Functional Analysis
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Lecture No. # 39

L ^{P} - Space

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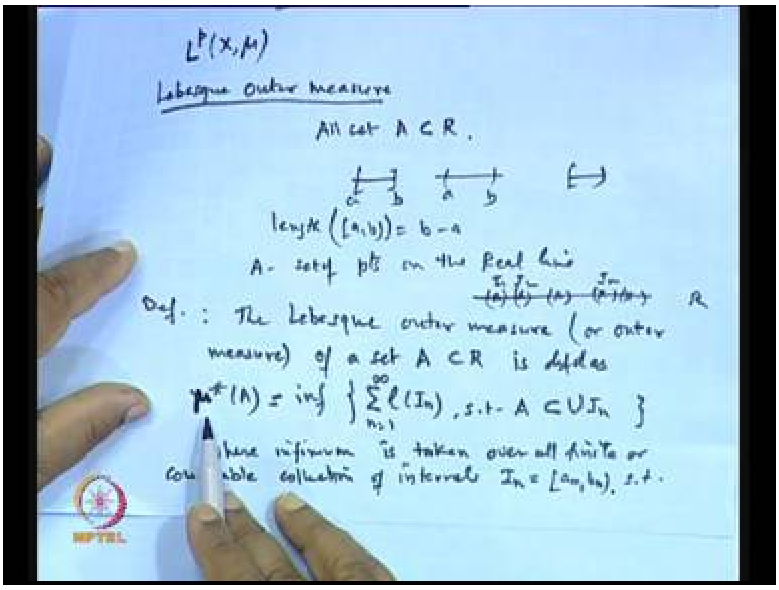

So, today we will discuss the space L p (x, mu) where x is an arbitrary set and mu is a major value. Now, this space L p consists of the measurable functions, which are p eth integral. So in fact, it requires the knowledge of the little bit about the measure theory. What are the measurable functions? What is the measure? How to introduce the concept of their sigma-algebra? These all these things we will require to introduce this ... In fact before going, let we see the concepts which are needed for this space to be discussed.

The first concept which we wanted to introduce is the Lebesgue measure-Lebesgue outer measure and Lebesgue measure, Lebesgue outer measure. Now, here we will deal all the sets. Whatever we deal, they lie in a real line or they are the part of the real line R, real line R. So, when we say the interval a b, whether it is a semi closed interval or closed interval or maybe, half closed and open interval like this. Then the length of the interval is nothing but, the difference between b minus a. Now if the set a, if the set of points on the real line real line, but they are scattered point say this is real line R, points are somewhere here like this. So, connection of these points is suppose a. We wanted to know the corresponding length of this or the measure of the set a.

So, the idea of the measure is the generalization of the concept of the length of an interval, whether it is open, closed, or semi closed interval. So, in order to get the measure of the set a, we introduce first the concept of the Lebesgue outer measure and then slowly, we will go for the measure for the set. So, what we do is here? That the outer measure, we define the Lebesgue outer measure of a set as follows: the Lebesgue outer measure or simply an outer measure, outer measure of a set a, of a set, say A, which is a part of a real line. Set A is defined as mu star, let it be mu star.

A is the infimum of sigma, length of I n; n is 1 to infinity, where the infimum is taken over infimum is taken over all finite or countable or countable collections of intervals intervals I n, which is of the form say a n, b n type, this type a n, b n type. Such that the countable union of this, such that the countable union of I n's covers A, such that countable union of this one covers... So, what we do is, we are enclosing the points of the set a by means of the intervals I 1, I 2, I n, and so on. Finding out the length of these intervals, taking the sum then change the length of intervals, again I 1, I 2, as replaced by I 1 days, I 2 days, etcetera, and again find out the sum of the length of this. Choose the infimum value where infimum is taken over all such possible intervals. Then if this infimum exists, we say it is the outer measure of the set A.

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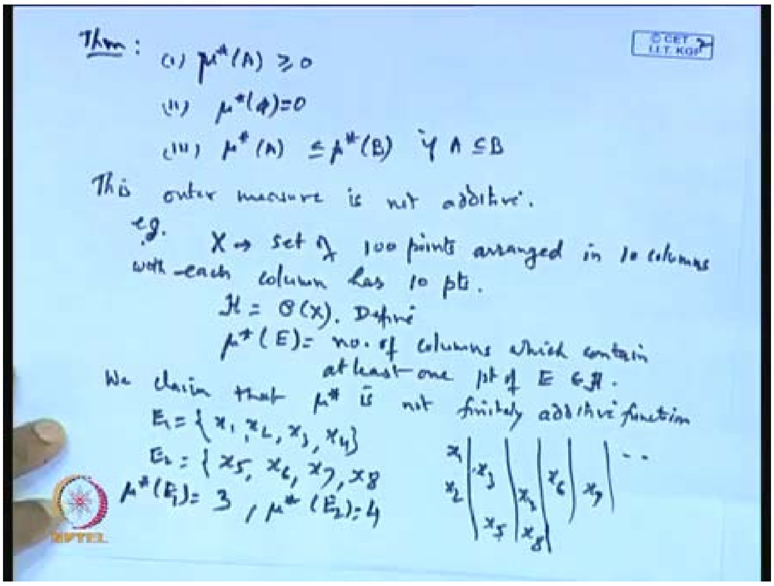

Now, this outer measure satisfies the following conditions: The result is our theorem, the outer measure mu star of A will always be non-negative. The outer measure of an empty set is 0. Outer measure of a set A is less than equal to outer measure of a set B, if A is contained in B; this always be satisfied. Now, this outer measure is not an additive function. In fact, it is a sub additive function. The outer measure outer measure is not additive even countable, finitely additive is not there. For example, if we take this set X. As a set of X, as the set of hundred points arranged in ten columns, arranged in ten columns such that each column each column has ten points each column has ten points with each column ten points.

And let suppose H is the power set of X plus all the subsets of X. Define mu star of E as the number of the columns, number of columns where number of columns, which contains at least one point of E which contain at least one point of E, where E is an element of H E is an element of H. Now, this mu star E clearly satisfies this condition; mu star A greater than equal to 0, mu star phi will be 0; mu star A is contained in mu star B and so on. So, this is an outer.

This satisfies the all the three conditions, but we claim that mu star is not finitely additive function. Because, if we chose the two sets E 1 and E 2, let E 1 is suppose x 1, x 2, x 3, x 4; E 2 is suppose, this is suppose these are the ten columns. So, E 1 element lying here x 1, x 2, x 3, say x 4 and E 2 is suppose x 5, x 6, x 7 where the x 5 is here; x 6 is here; x 7 is somewhere

here, and x 8 let it be here. So, these are the some ten columns where the elements E 1, E 2 lie in this column.

Now, if we take the mu star of E 1, then mu star of E 1 has occupied basically three columns one, two, and three, because x 1, x 2, x 3, x 4 lies in three columns. So, this is 3 mu star of E 2. The elements x 5, x 6, x 7 it lays between one, two, and three, four so, this is the four columns are required to cover the element of this. If I take the E 1 union E 2, then what we get is E 1 union E 2 is x 1, x 2, x 3, x 4, x 5, x 6, x 7, x 8 and their intersection is empty, because x 1, x 2, x n, they are different elements; these are all different, x i is not equal to x j.

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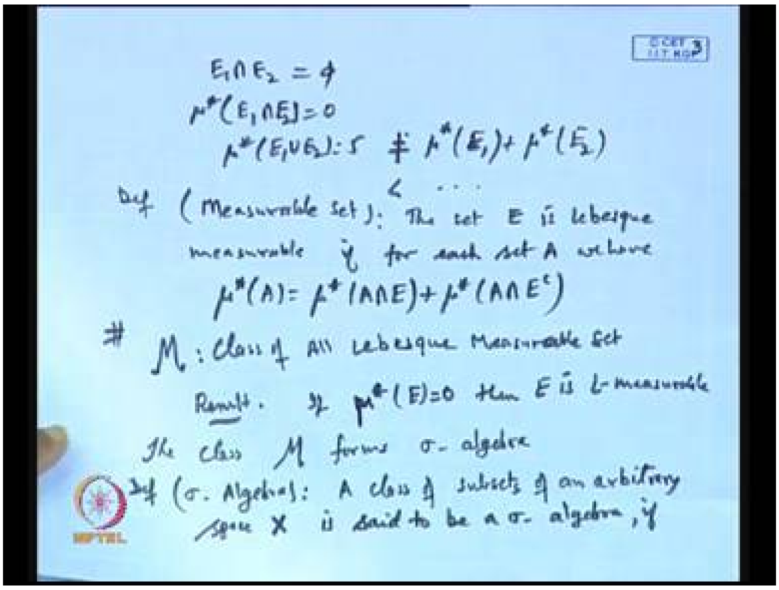

So, if we take this, then E 1 intersection E 2 is empty, they are disjoints and E 1 union E 2 requires one, two, three, four, and five columns. So, measure of this outer measure of this is E 1 intersection E 2, this is 0 while the mu star E 1 union E 2 is 5, which is not equal to the mu star of E 1 plus mu star of E 2 large then, it is strictly less than it is strictly less than this value. So, in spite of the E 1 E 2 are disjoint, the mu star of E 1 union E 2 is not equal to the mu star E 1 plus mu star E 2.

This shows that mu star is not finitely additive function. So, though we have introduced the concept of the outer measure, which is the Lebesgue outer measure, but this has a drawback that this is not an additive function. So, what we are interested in identifying those elements of X, those outer measures Lebesgue measure functions which decompose the each set in such that outer measure becomes additive, and those sets we call it to be a measurable sets.

So, we define the measurable set, the set E is Lebesgue measurable or simply a measurable set.

Lebesgue measurable for each set A, if for each set A for each set A we have mu star of A equal to mu star of A intersection E plus mu star of A intersection E compliment. Then such a set E is said to be a Lebesgue measurable set. And the class of all Lebesgue measurable sets Lebesgue measurable sets, we denote this by capital M; class of all Lebesgue measurable sets. And it is very easy to show that if a set whose outer measure is 0, then this set E is Lebesgue measurable, Lebesgue measurable sets.

And this class of Lebesgue measurable sets forms sigma-algebra, the class M forms sigma-algebra. What is the meaning of sigma-algebra? We define the sigma-algebra as a class of subsets, class of subsets of an arbitrary of an arbitrary space X is said to be is said to be a sigma-algebra sigma-algebra, and if X belongs to if X belongs to the class.

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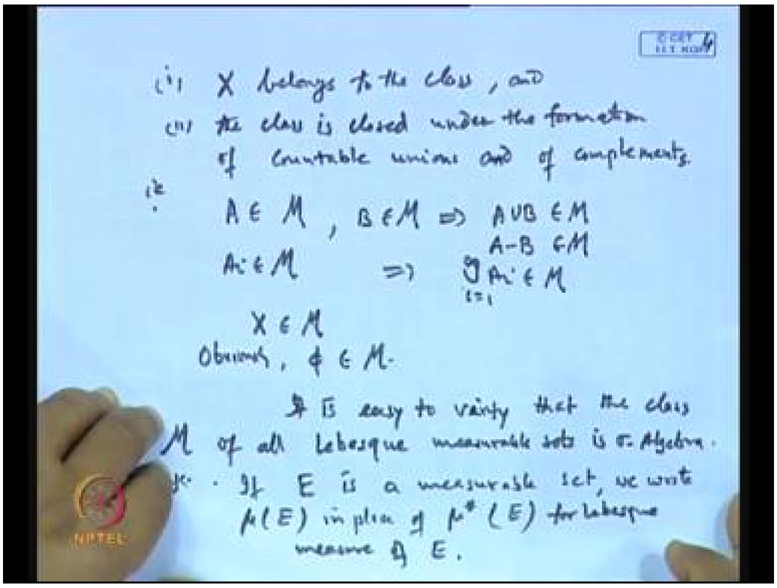

And second one is and second one is, if X belongs to the class and the class is closed under the formation of countable union and of compliment. And the class is closed under the formation of under the formation of countable union's countable unions and of compliment compliments. That is a class is said to be sigma-algebra, if A belongs to the class say I am taking M here; B belongs to the class M. Then, A union B must be in M; A minus B must be in M. And not only this, if A i belongs M, then countable union of A i is… i is 1 to infinity must be in M, and further the entire space X should also be in M.

So, obviously empty set phi is obviously in M. Now, it is easy to verify that empty sets, they are Lebesgue measurable sets; X is also Lebesgue measurable sets. And if A, B is there, then A union B will be the A minus B will be available in Lebesgue measurable set and countable. So, Lebesgue measurable set, class of Lebesgue measurable set M is basically a sigma-algebra. So, it is easy to verify. I am just writing I am not giving the complete proof. It is easy to verify that the class M of all Lebesgue measurable sets is sigma-algebra.

So, this concept we will require here the concept of the sigma-algebra. Now as a remark, we will just put it here. If E is a measurable set if E is a measurable set, then we write, we write mu of E in place of mu star of E for the Lebesgue measure of E, for Lebesgue measure of E. That is our concept on the sigma-algebra of measurable set and this we use. Now, every interval is a measurable sets so, that is nothing to here prove it.

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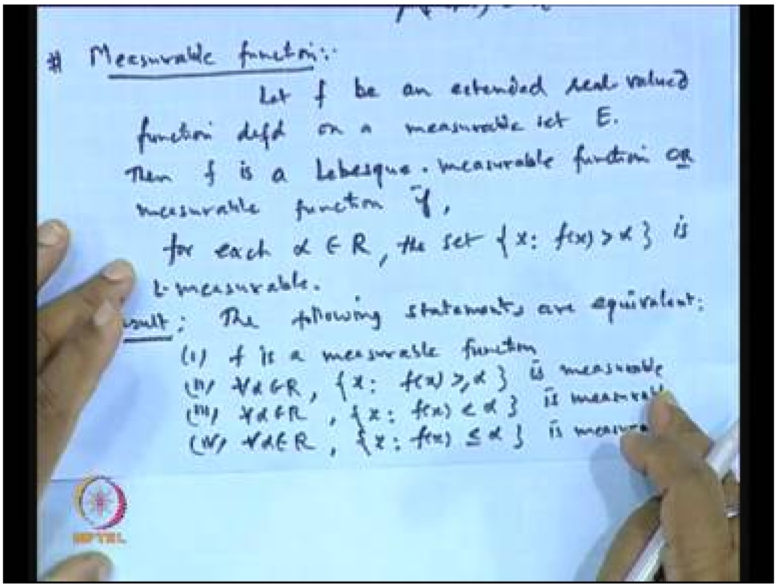

Let me take an example here. Suppose, we consider the class X as a class of finite union of the interval of the form, semi closed interval (a i, b i). If we consider the class of finite union of (a i, b i), then it can be shown that it forms a sigma-algebra. All these properties will be satisfied and then of course, mu of (a i, b i) that is the measure we can write it b i minus a i. So, this is example we can get it and so on. This forms the Lebesgue measurable that algebra– sigma-algebra of the sets.

Now, there is another concept which we need it now, the concept of the measurable functions. I am going in very short, just reviewing the whole thing, because this is a separate discipline measure theory, where you will see in detail.

But this terminology we will use in establishing capital L p as a Banach space as a normed space. So, to show this one, we require these concepts of measurable function, integration of the p eth integral functions and like this. So, that is why we require this. The measurable functions, let us introduce this measurable function as let f be an extended an extended real valued function defined on a measurable set E, measurable set E. Then, we say f is a Lebesgue measurable function, Lebesgue measurable function or simply a measurable function, simply a measurable function; Lebesgue measurable function simply a measurable function. If for each alpha, for each alpha belongs to R is a real, the set X such that f(x) is greater than alpha is measurable is Lebesgue measurable is measurable.

So, what we see here is a function f is said to be a Lebesgue measurable function. If the corresponding set, this set of those points where the f(x) is greater than alpha for each alpha is a measurable set. So, this set is measurable means, it decomposes any other set into a two disjoint sets. So, that the outer measure becomes additive. That if I take this set to be say E, then M star of A equal to M star of E, A intersection E plus M star of A intersection E compliment. If it satisfied M star or mu star, then this set will be a Lebesgue measurable set. So, in order to show the function to be Lebesgue measurable function, we have to show the corresponding set is a Lebesgue measurable set.

And this definition is equivalent to the following conditions are equivalent the following statements, statements are equivalent. The first is let us said f is a measurable function. It means this set is a Lebesgue measurable set is measurable or Lebesgue measurable. So, second condition is that for every alpha belongs to R, the set where the f(x) is greater than or equal to alpha is a measurable set.

Third is for every alpha belongs to R, the set X such that f(x) is strictly less than alpha is measurable set. And fourth is for every alpha belongs to R, the set X such that f(x) is less than equal to alpha is measurable set. So, all these conditions are equivalent. So, in order to test the function f to be measurable, we can prove any one of the set, if it comes out to be a Lebesgue measurable set. Then the corresponding function will be a measurable function, Lebesgue measurable function.

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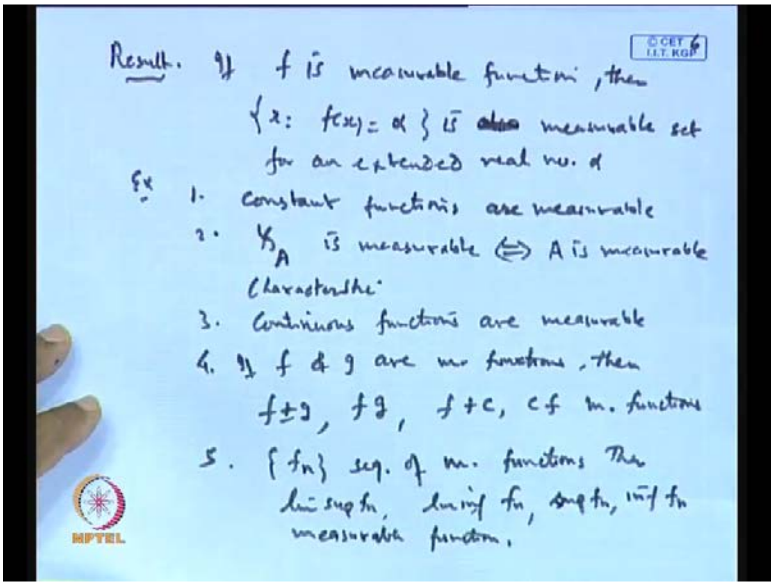

Further, it can be shown easily that a function, if the function f is measurable. If f is measurable function, then the set of those points, where the f(x) becomes constant alpha is also a measurable function is also, is a measurable function is a measurable set. This is a measurable set for any extended real number alpha for an extended real number alpha. So, even the set F where the f(x) becomes constant. These are all measurable sets, if f is a measurable function. Every constant function is measurable, there was an example. The constant functions are measurable functions, and then characteristic function psi of A characteristic function, characteristic function of a set A is measurable is measurable if and only if A is measurable. This can be as this shown so, we are not good.

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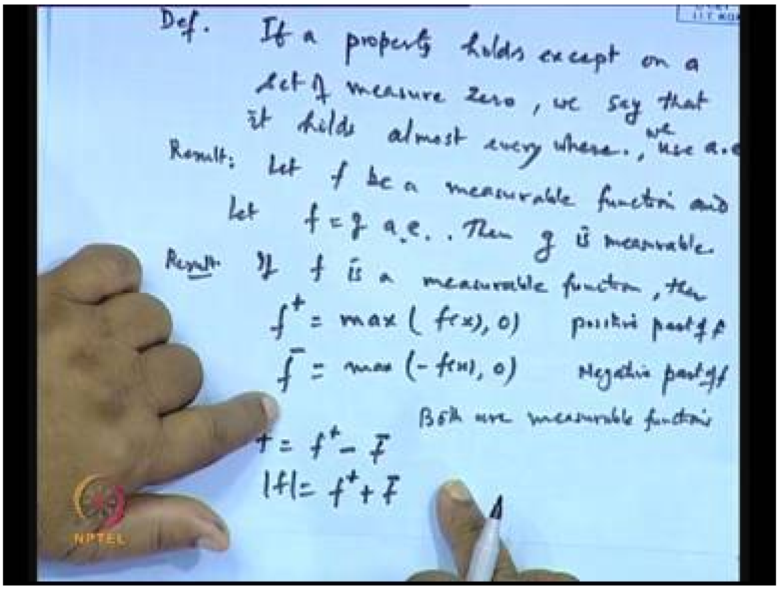

Continuous functions are measurable, other examples are measurable functions. Similarly, satisfy the all the algebraic property say, if f and g are measurable, their addition, subtraction, f plus g, f minus g, f plus constant g; these are all measurable functions. So, if f and g are measurable functions, then f plus minus g are measurable, f g is measurable, f plus c is measurable, c f is measurable. These are all measurable functions. Similarly, if a sequence f n is a sequence of measurable functions, then the limit superior f n, limit inferior of f n, supreme of f n, infimum of f n; these are all measurable functions.

So, class of measurable functions is a very big class and we wanted to make use of this class, and introduce the concept over L p space over this class. So, that is another concept, which we need is almost everywhere. If a property, if a property holds if a property holds, except on a set of measure zero, measure zero then we say that it holds, it holds almost everywhere almost everywhere. And we use the abbreviation a dot e almost everywhere. Now, the result one result we will make use the result is let f be a measurable function measurable function and let f equal to g almost everywhere. Then g is a measurable function means, if a function g which is equal to f almost everywhere that is except the point, where the function differs from g and that set forms a measure zero. Then, we say f equal to g almost everywhere.

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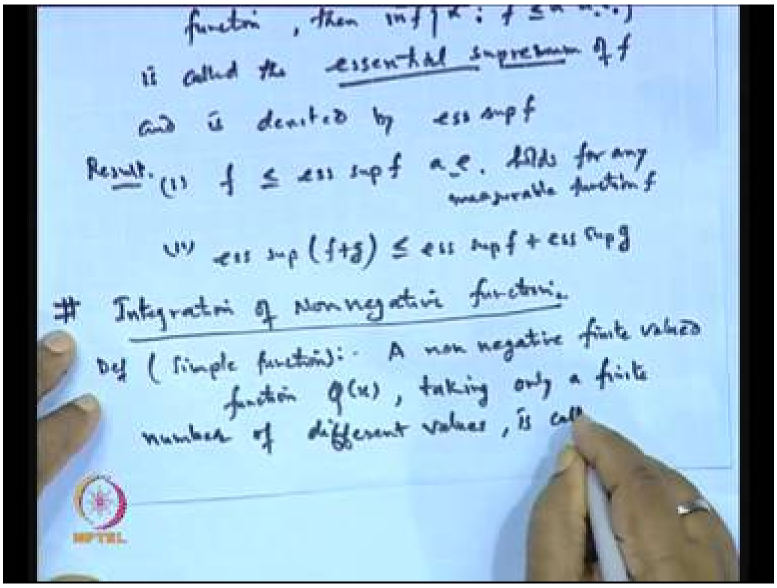

So, such a function g will also be measurable, if f is measurable. So, any function which is equal to g, almost everywhere almost equal to a measurable function, almost everywhere will be a measurable function will be a Lebesgue measurable function. Then, another result, which we need also suppose f be a measurable function measurable function. Then, the positive part of f, which we denoted by f plus and it is the value f(x) and 0. And the negative part of this, which is denoted as maximum of (-f(x), 0).

Then the positive and negative part; positive part of f, this is the negative part of f. Both are measurable function; both are measurable function positive, negative part. The advantage of writing this is that if f is any arbitrary function, then we can write it f as f plus minus f minus. And mod of f we can use it, f plus f minus, and f plus and f minus both are non-negative; both are non-negative functions. So, we can make use of this. Another concept is essential supremum. Supremum if let f be a measurable function, then the infimum of alpha where f is less than equal to alpha almost everywhere is called, is called the essential supremum, essential supremum of f.

And we denote this by and denote, and are denoted by essential supremum of f like this. Now in this, we have one or two results. The first result is for any measurable function, f will always be less then equal to essential supremum of f almost everywhere this holds, for any measurable function f. The second result which we required also, the essential supremum of the sum f plus g is less than equal to essential supremum of f plus essential supremum of g. The similarly, we can have a concept of the essential infimum concept and that will be...

Now, once we have introduced this, then we require the concept of the integration. The concept of the integration is integration of functions of a non-negative functions non-negative functions, infinite level. So, here, first we will see the simple function. Simple function is a non-negative finite value, a non-negative finite valued. Function phi x, a non-negative finite valued function phi x taking only a finite, only a finite number of different values, different values is called a simple function, Simple function is called a simple function.

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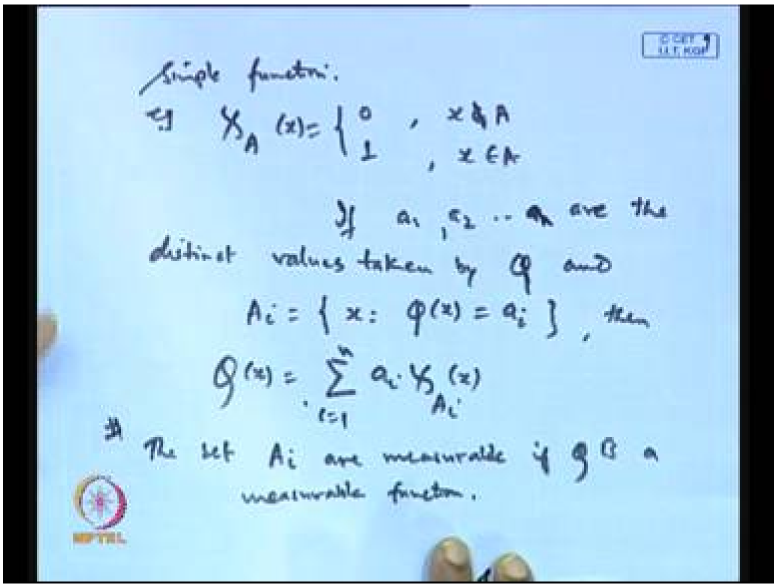

For example, a characteristic function psi, the chi of A has a value 0 and 1 only if x does not belong to A and when x belongs to A, the value is 1. So, such a function is a simple function, because it has the only the finite number of values, that is, only two values is there 0 and 1. So, in generally, if suppose a 1, a 2, a n these are the values are the values are the distinct values taken by the function phi, by the function phi. And suppose A i is the set of those point where the phi attains the values small a i. That is we are separated out the where the point phi x attains the value a i.

So, a 1 is the set of those points where the phi attains the value a 1; a 2 is the set of those point where the phi attains the value a 2, and like this. So, a 1, a 2, a n are distinct values. Then, we are getting the corresponding sets a 1, a 2, a n which are also be disjoint sets. So, a 1, a 2, a n, then…Hence, phi can be written as, then phi of x can be easily be written as in the form of the finite sum i is 1 to infinity; a i characteristic function of A i x, because when x belongs to a i. The value of this comes out to be 1 and only a i will be there, rest will be zero. So, when x belongs to this phi of x is equal to a i; it (( )).

So, phi x can be written as a finite sum of this in the form of the simple functions psi of A i is form of the characteristic function. And this set A i, the set A i are measurable, if phi is a measurable function. Because this if you remember, we have already shown one result that if phi is a measurable function, then the set of those points, where phi attains the constant values will also be a measurable function set.

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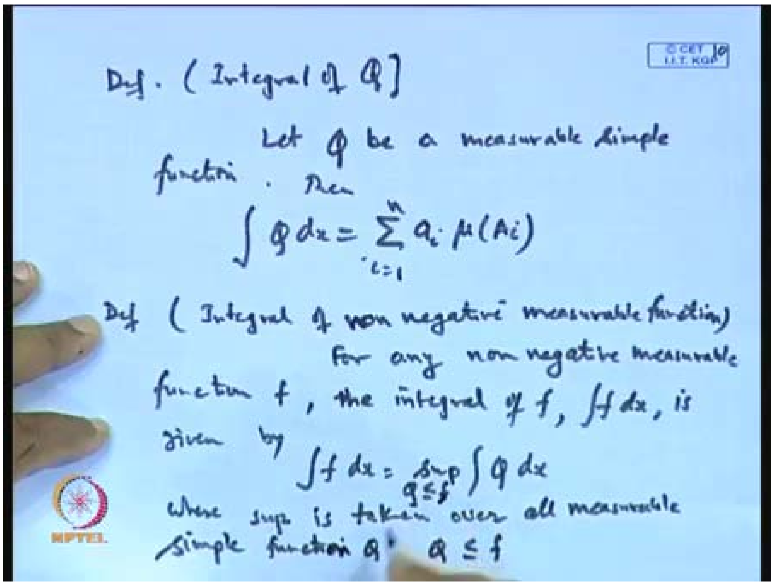

So, this set will be measurable if phi has is a measurable function. So, A i is are measurable, if phi is a measurable. So, now we take the integration introduce that of a simple function, integration of the simple function phi. Let phi be a measurable simple function simple function, then the integral of phi d x is defined as i is 1 to n, a i into measure of A i, because A i is a measurable set. If phi is a measurable function, A i is a measurable set. So, measure of the set A i can be computed and measure of A i. When multiply the corresponding values A i take the summation, it will give the value of the integral phi d x over the range.

So, where a i and A i is done. Then this we call it as a integral phi. Now, with the help of this simple function, we can now introduce the integral of a non-negative measurable function so, integral of non-negative non-negative measurable functions. So, for any non-negative measurable function, measurable function f, for any non-negative measurable function f, the integral the integral of f denoted as integral f d x is given by is given by integral f d x is the supremum of integral phi d x, where the supremum is taken, where the supremum is taken over all measurable all measurable simple functions simple functions, f measurable simple functions phi.

Such that phi are less than or equal to f, means supremum is taken over all phi, which are less than equal to or simple measurable function phi, which are less than or equal to f. That is the…. So, this is the way we introduce the concept of the integral, Lebesgue integral of the function f, when f is a non-negative measurable function. This we call it as a Lebesgue integral, Lebesgue integral of this.

Now, if the function is defined over the set E, then for any measurable sets, for any measurable set E, and any non-negative measurable function f, and any non-negative measurable function f any non-negative measurable function f, the integral of E f over the set E denoted as, integral f d x over the set E is nothing but, integral of f into characteristic function E d x. And this is as good as our saying integral of f d x. So, we will say this is the supremum over psi; psi is less than equal to f into characteristic function of E integral of psi d x. So, this way, we can introduce the concept of… and this integration we call as a Lebesgue integral of f over the set E, Lebesgue integral over the set E. Now here, we have taken the f to be non-negative measurable functions.

Now if suppose, f is not a non-negative measurable function is a general, general function, then we introduce the Lebesgue integral of the general function as… the Lebesgue integral of general function. So, let f be a measurable function measurable function f be a measurable function, need not be a non-negative need not be non-negative, need not be a non-negative function. And if, the positive part of this function is finite, negative part of this function is also finite. And then we say that f is integral, Lebesgue integral Lebesgue integral.

And the integral of f d x is integral f plus d x minus integral f minus d x. Now, both are non-negative, f plus and f minus both are non-negative. So, we can introduce this, we can write the integral f plus d x in terms of the supremum of a supremum of a simple function. So, this can be written as a supremum simple function; this can also be written as supremum simple function. Hence, integral of f will be well defined. Now, this even idea can be extended to an extended valued function, provided the one of them means, it should not be infinity minus infinity, this should be well defined. So, except that point, otherwise those points where it behaves and it must have a measure zero. So, we get this one.

Now, this is all about this measure theory part. And with the help of this, now we can introduce the concept of our L p space. So, let us see now the concepts L p space, because this is very much required for introducing the concept of L p spaces. So, let X, S and mu, this triplet be a measure space measure space. What do you mean by measure space? That, I will explain later, and p is greater than 0 and p n, we define capital L p x mu or simply capital L p mu, capital L p mu to be the class of measurable functions.

Here when we say the measurable means, it is a Lebesgue measurable function, functions f such that integral of mod f to the power p; the Lebesgue integral of mod f to the power p is finite. So, basically what we there, we are choosing those measurable functions, which are p eth integral. That is the Lebesgue integral of power p eth function mod f to the power of p is finite. With the convention that any two, any two functions equal, almost everywhere almost everywhere equal almost everywhere specify the same element, same element of capital L p. What do mean? It means that in the class L p, we are not choosing the functions; L p is a class of functions. Such that each class, in each class, the elements are equal almost everywhere.

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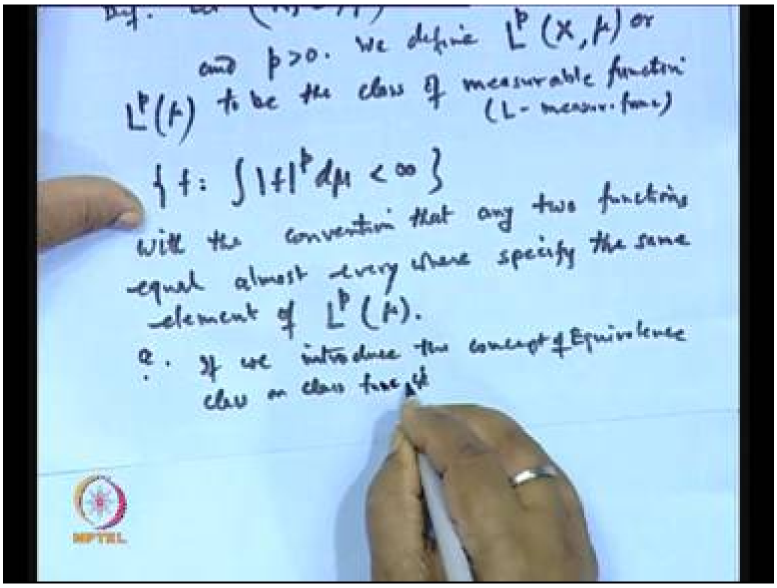

Say for example, if we take the zero element of L p mu, it means the class of those function where f equal to 0 almost everywhere. Then such a class, the collection of such class we are denoting by L. So, what we are doing is, we are taking capital x as an arbitrary set. S is a sigma-algebra, which we generated by a power set of X and mu is a measure on S. So, this triplet X, S, mu will know as the measure space. So, when we take the measure space, let f be a function defined on X, measurable sets.

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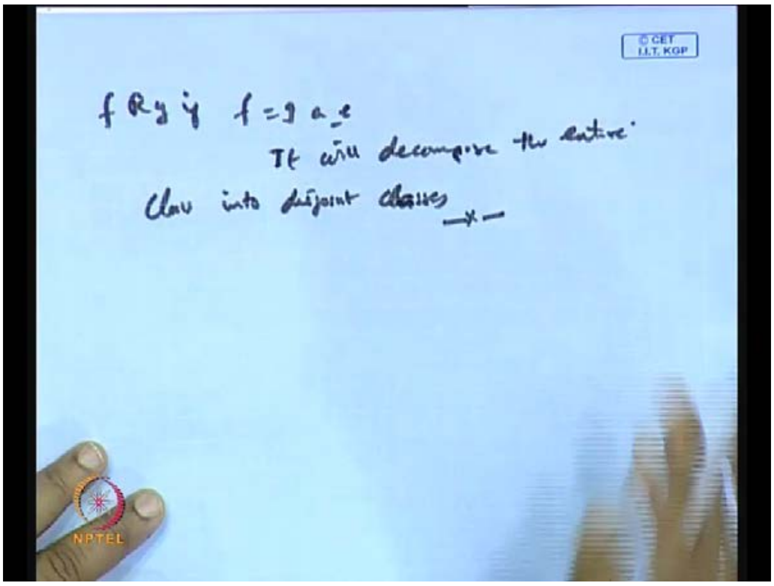

Then, this function measurable function, if it is p eth integral function, then we say this is an element of L p, means it belongs to one of the class of here. Because L p mu is the collection of the classes say, if we introduce the concept of the equivalence relation. The two elements f and g are equivalent, when f equal to g almost everywhere, f is related to g almost everywhere. Then, this will give decompose the whole class into equivalence classes. That is, that is if we introduce this concept of the equivalence class.

That is, if we introduce the concept of equivalence classes on f, on the class f. That is class of functions that is f is equal to g almost everywhere that is f is related to g. If f is equal to g almost everywhere, then it will decompose the entire class into disjoint classes disjoint classes and these classes will form the L p space. So, we will discuss it later on after next. Thank you very much. Thanks.