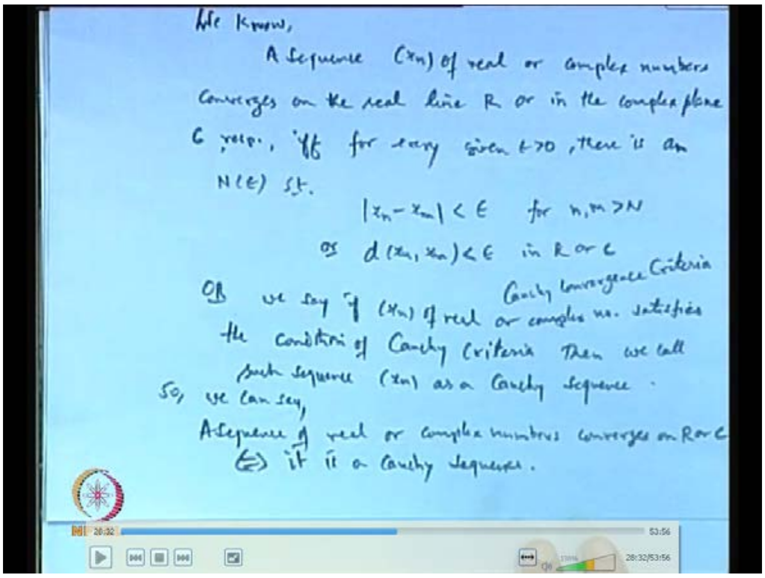
Now, another concept which we wanted to introduce is the completeness. The completeness concept is not dry, with the help of those four properties, which the metric joined, that is, the early metric space d of x y is greater than equal to 0 real non negative and positive real finite numbers. It is 0 when it is x equal to y and vice-versa d of x y equal to d y x and d of x y is less then equal to d x z plus d of z.

So, these properties are satisfying, then only we say the metric is the d is a metric on x, but, it does not give any information. It does not say whether the metric is complete or not. So, before going to the definition of the complete, and let us see what is the meaning of these completeness in a general metric space and how can we show, how can we prove the metric is a complete metric space.

But this is a different property and this is an extra property which the metric is joined. Some of the metric spaces are complete, where there are examples, where the metrics are also incomplete metrics space.

The concept of the completeness is taken basically form the sequence of the real number or sequence of the complex number. We know that sequence of the real number or complex number, if it is convergent, it will satisfy the Cauchy convergence criteria. What is that Cauchy convergence criteria?

(Refer Slide Time: 24:31)



So, let us say, a sequence x n. We know a sequence x n of real or complex number converges on the real line R or in the complex plane C if and only if if and only if, for every given epsilon, there is an N which depends on epsilon such that mod of x n minus x n is less than epsilon for all n and M greater than N.

Now, this mod of x n minus x m is less than epsilon, this is equivalent to basically the distance between x n and x m in a real line or complex plane, because, distance, now, some of distance on real or complex plane is defined as the usual way mod of x n minus x m. So, when we say a sequence x n of real or complex number is convergent, if and only if, for any epsilon, they greater than 0, there exist in such that the distance between x and m is less then epsilon and this criteria is known as the Cauchy convergence criteria ok.

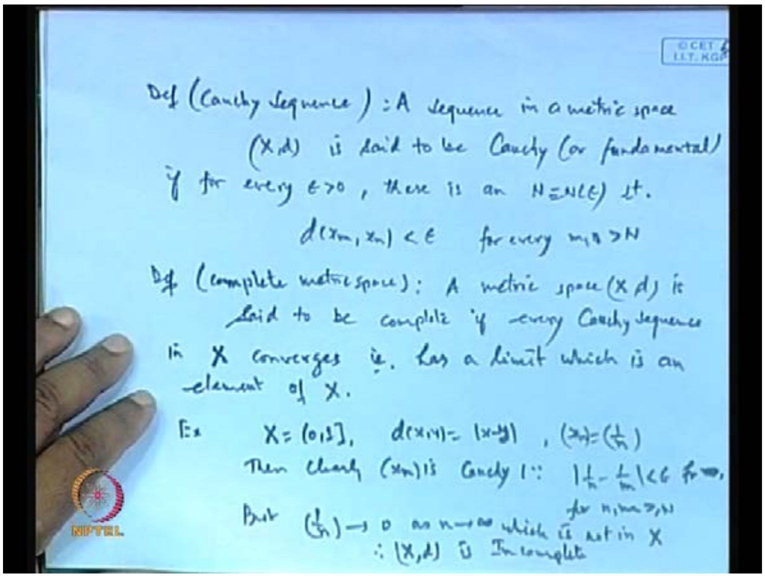
Or, we say if a sequence satisfies this condition, then we say it is a Cauchy sequence or we say if a sequence x n of real or complex number satisfies satisfy the condition of Cauchy cauchy criteria, then such a sequence, we call such sequence x n as a Cauchy sequence. So, we can also say real sequence of a sequence of real or complex number converges.

So, we can say. So, we can say, a sequence of real or complex number number converge, numbers converges on R or C if and only if it is a Cauchy sequence, it is a Cauchy sequence. So, this is the criteria, is well known and we know all the sequences of real or complex number, if convergent, it will be Cauchy and vice-versa and that is why, we say the real or complex number is a complete metric space.

But, in general metric space, this criterion may not be hold may not hold good. A sequence x n in x may be a convergent sequence or may be a Cauchy sequence, but need not be convergent. For example, if we take the earlier sequence x n to be 1 by n which we have discussed and metric, if I take the usual metric and x to be 0 1 where, 0 is a metric, then, we have see the sequence is not convergent. However, that can be shown to be a Cauchy sequence. So, such a space, we cannot say it is a complete.

So, we define the completeness related to this, a sequence which is Cauchy or not Cauchy sequence. So, before going, this let us see the definition of the Cauchy sequence in a general metric space.

(Refer Slide Time: 29:39)



A sequence in a metric space x d is said to be Cauchy or we also say, fundamental sequence.

If for every if for every epsilon greater than 0, there is an n depends on epsilon such that, distance d of x m comma x n is less than epsilon for every m n greater than m and the space x is said to be complete definition of a complete metric space. A metric space x d is said to be complete if every Cauchy sequence in x converges, that is the limit point, that is has a limit point has a limit, which is an element of x of x, which is an element of x.

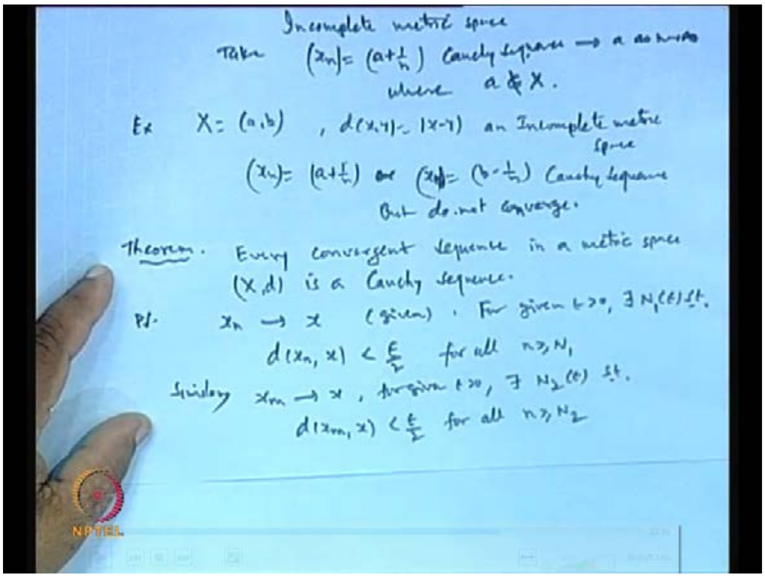
So, a metric space is said to be complete if every Cauchy sequence in x convergent, that it has a limit, which is an element of x. So, obviously, with this definition, 1 can say the set of real number or set of complex number under the usual metric is a complete metric space, because, a Cauchy convergence criteria says every sequence of the real or complex number is Cauchy; of if a sequence is Cauchy, it must be convergent, vice-versa for them clear. So, that is clear.

But in general, metric space, whether it is true or not, let us see. So, example is, if I take a metric space x to be 0 1 and d of x y as mod of x minus y and take a sequence x n to be 1 by n of real number, then, clearly sequence x n is a Cauchy sequence, because, the difference between because difference between two terms of the sequence can be made less than epsilon after certain integer n m greater than for n n m greater than equal to n, it can be shown ok.

1 by n minus 1, when n m is sufficiently large, it basically reducing to 0. So, we can identify the capital n such that, after certain stage, the difference between these 2 terms can be made as small as we pleased. So, it is a Cauchy sequence, but the limit of the sequence x n 1 by n, this tends to 0 as n goes to infinity, which is not which is not in x because, x is the semi closed interval 0 and one.

Therefore, this Cauchy sequence is not convergent. So, this space x under d is incomplete metric space incomplete. In fact, those space, where all the Cauchy sequences are not convergent, the space will be incomplete, that is, there exist even a single Cauchy sequence which is incomplete, which is not convergent, then it will be a incomplete metric space ok.

(Refer Slide Time: 34:47)



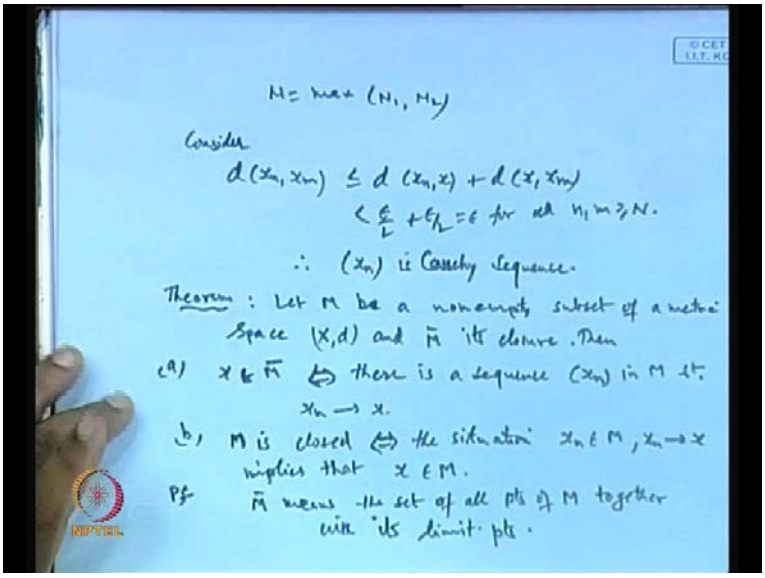
There are many examples, others also for incomplete metric space. If I take the space x to be r minus a and the notion of the distance d x y, if I choose to be mod of x minus y, then, we say, we see that this space is an incomplete metric space, because, we can choose a sequence x n which is of the form, say, a plus 1 by n, then, this sequence is a cauchy sequence which converges to a as n tends to infinity. While a is not a point, is a does not belongs to the spaces, where a does not belongs to this. So, it is incomplete metric space.

Then, another example we can choose from the set of real number. If I remove, take out all the rational point, then the set of rational number is an incomplete metric space, even the open interval under this metric d x y as mod x minus y with a induced topology will be an incomplete metric space. So, here, I can choose the sequence x n as a plus 1 by n or x n, we can choose to be b minus 1 by n. This type of sequence if I take, then, what all Cauchy sequence says, but but does not or do not converge converge. So, that is why, it is a not.

Now, what is the relation between convergence and the Cauchy sequence? Every convergent sequence is a Cauchy sequence, that we will show and Cauchy sequence need not be convergent. We have already take seen by many examples, where the sequence are Cauchy, but not convergent. So, we have result every convergent sequence in a metric space x d is a Cauchy sequence, is a Cauchy sequence the proof is very simple; suppose we have a sequence x n which converges to x.

So, by a definition, for given epsilon greater than 0, there exist an n depending on epsilon such that, such that the distance between x n comma x can be made less than epsilon or epsilon by two, for all n greater than equal to, say, capital n 1, then, similarly, we can say a sequence say x n converges to x. We can say for the given epsilon greater than 0, there exist say here n one. So, I am taking here n 1, here n two, depending on epsilon such that, d of x m comma x is less than epsilon by 2 for all n greater than equal to.

(Refer Slide Time: 39:09)



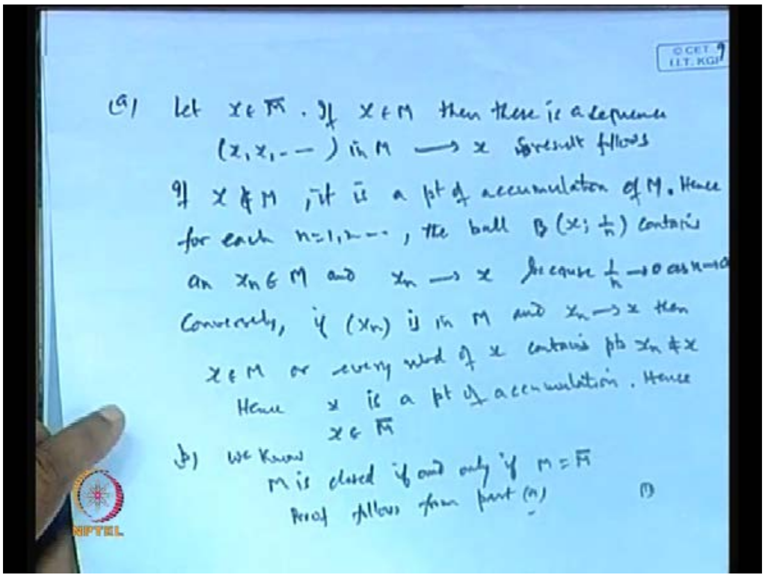
Now, if I picked up N to be maximum of N 1 comma N 2, then both these conditions hold for all n m greater than equal to capital n. This is also less than epsilon by 2, this is also less than. So, consider d of x n comma x m, now this will be less than equal to d of x n comma x plus d of x comma x m.

Now, this is less then epsilon by 2, this is less than epsilon by 2 for all n m greater than or equal to capital N. Therefore, this whole thing will be less than epsilon, hence, we say the sequence x n is a Cauchy sequence Cauchy. So, this completes the proof converse we have already discussed.

Second result which gives the relation between the closer and the convergence, the result is, let m be a nonempty subset of a metric space x d and m bar its closure, closure then, the following result hold x belongs to m bar if and only if there is a sequence x n in m such that, x n goes to x yes b part m is closed if and only if the situation x n belongs to capital N x n converges to x implies implies that x is a point of m, the proof is like this.

So, if x is an m closure, m closure means, set of all the points m together with its limit points, m closure means means the set of all points of m together with its limits points all limits points, then this is the m closure.

(Refer Slide Time: 43:07)



So, we will prove the first part that is, part a. What we want is, x belongs to m bar, then, there is a sequence x n n m such that. So, let us suppose x is a point in the closure of this ok.

So, the possibility is either x will be a point in M or x may be limit point. So, if x is in m, then there is a sequence of the type x comma x etcetera in m which converges to x. So, our result is follow. So, result follows, that is what we wanted to show there will be a sequence x n n m, which goes to x. There will be a sequence x n n m which converges to where x belongs to m so, obviously, this x will be in. So, this follows.

Now, if as x is not in m, then it will be a limit point. Then, it is a point of accumulation of m. So, if it is a point of accumulation, then, for each n for each n say, 1 2 3 and so on, the ball b centered at x with a radius say, 1 by n contains n and x n belongs to m and this x n sequence will go to x because, 1 by n is tending to 0, as n tends to infinity. So, again, the session is complete. So, if x m, is then, there will be a sequence n, such that x n converges to x. So, that will.

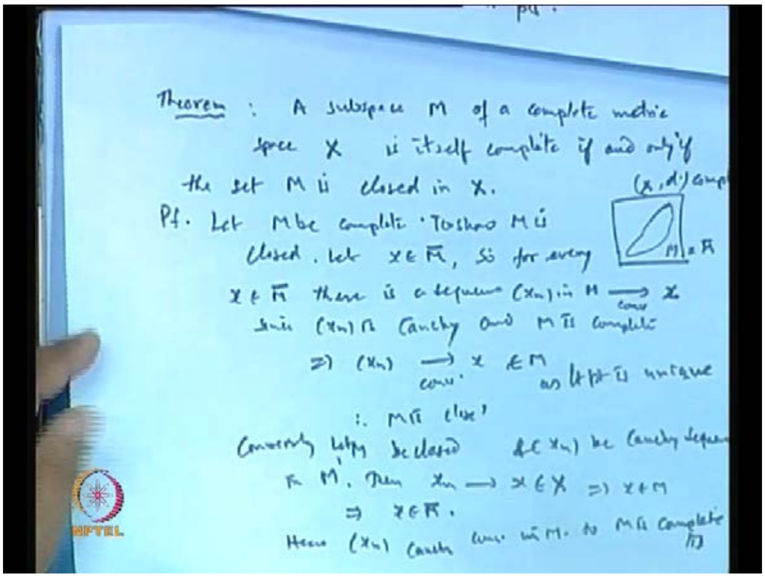
So, this part, the part conversely be, now, I have this result suppose, a sequence x n in m that goes to x, then, we prove that x belongs to m closure. So, conversely, if the sequence x n is in m and x n goes to x, then there are 2 cases. Either x will be a point of m or every neighborhood of x or every neighborhood of x contains points x n other than x, because, this is a sequence goes to x. So, either x will be a point in m, because, x n all in m or if it is not, then, every neighborhood of x will contain the point x n, which is different from x.

So, this is this shows that x must be an accumulation point. Hence, x is a point of accumulation. So, it must be a point in m bar, hence, x belongs to m closure m bar, by definition, closure of M. So, this comes the proof for the part b is very simple, it follows form part a.

We know that m is closed if and only if if and only if m is equal to m closure, this we know now if m is equal to m closure. So, let us see the part n. Now, when m is closed, it means m is equal to m closure. So, if m is m closure, then according to the part a, there will be a sequence x n which goes to x. So, we get situation x n belongs to x, but the point x will be in m, because, m is closed. So, it will be the point of m bar, means, it will be in m and vice versa. If I take this part, suppose x n is the x n converges so that x belongs to m, then, the x, either it will be limit point or will be a point of the m itself. So, it must be in m closure.

So, m is called to m closure and m this 1. So, all the limits points lies in m, it means, m is equal to m closure, it means m will be closed. So, this completes the proof of this, nothing to prove. So, it proof you can say, proof follows from part a, that is complete ok.

(Refer Slide Time: 48:20)



Now, there is another result, a subspace space M of a complete metric space X is itself complete if and only if the set m is closed in X. So, what is the meaning of this? A subspace M of a complete metric space X is itself complete if and only if the set m closed. We know by definition of the completeness, if x d is a metric space and here is the set m, and say, this is under the induced metric is a metric space, then, we say this set m is a complete metric space, if every Cauchy sequence is a convergent 1.

Now, instead of proving this part, we can simply say, if a subset subspace m of a complete metric space, this is a complete given is closed. That is, if I simply prove m is equal to m closure, then, automatically it will give you the conclusion that m will be a complete subspace of x.

So, you need not to consider the Cauchy sequence. So, that simply says, prove all the limits points of the m lies in that time itself. So, that is very good proof, let m be complete suppose. So, what we wanted to show is that, m is closed. If it is complete, means m is closed, this is our proof. Now, to show m is closed, then, what we show here is that, it contains all of its limit point.

So, let x belong to the closure of this. Now, if x belongs to m, then, automatically it will be a closed set, this proof. So, suppose x be a point in that, then, because x is in m bar, by the previous result if x belongs to m bar, then, there is a sequence x n in m. This result is, if x in it, there is a sequence in it such that x n converges to. So, by the previous. So, for every. So, for every x for every x belongs to m bar for every x belongs to m bar, there is a sequence x n in m, which converges to x.

But every convergent sequence is Cauchy. So, x n is Cauchy is a Cauchy sequence and m is complete. So, every Cauchy sequence is convergent. So, this implies that x n converges to x is a convergent sequence, because, it is Cauchy and the limit point is unique. So, it belongs to m, as limit point is unique. So, this proves m is closed.

Now, conversely, if we take m to be closed, closed and x n be a Cauchy sequence in m, then, because, Cauchy sequence x n is a Cauchy sequence. So, we say x n converges to, and this sequence converges to x n, which is a point in x. Suppose, now we wanted to x belongs to m bar, now this follows x belongs to m bar. By this property again, fourteen a if m is closed, if m is closed then, the situation x n converges to x implies x belong. So, this belongs to m bar. So, this shows x belongs to m, but m is equal to m bar. So, we get because, it is closed set. So, it belongs to m bar.

Hence, by assumption, arbitrary Cauchy sequence converges hence arbitrary Cauchy sequence converges in m. So, m is complete, this completes the proof, thank you, thanks.