Functional Analysis

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Lecture No. # 38

Convergence of Sequence of Operators and Functionals

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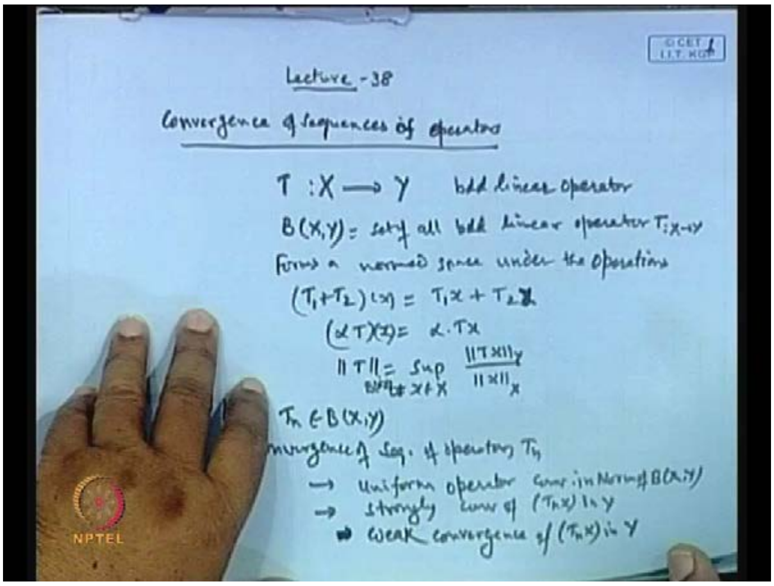

Yesterday, we have discussed the weak and strong convergence. Today, we will see convergence of sequences of operators. If X and Y are two normed spaces, then an operator T from X to Y, if it is a boundary linear operator, then the class of all such boundary linear operator is denoted by B of X, Y, a set of all boundary linear operators T from X to Y. And this class forms a normed space under the operations that the addition is defined as T 1 plus T 2 X is the T 1 X plus T 2 X (( )) alpha T X is alpha times of T X and then norm of T is defined as supremum of norm of T X over norm of X, where the X belongs to the domain of T capital X not equal to 0.

Now, this is the norm of Y, this is the norm of X and this is the norm of B X, Y. This is what we have. So, basically, when we construct the normed space with the help of these operators, then, we require the three norms. One is the norm in X, norm in Y and these two, together gives the norm of B X, Y. If we take any sequence belonging to this class B X, Y, then convergence of sequences of operator T n will depend on the norm whether we are taking norm of B X, Y or we are taking norm of T X or whether it goes to in the usual way in the norm of this. So, with respect to this norm, we can classify this convergence of sequence of operators in three categories.

These categories are; uniform operator convergence in the norm of B X Y, strongly convergence of T n X in Y and third is weak convergence of T n X in Y. So, we will have three categories for the convergence of the sequence, uniform convergence, strong convergence and weak convergence. Now, we will first define these concepts and then find out the relation whether 1 implies 2, 2 implies 2 or vice-versa is also 2.

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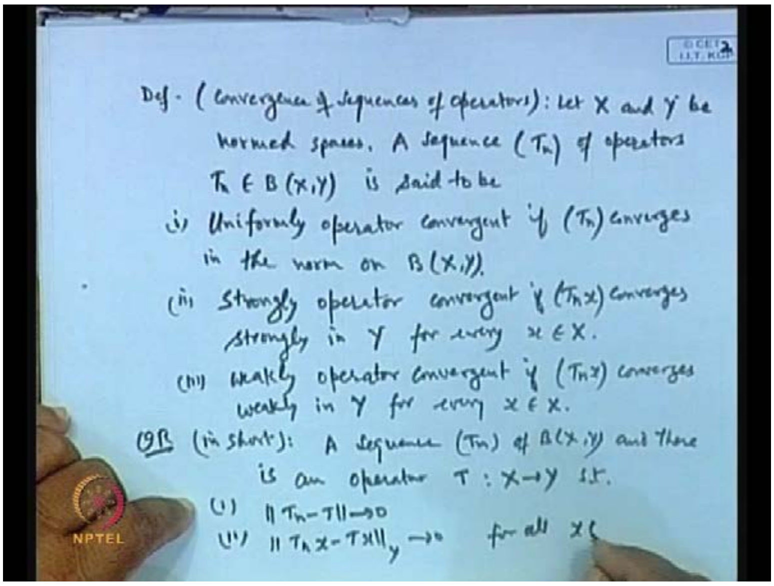

So, let us see how to define convergence of sequence of operators. Let X and Y be normed spaces. A sequence T n of operators, T n belonging to B X Y means bounded linear operator from X to Y is said to be uniformly operator convergent if sequence T n converges in the norm on B X Y, that is, the norm of T n minus T must go to 0 when n tends to infinity.

Then second is, this sequence T n is said to be strongly operator convergent if the sequence T n X, that is, the element norm in Y converges strongly in Y for every X in X because Y is a normed space. So, norm of Y is there. Therefore, this sequence converges strongly in y. So, in normed space, we have seen two, one is the strongly convergence, another one is the weak convergence. Strong convergence means, the sequence convergence to X under the norm norm of X n minus X goes to 0 and weak convergence means when f of X n goes to f of X for every f, then it said to be the weakly convergence sequence for f belongs to X Ts. Third is weakly operator convergent. Sequence T n X, this sequence converges weakly in Y for every X belonging to X. So, this is known as the weakly operator convergence.

Notation wise, we can write a sequence T n of B X Y, B X Y, a sequence T n belongs to B X Y and there is an operator T from X to Y such that one - norm of T n minus T tends to 0, second is norm of T n X minus T X that is the norm of Y goes to 0 for all X belonging to capital X.

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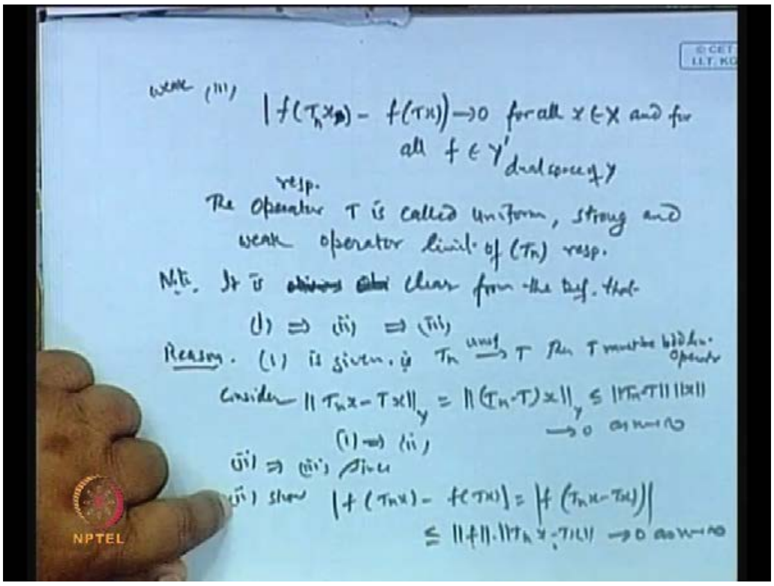

And third is modulus of f T X n minus f of T X tends to 0 for all X, belonging to capital X and for all f belongs to all f belongs to the dual space of y respectively respectively.

Then T is called the uniform. This is uniform.

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This is strong and this is weak. The operator T is called uniform, strong and weak operator limits limit of T n respectively. So now, we have seen the three ways of convergence of the sequence of beta, uniform operator convergent, strong operator and weak operator.

Now, there arises question that sequence T n is a bounded linear operator, but we have not told anything about T whether this T remains bounded linear operator, whether T is a point of X Y. What will be the nature of T in case of this uniform, in case of this strong or in case of the weak. So, we will discuss that whether T n converges to T uniformly, T has to be bounded linear operator or not and second, in case of the strong convergence, whether the limiting point T is bounded or unbounded. Similarly, it is for the weak convergent. This is the first case.

Second point is, what is the relation between these three? It is clear from the definition that 1 implies 2 and 2 implies 3. Every uniform operator convergent implies the strong operator convergent and strong operator convergent implies this. Why because suppose if uniform operator is given, it means that norm of T n minus T goes to 0. So obviously, the reason is 1 is given suppose, then what is to consider norm of T n X minus T X of Y. This is equal to norm of T n minus T X Y. Now, this will be less than equal to norm of T n minus T into norm of X. So, that will be T into norm of X.

Now, one thing which I point out here is that this is T. I think this is wrong which I did. It is T n X n. Please make the correction. T n X minus f of t X. Now, here one thing is, when 1 is given that is if the sequence T n converges to T uniformly then T must be bounded linear operator. Otherwise, this meaning of norm T n minus T does not carry any meaning because T n, if it is unbounded, the difference norm of T n minus T cannot go to 0. So, if T n is a sequence of the operator that converges in the norm of T uniformly, their limiting point must be a bounded linear operator. So, T n minus T becomes bounded. Therefore, by definition of the bounded operator, norm of T X is less than equal to norm T into norm X. Now, this goes to 0 as n tends to infinity. Therefore, this will go to 0. So, it implies it. So, first implies second.

Now, second implies third. What is second? second Second shows that mod of T n X minus f of T X. This is equal to f of T n X minus T X mod of this f is giving to be a bounded linear functional on Y. So, mod of f is less than equal to norm of f into norm of this. Now, this is giving to be 0, 10 into 0, this is bounded. So, it goes to 0 as n tends to infinity. So, second implies third. So, first implies second, second implies third is obviously true.

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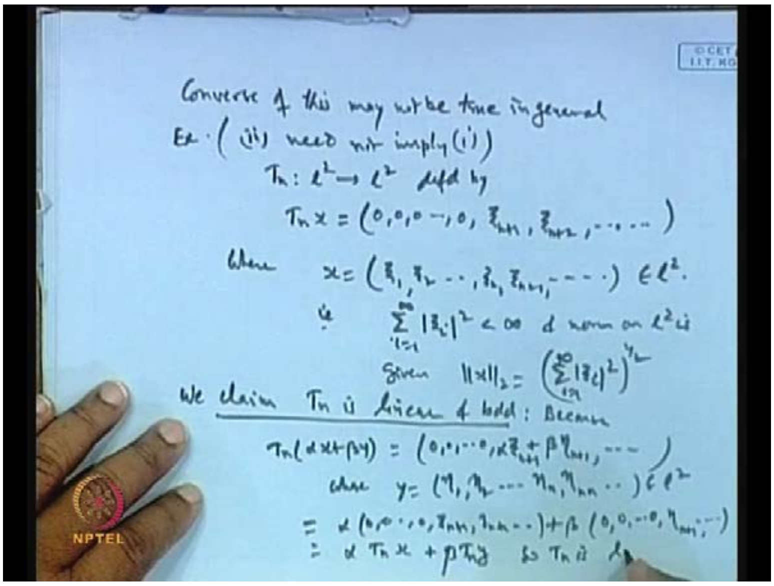

In general, the converse of this may not be true. Let us take the counter example first. Let us see first example. The sequence converges strongly, second need not imply first. This is an example for this. It means that a sequence of the operator T n converges strongly, but not uniformly. So, Let us take T n as a sequence of the operator from l 2 to l. Let T n be a sequence of the operator from l to l defined by T n X is 00 0 0. Then, xi n plus 1 n plus 2 and so on where X xi 1, xi 2, xi n, xi n plus 1 and so on. This is an element of l 2.

Now, we know that L2 is also a Hilbert space and norm of l 2, we know this l 2 X belongs to l 2 means that is sigma of mod xi I whole square, I is 1 to infinity. This is finite convergent and norm on l 2 is given as norm of X 2 is sigma xi I square power half. This is the norm on l 2. So, under this norm, we can say T n X is a point in this.

Now, the operator T n which is defined above, we claim that this operator T n is linear and bounded. Why because, T n alpha X plus beta Y that will be by definition 0 0 0 xi and alpha times xi I n plus 1 beta times say eta n plus 1 and so on, where y is equal to eta 1, eta 2, eta n, eta n plus 1 and so on belongs to l 2 and X is our xi 1 xi 2 belongs to l 2. So, alpha X plus beta means, this will come. Now, this can be written here as alpha times 0 0 0 xi n plus 1 xi n plus 2 plus beta times 0 0 0 eta n plus 1 and so on and that will be equal to alpha times T n X plus beta times T n Y. So, T n is linear.

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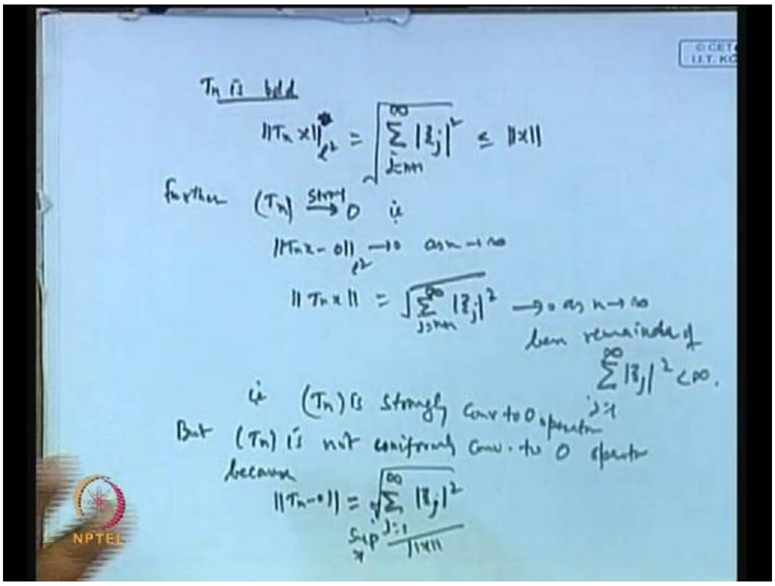

T n is bounded also. Why it is bounded, because what is the norm of T n X? This is the norm of l 2. So, by definition, this will be sigma mod of xi j j equal to n plus 1 to infinity under the norm. So, let it be square. Now, this will be less than equal to norm of X 2 because this is power under root half. So, you can take this. Let it be removed. Now, j equal to 1 to infinity. So, T n is bounded. Therefore, T n is a sequence of the operator which is linear and bounded.

Further, sequence T n converges to 0 strongly. This sequence converges to 0 strongly. That is norm of T n X minus 0 under l 2 must go to 0 as n tends to infinity. What is the norm of T n X T n X minus 0?It means, this is equal to under root sigma j equal to n plus 1 to infinity mod of xi j square.

Now, this is a series after n th term. So, it is a remainder of a convergent series. So, this goes to 0 as n tends to infinity because this is the remainder of the series sigma mod xi j whole square j equal to 1 to infinity which is convergent.

Therefore, it will go here. So, T n X converges to 0 under this norm. It means T n sequence is strongly convergent to 0. This is the 0 operator operator, but this T n sequence is not uniformly convergent to 0 operator. It is not uniformly convergent because what is the norm of t n minus 0 operator. Norm of t n minus 0 means j equal to 1 to infinity. What is this? By definition, this will be equal to norm of T n X. Let me write this thing supremum over taken whole x.

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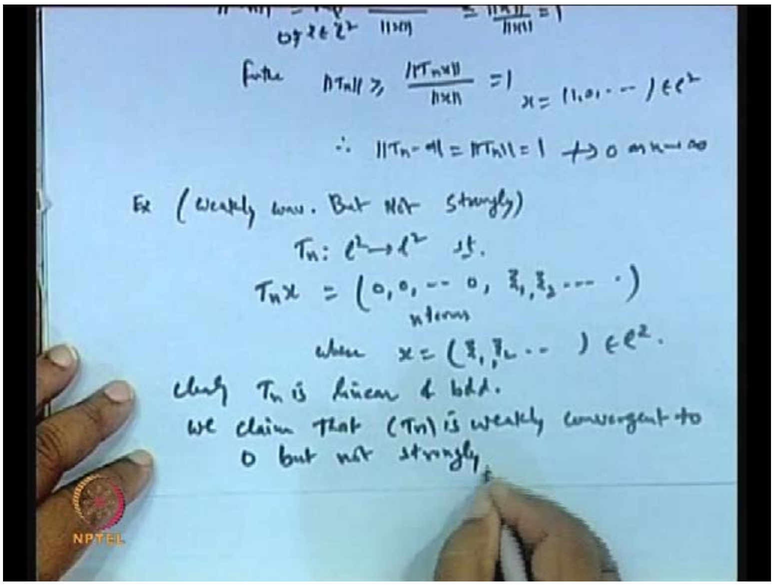

Norm of T n is the supremum of norm of T n X over norm of X, X is belonging to our l, 2 space not equal to 0. Now, T n X is this. So, this is less than equal to norm of 1, norm of X, norm of X that is 1; supremum of this thing is less than equal to 1.

Now further, norm of T n X T n is greater than equal to norm of T n X over norm of X for particular X. If I take X to be 1 0 0 0, this is an element of l 2. So, what is this norm? t n X will be 1, norm of X will be 1. So, it means norm of T n minus 0, this is coming to be norm of T n which is 1. So, it cannot go to 0 as n tends to infinity. Therefore, this sequence does not converge uniformly. So, this is an example where the sequence of the operator converges strongly, but not uniformly. Now, second example when the sequence of the operator convergent weakly, but not strongly.

Let us take the example. So, define an operator T n from l 2 to l 2 such that T n X T n X equal to 0 0 0. This is up to n terms; and then xi 1, xi 2 and so on. X belongs to e 2.

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It is a set operator and n digits are separate. Now, this operator is clearly linear. T n is linear and bounded. I think there is nothing to prove here. We claim thus that this sequence T n is weakly convergent operator convergent to 0, but not strongly. This operator converges weakly to 0 operator, but not a strong convergent.

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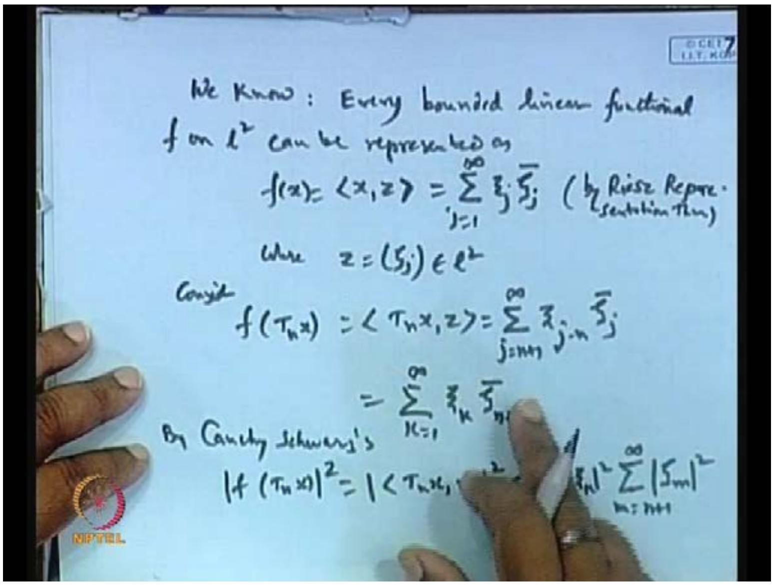

We know that every bounded linear functional f on l 2 can be represented as in terms of the inner product f X has inner product of X Z is uniquely determined by f and norm of z equal to norm of f and it will be in l 2. So, the inner product will be I j is equal to 1 to infinity xi j zeta j conjugate, where z is zeta j is an element in L 2. This is by Riesz representation theorem. This can be written as j equal to 1 to n and so on. Let us take f of a. So, consider f of t n X. This is by definition inner product of T n X Z, but T n X is shift operator. We shifted the point first position to n plus 1 th position. So, it means the first few n terms will be 0 basically. So, we can start with this n plus 1 j n plus 1 to infinity.

Then, xi 1 will be multiplied by zeta n plus 1. So, we can write this thing as j n plus zeta zeta j minus n and xi j minus n into zeta j bar because when j equal to n plus 1 this xi 1 is multiplied by zeta n plus 1 conjugate. So, this can be written as sigma k equal to 1 to infinity xi k zeta n plus k conjugate.

Now, apply the Cauchy Schwarz inequality. So, by Cauchy Schwarz inequality, modulus of f T n X square will be equal to modulus of inner product of t n X Z square and that will be less than equal to sigma of mod xi k square. k is 1 to infinity into sigma say here, m to m is n plus 1 to infinity mod of zeta m whole square. By Schwarz inequality, modulus of this thing is less than equal to mod of this, all is called raise to the power half into mod of this. So, half is out now. This will be less than equal to… Now, we claim that, this goes to 0 as n tends to infinity, why because, this is finite, this part is the norm of X and this part is the remainder of a convergent series.

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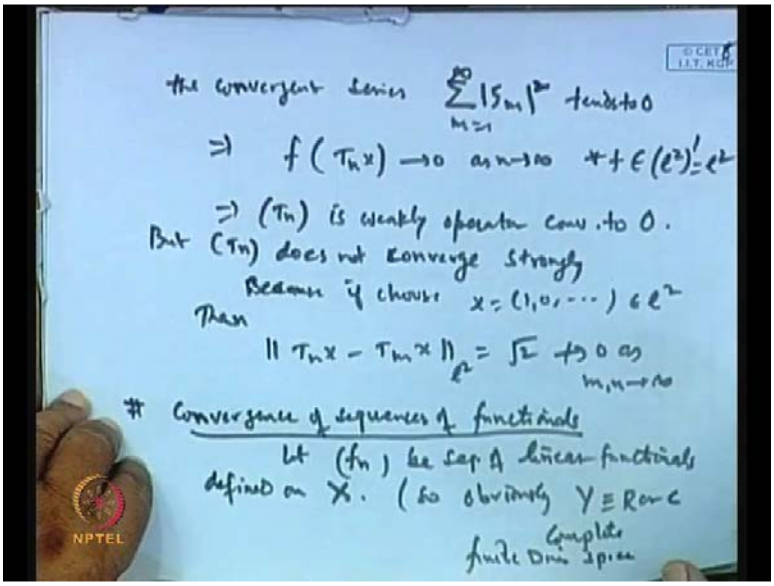

So, it will go to 0 as the remainder of the convergent series sigma mod zeta m is 1 to infinity, remainder of this convergent series. This convergent series tends to 0. So, it will go to 0. Therefore, this implies f of T n X tends to 0 as n tends to infinity and this is true for every f belongs to the dual of l 2 dash; that is l 2 dual of l p is l q.

Therefore, this sequence T n X will converge to 0 weakly. Therefore, sequence T n is weakly operator convergent to 0 operators, but this T n does not converge strongly. Why because norm of T n is 1, norm of T n X is not strongly convergent, because if we choose X to be 1 0 0 0, which is in l 2; what is the norm of T n X minus T m X. This is norm of l 2. is It is not under root 2, because this difference will be 1 comma. It will go to root 2, does not tend to 0 as m n goes to infinity. So, this sequence T n is not a Cauchy sequence. Therefore, it cannot be convergent.

This is not a convergent sequence, because if it convergent, it has to be Cauchy. So, we contradict the theme, therefore, this term. So, this is an example where it is weakly convergent to 0, but not a strongly convergent. Same thing will be continued if I replace operator by a functional, because, when we say T n is a sequence of the operator from X to Y. Then, both Y and X are normed space.

When we replace Y by r over c, then this sequence operator shows to be a functional. So, same type of criteria is there. Let us see convergence of sequences of functional. Let f n be a sequence of linear functionals defined on X. Since they are functional, obviously, Y is r or c because it maps the point X to the real or any scalar quantity complex number and Y n r and c they are complete metric space. So, if they are complete in that case, r and c is also dimension of for one. So, these are the finite dimensional space.

If they are finite dimensional space, then weak and strong convergence is the same. Therefore, the concept of the three types of convergence which we have earlier in case of the operators, this concept is uniform, strong and weakly operator convergent. So, second and third will coincide. We will not get these two different things. What we get is uniform convergence and one of them is weakly convergent, because weak and strong convergence are identical. Convergence converges in the limit strongly and weakly. So, here we will instead of weakly, we say weakly star convergent . So, now, we define the concepts as…

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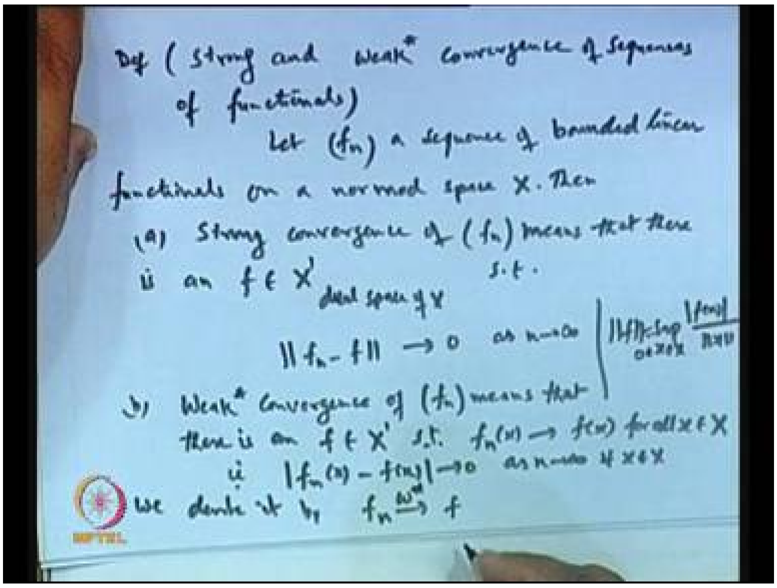

Now, we define the concepts as: strong and weak star convergence of sequence of functional. Let f n be a sequence of bounded linear functional defined on a normed space X. We say a strong convergence of f n means that there is an f belonging to the dual space of X such that norm of f n minus f tends to 0 as n tends to infinity.

What is the meaning of norm here? The norm means norm of f is supremum mod f X over X belongs to the domain of this. So, X belongs to domain X non 0. So, this goes to 0 in the norm and weak star convergence of f n means that there is an f belongs to X dash such that f n X goes to f X for all X. Remember here, for all X belonging to capital X, that is a modulus of f n X minus goes to 0 as n tends to infinity for all X belongs to x. We denote it by f n converges to f weak star. There w l, here w star, weak star convergence. So, correspondingly we have this. The first question which we raise is that whether the limiting operator t remains bounded or not.

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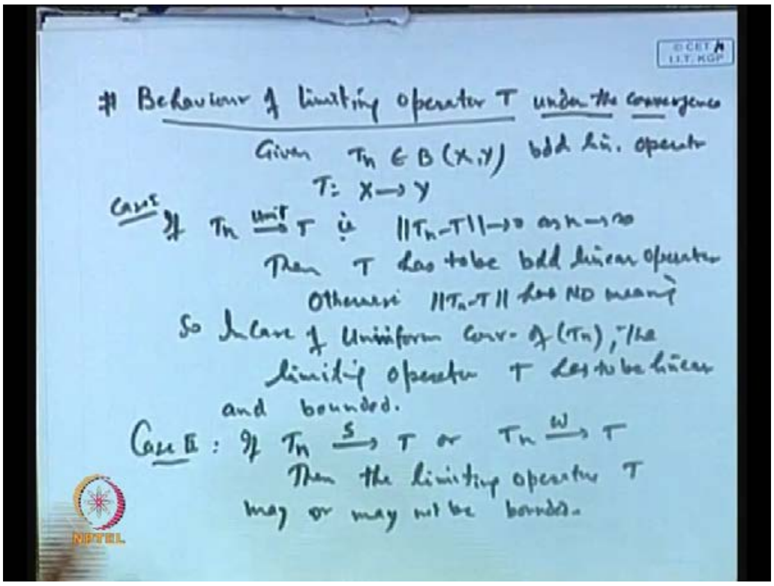

We wanted to discuss the behaviour of limiting operator T under various type of convergence.

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This is for general operator not for functional.

So, the very first question T n is a sequence of bounded linear operator. T n belongs to B X Y and T is an operator from X to Y. What is the behaviour of this? this is a bounded linear operator and T is any operator. Now, the question is that if T n converge T, whether T is bounded linear or not. So, first is if T n converges to T uniformly, then that is norm of T n minus T goes to 0 as n tends to infinity. Then T has to be bounded linear operator. Otherwise, difference of this has no meaning. It cannot go to 0.

In case of uniform convergence of T n, the limiting operator T has to be linear and bounded. So, this is the case 1. Case 2: If T n converges to T strongly or T n converges to T weakly, then the limiting operator T may or may not be bounded. It will be linear, but may not be bounded.

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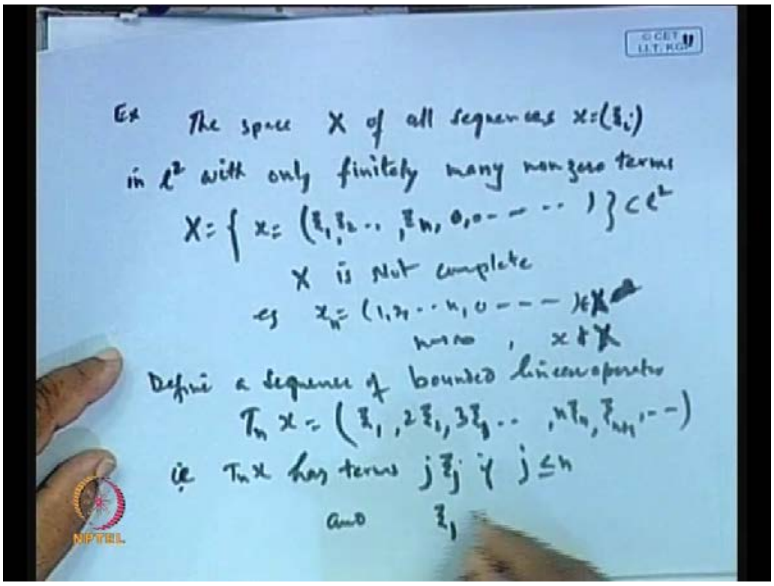

For example, suppose, we have the space X of all sequences X is equal to xi I in l 2 with only finitely many non zero terms, that is X is the set of those sequence is X xi 1, xi 2 say xi n and rest are zeros like this with only finitely many non zero terms and rest are zeros. (( )) Now, this set X which is a subset of l, this is basically set of X belongs to l 2. So, this will be a subset of l 2 scalars of all such sequences. This X is not complete because if I take a sequence 1, one by 2, 1 by 3, 1 by n and then 0 0 0 then sorry any sequence 1 2 3 first n terms and then belongs to l 2.

When n tends to infinity, then large numbers of these terms are non zero. So, basically, it will be a point or may be point of l 2, but may not be point in X because it will not be in X because X contains only those sequences which have only finite number of nonzero term. Rest are 0. So, for example, this sequence if I take X to be 1 2 say 0 0 0, this is in l 2 finite, but this is X n.

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When n tends to infinity, then X does not belong to capital X. It is not a complete metric space. Now, we will define a sequence of bounded linear operators T n X as xi 1, 2 xi 2, 3 xi 3 and then n xi n, after that xi n plus 1 and so on. It means that first n terms are multiplied like this 1, 2, 3, n and rest are 0.

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T n X has these terms j xi j if j is less than equal to n and equal to xi j if j is greater than n.(( )).

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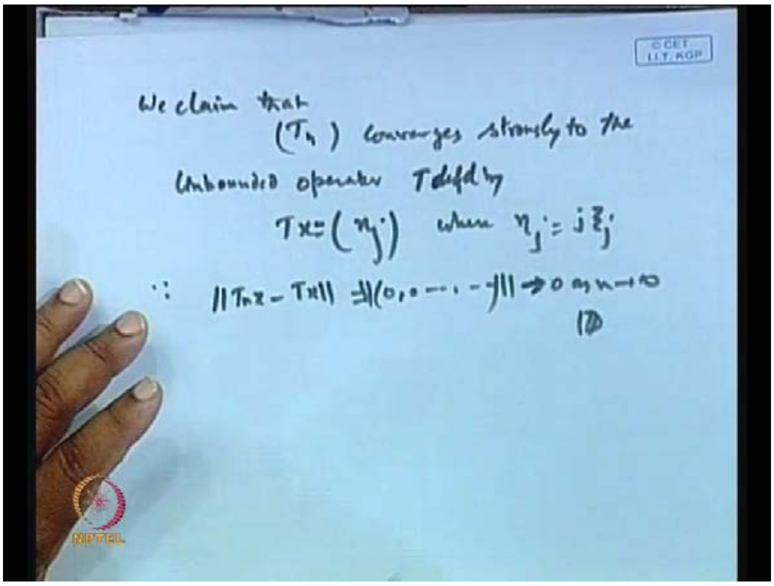

We claim that this sequence T n converges strongly to the unbounded operator T defined by T X as eta j, where the eta j is j times xi j. Obviously, this T n X, when n tends to infinity, it is of the form j type this xi j. This is an unbounded operator, it is a linear operator. So, it will be unbounded. So, T n X converges to it strongly. Why this t n X converges strongly to the unbounded over strongly means norm of T n X minus this (( )). So, what is this norm of T n X minus this T X is basically the 0 0 0 0 norm of this. So, it is basically tending to 0 as n tends to infinity. That is why it converges. So, it converges strongly of this, but is not unbounded. So, clearly T is unbounded. Thank you.