Functional Analysis

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Module No. # 01

Lecture No. # 40

L^p - Space

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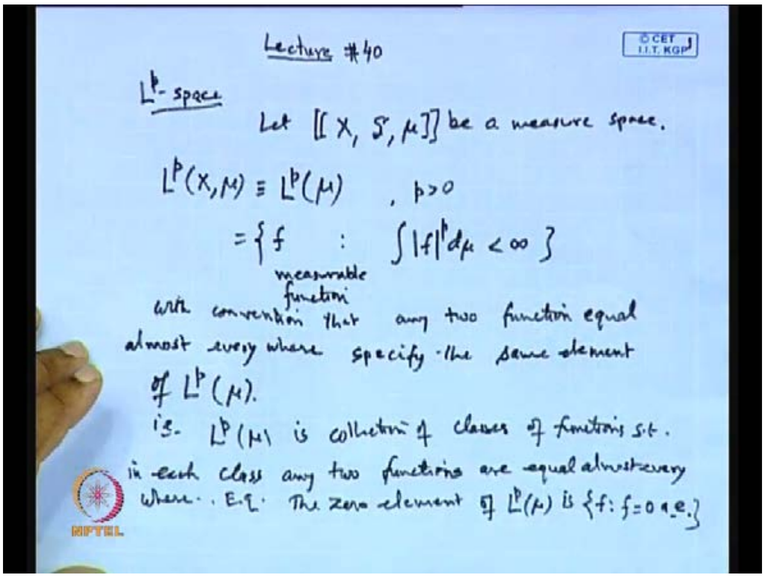

So, we have discussed the L p space. We have defined the L p space as follows: let X, S, mu this be a measure space, where X is a non empty set remains, capital S is a sigma-algebra of the subsets of X, and mu is a measure on it.

Then, capital L p space we denote (( )) capital L p (X, mu), or briefly, we can also say write capital L p mu, where the p is greater than 0– p is greater than 0– is defined as set of all measureable functions f such that…, such that integral of mod f to the power p d mu is finite.

So, basically, the capital L p space– the collection of all such measureable function where integral of mod f to the power p d mu is finite, but with a convention– with the, with convention, convention that that any two function equal almost everywhere, equal almost everywhere, specify the…, specify the same element of capital L p mu. It means, the capital L p is not a collection of a… the function; it basically a collection of classes where the any two elements of the same class are equal almost everywhere.

So, it is just like that, that if we have a collection of the functions, and suppose we introduce the relation on the function f, say f is related to g, if f equal to g almost everywhere, then it will decompose the whole collection into a disjoint classes, and those classes– union of those will be the entire space, clear? So, these classes– when we say the function f and g belongs to the same class, it means the they are equal almost everywhere. Only at the point where they differ has a measure 0.

So, that is the (( )). So, we can say that is capital L p space is, is basically that collection of classes of functions– capital L p space, the L p is the collection of classes of functions, that is the elements of this L p space are the classes of functions, such that in each class, any two functions– functions– are equal all most everywhere– almost everywhere– equal almost everywhere. For example, when we say the zero element of L m, for example, the 0 element of the class L p mu is the class of all functions f, where f equal to 0 almost everywhere.

So, that may be there, but in notation we say f belongs to L p means f is a measureable function, such that this condition holds that the it is be a integrable functions for it, ok?

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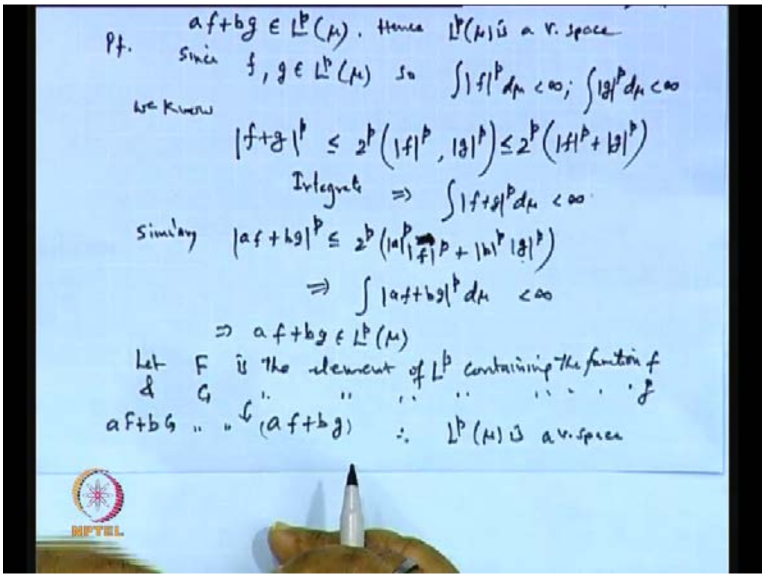

Now, this collection of these classes or the functions, if they form the vector space, and under the suitable addition and scalar multiplication, so, we say as an first result or lemma, let f and g belongs to L p mu, and let a and b are constant; let a, b be constant– be constants; then a plus a f plus b g– that belongs to L p mu. Hence, as a consequence, we can say, hence, capital L p mu is a vector space or linear space– proof is very simple. Since f and g are the elements of the L p mu, so, by definition integral, mod f power p d mu is finite, integral mod g power p d mu is finite, ok?

Now, we know this relation that if two numbers are there– f and g, then mod f plus g, we can write this less than equal to 2 to the power p mod f p comma mod g p. In fact, this is valid for any number a and b. So, this can be written as a plus b 2 the power p is less than equal two to the power p mod f to the power p, (( )) when a positive numbers. So, this is true, and which can be further less than equal to, you can say, 2 to the power p mod of f p plus mod of g power p.

Now, since f and g belongs to f, so, this is finite– this one is finite. Therefore, when you integrate both sides, we see that integral mod f plus g power p d mu is finite. It means, if f is an element of L p, g is an element of L p, then f plus g must be an element of L p. So, this will be there, and similarly, similarly, we can show that a f plus b g power p– this will be less than equal to what– 2 to the power p, within bracket, we can write mod a power p, into mod f power p, plus mod b power p, into mod g power p. So, again integrate it.

We get from here is integral a f plus b g power p d mu is finite, because this is finite– this integration is finite; integration of this is finite. So, a f plus b g belongs to the class. So, a f plus b g belongs to L p now; what, what we told is that L p is the collection of the classes. So, let…, let us take capital F is the element of capital L p containing the function, containing the function f, and capital G is the elements of L p containing the function g. So, all the function, which are equivalent to f, will belongs to capital F; all the functions, which are equivalent to g almost everywhere, belongs to G, ok?

Then we can easily see that a f plus b g– this elements is contained in a f plus b g. This class is the elements of L p containing the function a f plus b g; it can easily be seen. Therefore, L p mu is a vector space, ok?

You can choose that two classes f and g– collection of the class– and then all the elements are equivalent to this f, all the element, which are equivalent to g, belongs to G almost everywhere, and then we can write this problem. So, this is a vector space, clear?

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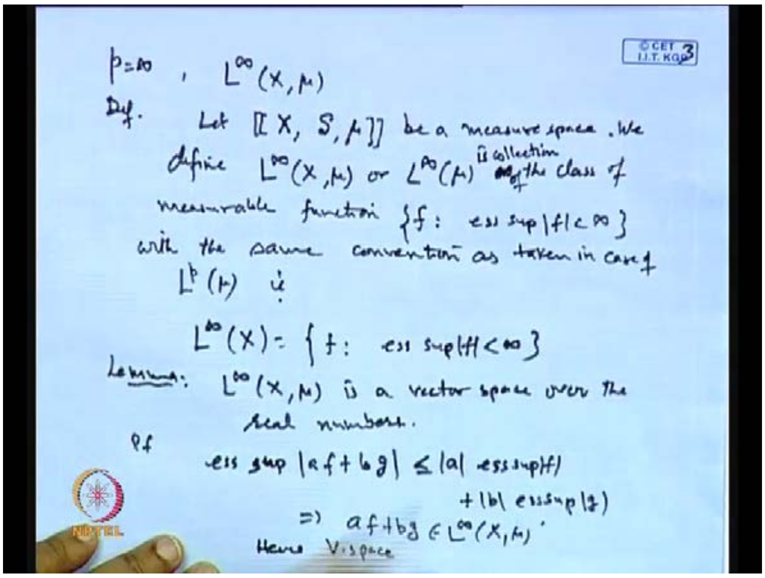

Now, similarly, when p is infinity– when p is infinity– then we introduce the concept of L infinity X or L infinity (X, mu). We define L infinity (X, mu) as let X, S, mu be a measure space; we defined L infinity (X, mu) as…, we defined L infinity (X, mu), or simply, L infinity mu, as the class of, as the class of measureable functions in the collection of…, in the collection of…, of the class, classes of measureable functions– classes of measureable functions f, such that essential supremum of mod f is finite with the convention– with the same convention– as taking in case of capital L p mu, that is the any two elements, which are equal almost to everywhere, belongs to the same class. So, that is the convention. So, that is capital L infinity X. In short, we say, it is the class of measureable function f such that essential supremum of mod f is finite, ok?

We have discussed the essential supremum already, earlier; that is the infimum of alpha, such that f is less than equal to alpha almost everywhere is the essential supremum of mod f; mod f is less than equal to alpha f. So, this…. Further, this is also a vector space, ok? This also is a vector space– L infinity (X, mu) is a vector space over the real numbers– over the real numbers– because the reason is we know the essential supremum of mod a f plus b g– this will be less than equal to mod a essentials supremum of mod f, plus mod b essential supremum of mod g, is it not?

Now, f belongs to the class L infinity. So, essential supremum mod f is finite if b g belongs to the class L infinity. So, essential supremum of mod g is finite. Therefore, the right hand side is finite; so, left hand side is finite; hence, this will be a this.

So, this proves the L infinity is a vector space. So, this implies a f plus b g belongs to L infinity (X, mu). Hence, vector space– hence, vector space the same way, which we did. Now, there is a some relation for that, that is inequality relation; in case of these functions– sequences– spaces, say, small L p space, that is the set of all sequences a n, such that sigma mod a n to the power p is finite; mod a n to the power p is finite.

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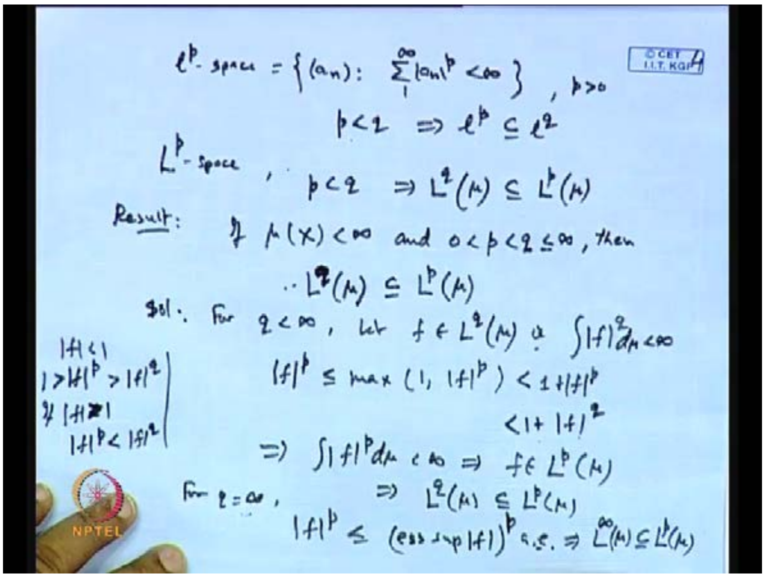

Now, this when p is, of course, greater than 0 or greater than 1, we can all say, now, in this case, if p is less than q, then it implies that L p is contained in L q, because once this series is finite means, then n a term will go to 0, so n a term will be less than 1. Therefore, when we raise the power the quantity opposite, and we get L p is less contained in L q. But in case of the L p space, the inequality reverses if p is less than q; it implies that L of q mod mu is contained in L of p mod L of p mu.

So, here, the inequality reverses. So, we have an example to show; in fact, the result is like this: if the measure of a whole space is finite, X is finite, and 0 less than p less than q less than equal to infinity, then L infinity, sorry, then L infinity, then L p, sorry. So, here is L q, then L q mu– L q mu– is subset of L p mu, and in fact, this will be the subset of, finally, if we go L infinity will be the subset of f (( )), ok? So, solution is simple– again, for q to be finite, for q, suppose, finite, and let f belongs to L q mu, that is integral mod f power q d mu is finite– this is given.

Now, we wanted this f is in L p mu. So, integral mod f to the power p d mu must be finite. So, we… if we take the mod f to the power p, now we know this is less than equal to maximum of 1, and mod f to the power p, depending on this, if mod f is less than 1, the maximum value will come out to be 1. When mod f is greater than 1, the maximum value will be mod f to the power p.

So, basically, this is less than always 1 plus mod f power p. Now, if p is less than q, and then mod f, f, mod f is greater than 1, then we can say mod f to the power p is less than mod f to the power q, but if p is mod f is less than 1, then it is less than equal to 1. So, we can say this is– total is– always will be less than 1 plus mod f power q, whatever the mod f may be; is it not?

Because the things are like this: if the mod f is less than 1, then what we get is the mod f to the power p clear, and mod f to the power q if p is less than q, then this order greater, is it not? Mod f is greater than p is less than, because this is less than inequality (( )), but if mod f is less– greater than 1 and p is less than q, then mod f power p is less than mod f power q, ok? So, that way, but when mod f is less than 1, this will always be less than 1. So, that is why it is always be less than 1. So, when p is less than q, this entire maximum value will always be less than 1 plus mod f to the power q. So, we get from….

Now, f belongs to L q (( )). So, integral of this is finite, and this mod f is here. So, integral of this finite– therefore, when you integrate it, we get mod f to the power p d; mu is finite– it implies f belongs to L p mu. So, f belongs to L p mu implies f belongs to this; it means L q mu is contained in L p mu, ok?

Now, in case of q to be infinity, if q is infinity, then what happens is that mod of f power p, which is less than equal to essential supremum mod f raise to the power p almost everywhere, is it not? Now, if f belongs to L infinity– L infinity mu– it means, this is finite. So, this has to be finite. So, this implies L infinity mu is contained in L p mu, like this. So, this follows the result. So, we get the things, ok?

Now, just like in capital L small L p space, we have tried few results, like this inequality, like Holder's inequality, and Minkowski inequality. The similar type of inequality holds here, and we also call it, call them as a Holder's and Minkowski, because they are derived by Holder's and Minkowski. So, let us see the various types of inequality for our space.

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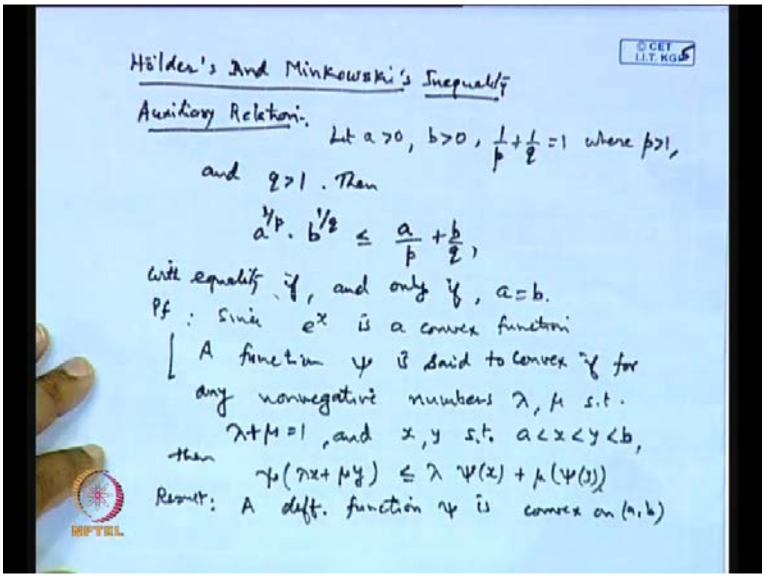

So, before going this first inequality– inequality of Holder's. In Holder's and Minkowski inequalities, we require some auxiliary relation. So, auxiliary relation, or auxiliary equation, you can say, let a is greater than 0; b is greater than 0 1 by p plus 1 by q is 1, where p is greater than 1, and q is greater than 01, then a to the power 1 by p into b to the power 1 by q is always be less than equal to a by p plus b by q, with equality– with equality if and only if– if and only if– a is equal to…, ok?

So, proof of this (( )), what we say is a and b are non negative numbers greater than 0; in fact, if 1 is 0 means, automatically, it is, obviously, 2, and p and q are conjugate numbers; 1 by p plus 1 by q is 1.

Then, p and q are both are greater than 1, then this relation holds. Now, in order to prove this relation, we make use of the convex function. We know e to the power x is a convex function– convex function– we mean a function is said to be convex when f of lambda x plus mu y is lambda of f x plus mu y. We say a function, that is a function psi is said to be convex, is said to be a convex function, if for any…, if for any non negative numbers, for any non negative numbers lambda and mu, such that lambda plus mu is 1, and x, y such that a is less than x, less than y less than b, then we have psi of lambda x plus mu y is less than equal to lambda times psi x plus mu times of psi y– mu times of psi y.

And if there is a (( )), we say it is function is a (( )) function. So, this is the definition of this, and in this one result, which is also true that if a function is differentiable function, a differentiable function psi is convex on the open interval, say, (a, b)– convex on the open interval if and only if, if and only if psi dash– means first derivative of psi– is a monotone increasing function– monotone increasing function– monotone increasing function.

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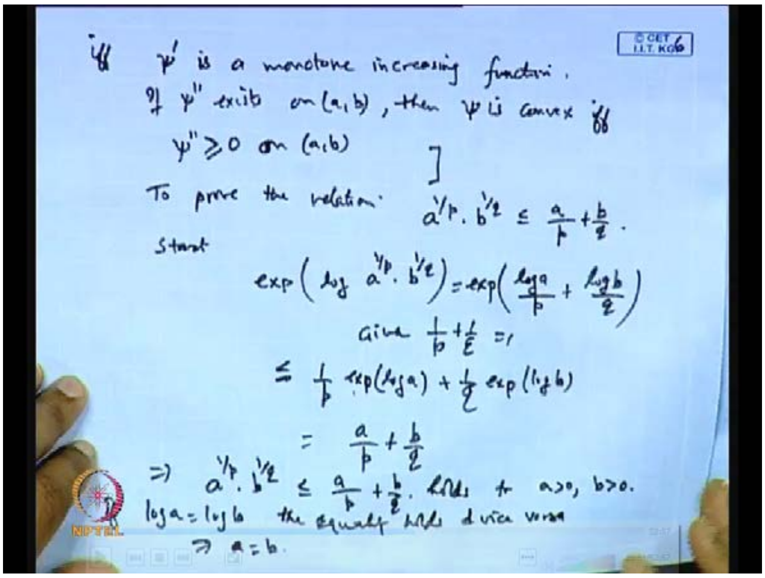

And if psi double dash exist, exist on the interval (a, b), then psi is convex if and only if psi double dash is greater than or equal to 0 on this interval (a, b). So, this is the criteria to test the… whether the given function psi is convex or not. If it is a differentiable function, the first derivative must be a monotonic increasing function, or if the second derivative also exist, then second derivative will be greater than equal to 0; e to the power x is a convex function because it is a differentiable function.

When we go for the first derivative– the e to the power x derivative is e to the power x, which is increasing function, where x 1 is greater than x 2; e to the power x 1 is greater than e to the power x 2 (( )), and second derivative comes out to be positive. So, e to the power of x is a convex function. So, let us say, e to the power x is a convex function; now, use this convex property of e to the power x to derive the relation, which we need it– the relation is a to the power 1 by p plus b to the power 1 by q is less than equal to this. So, consider this one to prove the relation a to the power 1 by p b to the power 1 by q is less than a by p b by q– this relation.

So, we start with this– each exponential of logarithmic of this term, because exponential log– they are inverse function. So, it will not affect the (( )) expression will remain a to the power 1 by p b to the power 1 by q, but the advantage of doing this thing is we can write expression exponential log a by p plus log b by q. Now, exponential function is a convex function, and 1 by p plus 1 by q is 1. So, basically, we have that lambda of x plus mu of y. So, y exponential property, 1 by p will be taken outside, and we get exponential log a plus 1 by q exponential log p.

And log a exponential will, and log gets cancelled, and basically, you are getting a by p plus b by q. So, what we get it from here is that, sorry, this exponential property. So, it is less than equal to…. So, you are getting finally, a by p plus b by q. It means, this term– a to the power 1 by p b to the power 1 by q is less than equal to a by p plus b by q– holds for any a positive, b positive, and that proves the result. Now, when the equality holds, in case of the equality, what happen is we need the equality sign here.

So, for equality sign, log a must be equal to log b (( )) they are equal; then only, 1 by p plus 1 by q becomes… where log a can be outside and the 1 by p plus 1 by q is 1. So, if log a equal to log b, the equality holds, and vice versa also, but log a equal to log b means a must be equal to b– a must be equal to b. So, this will be true alpha; hence, this completes the proof. Now, using this inequality, we can now derive our Holder's inequality.

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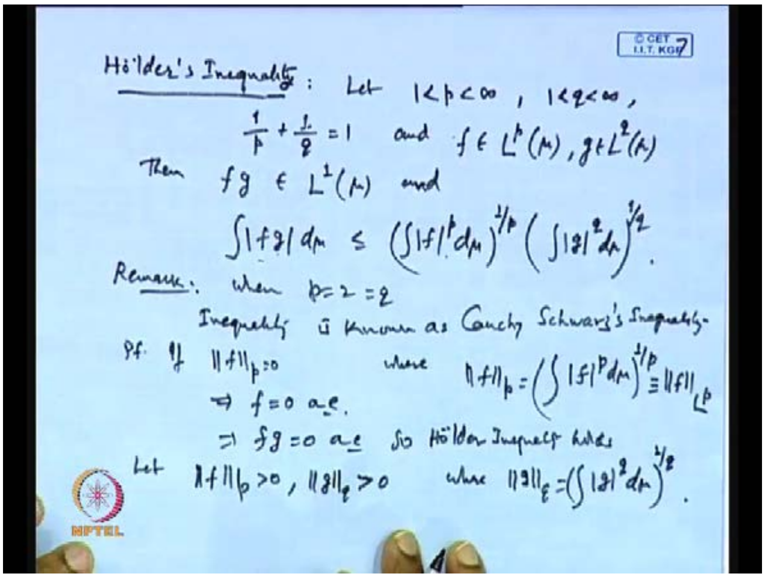

So, Holder's inequality – let 1 is p less than p is greater than 1; q is also greater than 1, and 1 by p plus 1 by q is 1, and let f belongs to L p mu; G belongs to L q mu; then the Holder's inequality theorem says that f g must be a point of L 1 mu, and not only this, it will satisfy this condition that mod, integral mod f g d mu is less than equal to integral mod f to the power p d mu power 1 by p, into integral mod g power q, d mu power 1 by q. This inequality known as the Holder's inequality. When p equal to 2, remark: when p is equal to 2 equal to q, the inequalities is known as Cauchy Schwarz's inequality– Cauchy Shwarz's inequality.

Now, let us see proof of this. So, f is a point in L p; g is a point in L q; and then, product will be in L 1; this we wanted to show, and second one is this inequality. So, in fact, if I prove this inequality, then this is automatically followed, because if f is in L p, it means this integral is finite; when g is in L q, means this integral is finite. So, the right hand side is finite; therefore, this integral must be finite.

So, f, g must be a point of L 1 mu. So it is, obviously, true. So, only we wanted to show– establish this inequality– when f and g are the point in L p and L q. Now, if our f is such that norm of f p is 0, where what is the norm p? In fact, this I will define later on, but right now, means integral mod f power p d mu raise to the power 1 by p– this we are denoting by norm f suffix p or L p. Also, somebody– some or we can also write this is norm f L p in order to differentiate with small l p, because this is our L p space, ok?

So, if suppose this is 0, it means from here, f must be 0 almost everywhere, because this integral is 0, then this integral is 0 means f must be 0 almost everywhere. So, when f is 0 almost everywhere, then f g equal to 0 almost everywhere. So, from here, this is integral is 0, and this is already 0– zero. So, equality holds. So, equation is, so, Holder's inequality is Holder's inequality holds. So, nothing to (( )).

So, let us assume, let f is such that norm of f suffix p is positive, norm of g suffix q is positive, where g means norm of g; q means integral mod g to the power q d mu power 1 by q. This is… So, suppose they are…. So, when they are non negative, we can divide it greater than 0 means we can put it in the denominator. So, let us choose the a and b in such. So, choose a to b mod f power p over norm of f p (( )) power p– this power p, ok?

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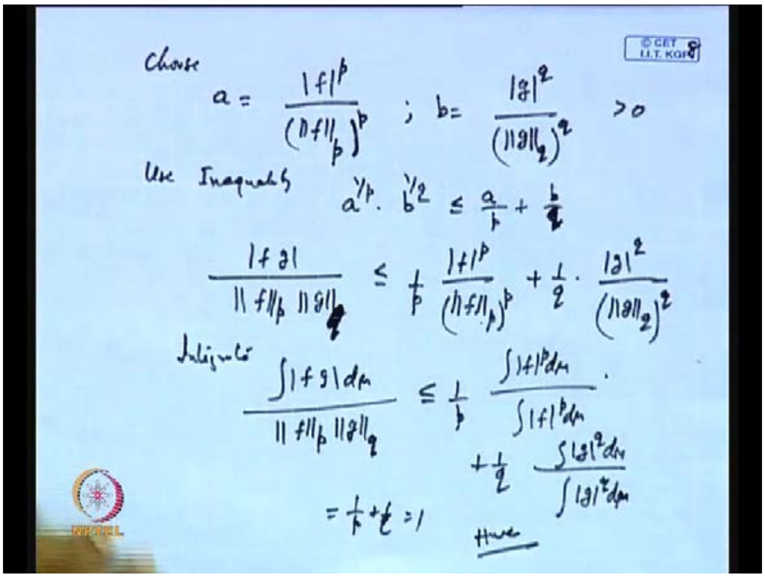

And b, let us take to mod g to the power q over norm of g q power q. Now, both are positive and well defined, because this is non-zero and this is non-zero, ok? Now, use this inequality a to the power 1 by p, b to the power 1 by q, is less than equal to a by p plus b by q– use this inequality. So, if we put these values, what we get is mod f g, because f and g power gets cancelled; divide by norm of f p into norm of g q– g q. Sorry, this is g q; norm of g q is less than equal to a by p. So, mod f power p over norm f p power p, and then 1 by p plus 1 by q mod g power q over norm g q power q; clear?

Now, take the integration. So, if you take the integration, both side, what do you get from here is this– integral mod f g d mu, and this is constant, basically. So, you can take outside– outside– and this is less than equal to when you integrate mod f to the power p. So, this is nothing but the norm f to the power p. So, it will be 1 by p, and this will remain because it is integral mod f p d mu– is it or not? And, then this will be when we open this integral, it is also same as this.

Similarly, 1 by q integral mod g to the power d mu, and this is also same as d mu. So, basically, they cancel and we get 1 by p plus 1 by q is 1. Therefore, multiply this and we get the answer. So, integral f g is less than equal to norm f suffix p norm q suffix p, and that is the same as this. So, we prove this Holder's inequality holds.

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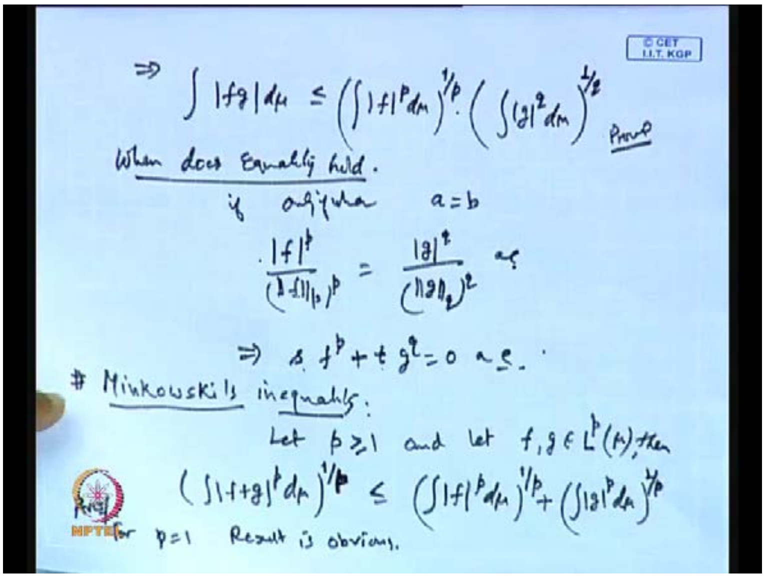

Now, when does this equality holds? So, we get from here, integral mod f g d mu is less than equal to integral mod f d mu power 1 by p into integral mod g q d mu power 1 by q. So, Holder is proved. Now, the question– when the equality holds– when does equality hold? So, in order to get the inequality, we need this– equal means, basically, this is equal, and this equal means we need the equality of this. So, for equality of this, we require a is equal to b. So, equality holds only when a is equal to b, is it not?

And when a is equal to b almost everywhere, this was the a equal to b. So, a means mod f to the power p, by norm of f suffix p power p equal to mod g, power q divide by norm of g q power q almost everywhere, and that will give you the condition that mod f, some constant times, s f to the power p, plus a constant times t g to the power q, equal to 0 almost everywhere, because just you take this constant, divide– multiply this, this multiply is, so it becomes s, and this minus this becomes t, becomes minus of this equal to, almost everywhere. So, this is the necessary and sufficient condition for this equality– Holder's; in fact, it is the if and only if– if and only if. So, here we get this one.

The another inequality, which we also have– the Minkowski inequality. So, second is Minkowski inequality. Let p is greater than or equal to 1, and let f and g belongs to capital L p mu; then, integral mod f plus g to the power p d mu power 1 by… 1 by p– power 1 by p– this will be less than equal to integral mod f power p d mu raised to the power 1 by p, plus integral mod g power p d mu raised to the power– whole raised to the power– 1 by p.

So, this result is true for p greater than equal to 1, and in fact, for p equal to 1, the result is obvious– for p equal to 1, the result is obvious. So, only we will prove for p equal to greater than 1, ok?

Now, this result is parallel to our results in case of the small l p space, but small l p and capital L p, because l p is the sequences and capital L p is the collection of measureable functions. So, both are having a different nature of this, but the corresponding inequality is almost same. Here, we have used the integration, then that is it. So, let us see the proof for this– proof of this Minkowski inequality.

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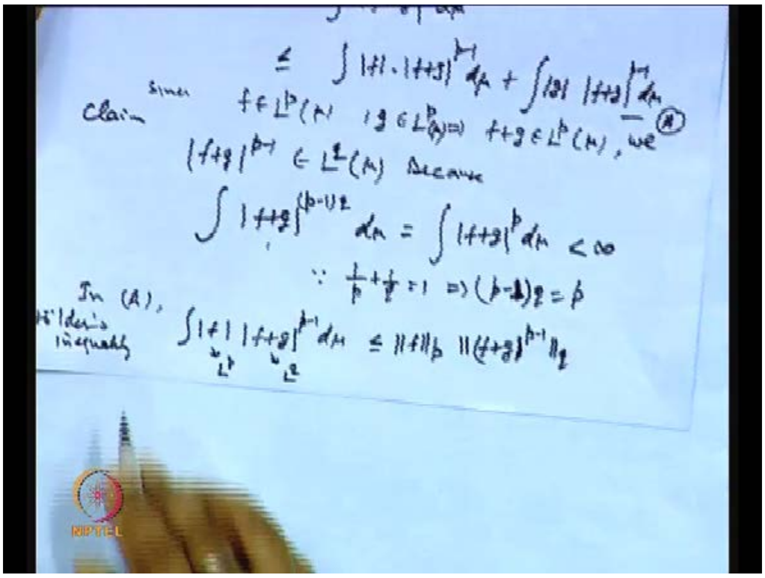

So, suppose start let norm of f suffix p– we denote by this integral f p d mu power 1 by p; let us start with this. When p is greater than 1, then consider mod norm of f plus g p, and raised to the power p; consider this.

So, basically, this is equal to integral mod f plus g power p d mu. Now, this can be breakup into two parts– integral mod f into mod f plus g power p minus 1 d mu, plus integral less than equal to, plus integral mod g mod f plus g power p minus 1 d mu, ok? Because this can be written a plus g into mod of f plus g into mod of f plus g power p minus 1, and then apply the triangular inequality. We get less than equal to mod f plus mod g, and (( )) like this.

Now, f is in L p space; g is in L p space. So, f plus g is in L p space, because it is a linear space– it is a linear space, ok? But what about the f plus g power mod of this power p minus 1? We claim that this is in L q space, because if we find the integral of this f plus g power p minus 1 and raise to the power q d mu, then what we get it integral mod f plus g power p d mu, because 1 by p plus 1 by q is 1. So, from here, we can say p minus 1– that is, p minus 1 into q is– p minus 1 into q is p. You can just transfer; here is 1 by p. So, p minus 1 by p is 1 by q– q– multiply.

So, this– it means this entire thing can be replaced by p. Now, f and g– both are in a f plus g is in L p space. So, this is finite; therefore, this integral whose q th power is integrable and finite, so it belongs to L q. So, this elements belongs to this. Now, from here, inequality– this is the first element, which is L p– this is in L q. So, by Holder's inequality, we can apply, ok?

So, let this is A. So, from A, in A, the integral mod f mod f plus g power p minus 1 d mu; apply Holder's inequality, ok? This is in L p; this is in L q. So, we can apply the (( )) Holder's inequality, and by this Holder's inequality, you are getting this is norm of f p, ok? And then, this will be equal to norm of f plus g power p minus 1 norm of q, is it not?

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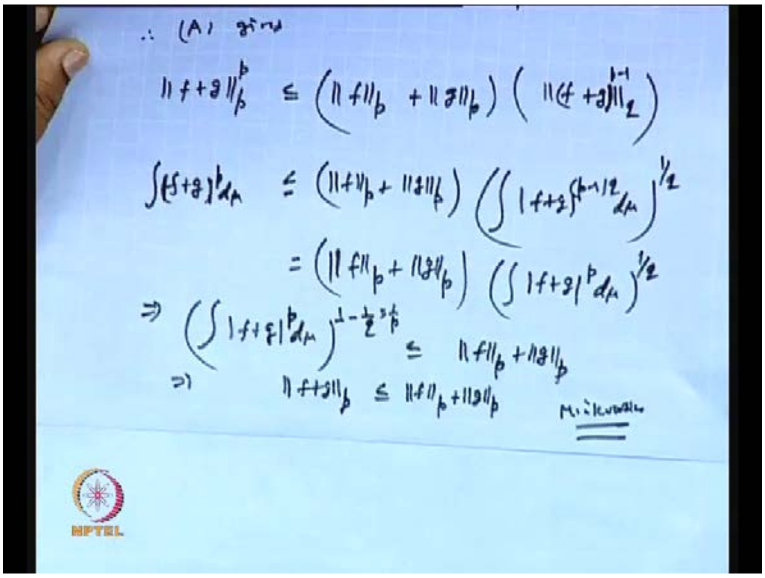

Similarly, for the second part– similarly, for second integral– second part is the mod g mod f plus g p minus 1 d mu. This is less than equal to norm of g p into norm of f plus g power p minus 1 q.

Therefore, A gives this thing– norm of f plus g power p is less than equal to norm f p plus norm g p, and within bracket, and inside, we get outside norm f plus g power p minus 1 power p minus 1 into norm divided by norm q. That is all; p minus into norm q is total.

Now, this will be equal to, when we take this one as a power p. So, here you are writing this p minus 1 q is p. Now, what you are writing here– this part– is this is…, now, if I open this part, what you get p minus 1q, and raised d mu, and raise to the power 1 by q, is it not? Raise to the power 1 by q. Now, this is equal to p, and then, now, p minus 1 into q is p– p minus 1 into q is p– that we have already seen.

So, this will be equal to p minus 1 is (( )). So, this will be equal to integral p d mu power 1 by q, and what is this? This is, basically, what? Integral f plus g power p mod of this power p, is it not d mu? That is all, because if power p gets cancelled, so you are getting this. So, this is less than equal to this.

Now, here this will carry. So, if we when take it from here, then we get integral mod f plus g power p d mu raised to the power 1 minus 1 by q is less than equal to norm f p plus norm g p, ok? And now, 1 minus 1 by q is 1 by p; this is 1 by p. So, basically, this is the norm of f plus g p, which is less than equal to norm f p plus norm g p, which is the Minkowski inequality. So, this completes the idea of the Minkowski inequality, ok? Then again, the equality holds when p is 1, and otherwise, that part, so, that can be seen. So, this will be 8, 9, 10.

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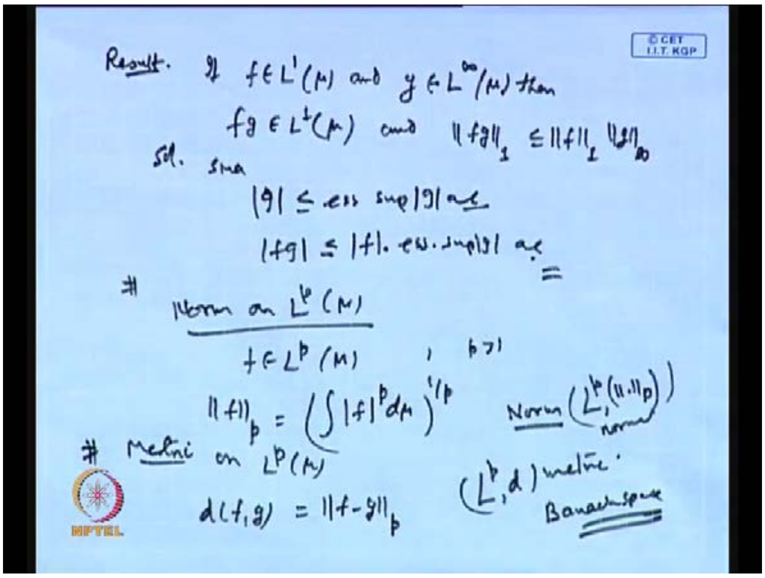

Now, there are some results– more, which is interesting. The result is– if f belongs to L 1 mu and g belongs to L infinity mu, then f g belongs to L 1 mu, and norm of f g under the L 1 norm is less than equal to norm f 1– f under L 1 norm– into norm g under L infinity norm. The reason is simple, because– because this is mod g is less than equal to essential sup of mod g almost everywhere. Therefore, mod of f g is less than equal to mod f into essential supremum of mod g almost everywhere; hence, when you take the integral, you get the….

Now, the lastly, we say we can introduce the concept of the norm on L p mu as– if f belongs to L p mu, then the norm of f p is defined as the integral f power p d mu power 1 by p, when p is greater than 1. This is the norm, and it satisfies all the condition of the norms one can verify using the Minkowski inequality. You can prove the triangular infinity, and further, the metric on L p can be introduced, also– d of f g is equal to norm of f minus g L p norm, and all the property of the metric is a. So, capital L p, under this, becomes a metric space. Also, capital L p under this norm becomes a norm space, ok?

And in fact, it is a Banach space that one can prove it; that is all. Thank you very much. Thanks.