Functional Analysis

Prof. P. D. Srivastava

Department of Mathematics

Indian Institute of Technology, Kharagpur

Lecture No. # 37

Strong and Weak Convergence

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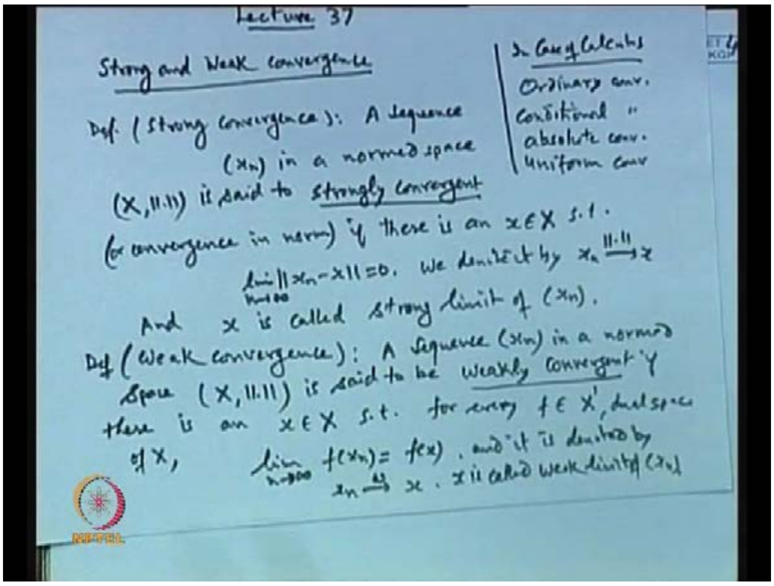

Today, we will discuss the strong and weak convergence in the norm linear space. We have seen that in calculus. There are so many ways of defining the convergence. Like we have ordinary convergence of a sequence, then conditional convergent sequence, absolute convergence of the sequence and uniform convergence; these concept we have gone through this concept in case of calculus. Ordinary convergence is simply x n converges to x mean mod of x n minus x goes to 0, conditioner or a series is ordinary convergent.

Conditionally convergent is when the series is convergent, but the absolute (( )) is not convergent, conditionally convergent of the series and absolutely convergent is when each term is replaced by the absolute value. And the series is convergent and uniform convergent is when the sequence of the functions is defined over a certain domain, then we say that the sequence of the function converges uniformly.

It means the epsilon should not depend on the point mod of f n x minus f m. It should remain less than (( )), whatever f n f x and f z, whatever x and z may be inside that. Epsilon should not depend on the point. But in case of the convergence, the points are important. So, these are the various concepts of convergence in case of calculus.

We have the same type of concepts in functional analysis also. We have these concepts in case of norm space. Apart from this, one more concept which we have in case of the norm space is a weak and strong convergence. In every norm space, one can find out the dual space of this, and then once you get the dual space, the f belongs to the dual set of all boundary linear functional. The convergence with respect to the boundary linear function plays an important role in the (( )).

So, that convergence is termed as a weak convergence and the convergence in the norm, is termed as a strong convergence.

Convergence in norm, we mean that if a sequence x n belongs to a norm space x t and we say that x n converges, it means that there must be some point x available in the x such that the difference between x n n x under the norm should go to 0 as n tends to infinity. That is the converge norm.

We define first the strong convergence as the convergence in norm. Strong convergence means, a sequence x n in a norm space, x norm is said to be strongly convergent or we can also say convergence in the norm, if there exist or there is an x belonging to capital X such that the norm of this x n minus x as n tends to infinity is 0.

Then, such a sequence we say, the sequence x n is strongly convergent in n. We denote this by saying that x n converges to x in this norm. It means that it is strongly convergent and x is called a strong limit of the sequence x n. ok

The concept of the weak convergence is as follows: A sequence x n in a normed space x norm is said to be weakly convergent. If there is an x belongs to capital X such that for every f belongs to its dual, this is the dual space of x for every a belongs to x is limit of this f of x n as n tends to infinity is nothing but f x.

It is denoted by saying x n converges to x weakly. So, x n converges to x. x is called weak limit of the sequence x.

Basically, the meaning of the weakly convergence is that x n is a point in x. This is in the normed space, a vector quantity. When f is the functional defined on x, a linear boundary functional, we say that x n converges to x, means that corresponding sequence of scalars and scalars is obtained by taking the image of x n and that f.

So, f of x n becomes a scalar when the sequence of scalars converges. Then, we say such a sequence x n as weakly convergent.

This limit f of x n equal to f x as n tends to infinity must hold good for all f belongs to x dash. Then only we say that the sequence x n converges to x weakly, means image under each boundary linear functional goes to corresponding f of x. Then sequence of scalar goes to this.

Now, in this weak convergence, there are various applications in analysis and most like a differentiation equation, general theory of differentiation equations. We require certain lemmas to go in deep to the results on weak convergence.

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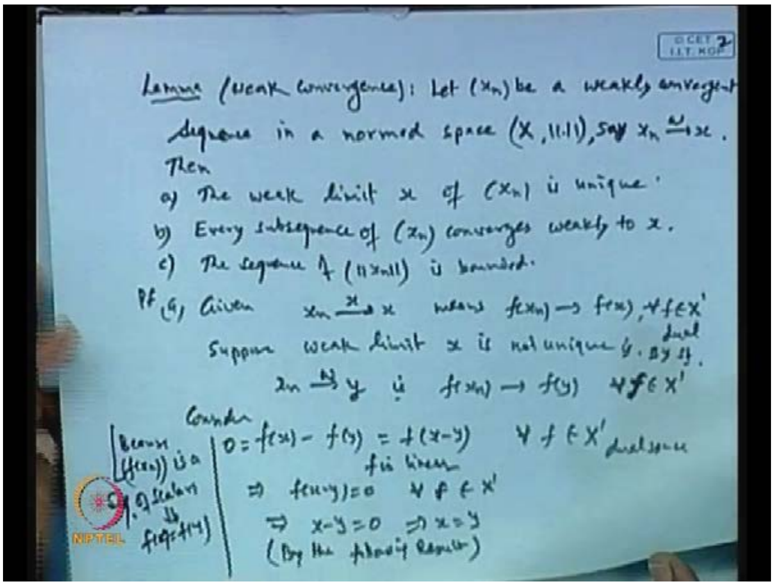

First lemma is that, on the weak convergence, the proof of these results on the weak convergence requires rigorously the application of uniform boundary theorem. So, this gives you results where the bounded uniform boundary theorem is used or in an application of uniform boundary theorem to analysis problem.

What is lemma? Let x n be a weakly convergent sequence in a normed space x norm, say weakly convergent.

Suppose, this sequence goes to x weakly, x n is a weakly convergent sequence, now, it means that there must be some point x available in x such that x n converges to x weakly or f of x n goes to f x for every f. Then the following results hold: the weak limit x of the sequence x n is unique. Just like that in the case of an ordinary convergence, if a sequence x n converges to x, alpha n converges to alpha, limit alpha will be unique. It means that we cannot take a sequence converging to two different limit points.

Then, we say that the sequence does not converge. So, just like a sequence of a scalar, the limit is unique. It is similar in case of this weak convergence or weak limit is unique.

Every sub sequence of x n converges weakly to x. Just like an ordinary sequence of a scalar, if a sequence converges, then, all of its sub-sequences will also converge. So, similarly, here also it will converge weakly.

Third is the sequence of norms is bounded.

The proof of this is easy. The weak limit of x of x n is unique. So, given that x n converges to x weakly. It means f of x n goes to f x for every f belongs to its dual. This is the dual space.

Now, f of x n is a scalar quantity. So, basically, what you are getting is that this sequence will go to f x scalar. Suppose, a weak limit is not the same, what is given is that we wanted weak limit of x is unique.

Suppose, the weak limit x is not unique, it means that there exist y such that x n also converges to y weakly. So, that is the meaning. f of x n goes to f y for every y, for every f belongs to its dual.

Now, f x n goes to f x for every f, f n x goes to f y for f. We want x equal to y. So, let us consider f x minus f y. f is a bounded linear functional. We can write f of x minus y as f is linear. ok

But, f of x minus y f x equal to f y is given. So, this must be 0 and this is 0 for every f belongs to its dual space. So, what this shows is that f of x minus y is 0 for every f belongs to its dual.

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Why is it 0? The reason is that f of x n is a sequence of a scalars and sequence of a scalar cannot have two different limits. So, this implies that the limiting point f x must be equal to f y.

x n converges to x means f of x n goes to f x, but f of x n is a sequence of scalars and we are assuming that this x n is not unique.

We are assuming another y, but basically by definition of b convergence f, image of x n under f will go to f of y.

Since f of x n is a sequence of a scalar, its limiting value f x and f y will not be different. If it is convergent, it is equal. Once they are equal, it will be 0. So, this is 0 for every f belongs to x.

Now, from this, we can say x minus y equal to 0. If this is 2 for every f, then, this must be 0 for every y and this implies x equal to y. (( )) Why is it 0?

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There is a lemma. This is by the following result. What is the result? The result is corollary of the Hahn Banach theorem.

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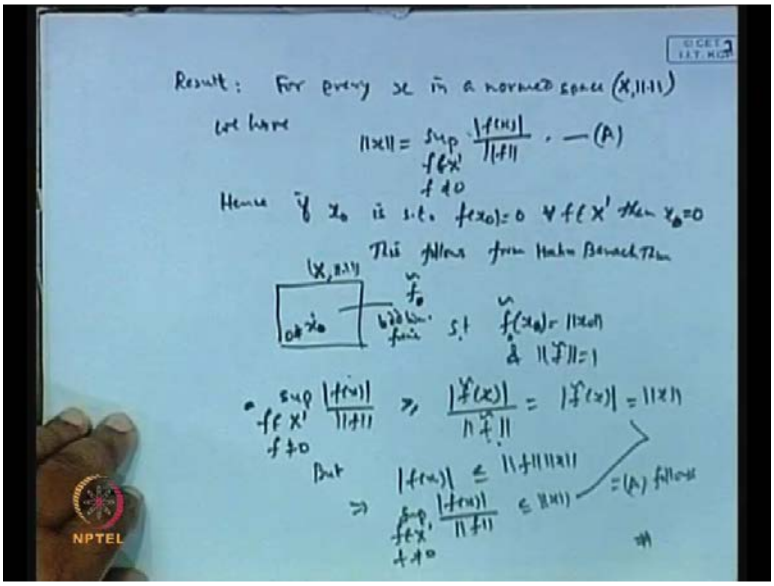

The corollary of the Hahn Banach theorem says, for every x in a normed space x norm, we have norm of x equal to supremum of mod f x over norm f belongs to x dash and f is not equal to 0.

Hence, if x 0 is such that f of x 0 is 0, for all f belongs to its dual, then x 0 will be 0. So, because of this result, we can say that f of x minus y 0 implies x minus y is 0.

Now, this result follows from Hahn Banach theorem. How does it follow from Hahn Banach theorem? What is Hahn Banach theorem? Hahn Banach theorem says that if x be a normed space and x 0, if I picked up any non zero point here, then corresponding to this x 0, we can find f or f 0 such that f 0 x 0 is 1.that

The Hahn Banach theorem is: Let x be a normed space and x 0 be a non 0 element of x, then there exist a bounded linear functional f delta. There exists a bounded linear functional f delta on x such that f delta x 0 is norm of x 0 and norm of f delta is 1.This is what we call as the Hahn Banach theorem in case of the normed space.

Let x be a normed space and x 0 be a non 0 point in this. Then there exists a boundary linear functional f delta such that image of this x 0 under f delta is norm of x 0 and the norm of this is 1.

So, using this, we can say the norm of x we wanted to show this. So, start with this supremum mod f x over norm f when f belongs to the dual and f is not equal to 0. Obviously, this will be greater than equal to this particular f delta. So, this is greater than equal to f delta x, f delta x over norm of f delta.

But, x is fixed. So, norm f delta we can find out if such f delta where norm is 1. So, this is equal to f delta x. But f delta x is equal to norm of x. So, this is norm of x.

So, this part is greater than or equal to norm of x, but mod of f x is less than equal to norm of f into norm of x. So, this implies mod f x over norm of f supremum is taken over all f belongs to this. f is not equal to 0 will remain less than equal to norm x.

So, combining these two, we get this lemma is 2, A is 2, A follows. Now, if A is 2, then, what he says is x 0 is such that f of x 0 is 0 for all x, then x 0 must be 0.Now, if this part is 0 for all f, then; obviously, the supremum will be 0. Obviously, norm of x 0 will be 0. So, it follows immediately and norm x 0 implies x 0 must be 0 because it is a norm.

So, this result, f of x minus y equal to 0 implies x minus y 0 follows from this result.

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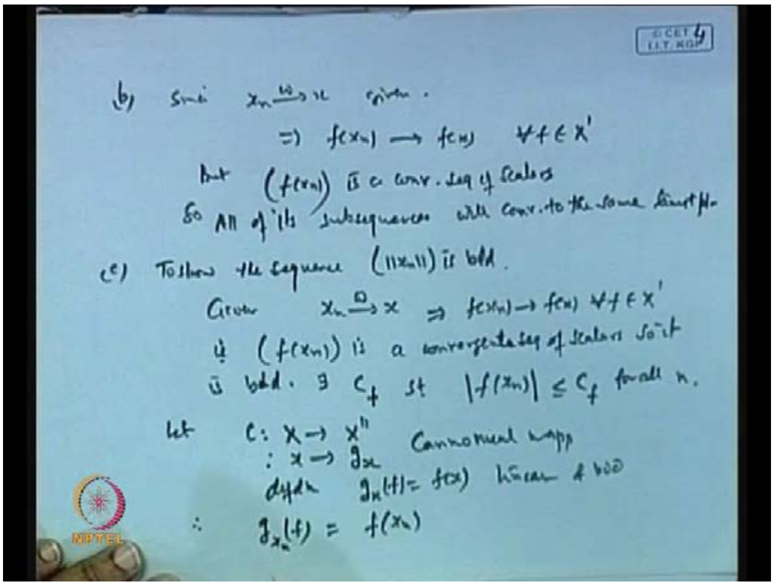

Now, let us see the second part of this. Proof of second part:

Every subsequence x n converges weakly to the same limit x. Now, this follows here, since x n converges to x weakly. This is given. So, this implies that f of x n converges to f x for every f belongs to x dash.

But, f of x n is a convergence sequence of a scalar and every convergence sequence is scalar, the subsequence is to converge to the same limit. So, all of its subsequences will converge to the same limit point. Hence, this follows.

Now, part c. What is part c is that sequence norm of x n is bounded.

To show that the sequence norm of x n is bounded, let us start with this given x n converges to x weakly. So, this implies f of x n will converge to f x for every f belongs to the dual. It means the sequence f x n is a convergent sequence.

Every convergent sequence is bounded sequence of scalars. So, it is bounded. Therefore, there exists a constant c which depends on f such that mod of f of x n, this will remain less than equal to constant c that depends on n for all n. c will not depend on n, it will depend on f.

Because if f changes, the corresponding constant will change. So, we get this. Now, let c be the mapping from x to x double dash which sends x to g of x.This is a cannonical mapping which we have already discussed.

So, the cannonical mapping is defined by g x f equal to f of x. It is linear. This is already shown to be linear and bounded also

So, let us consider g of x n f mod of this is equal to f of x n.

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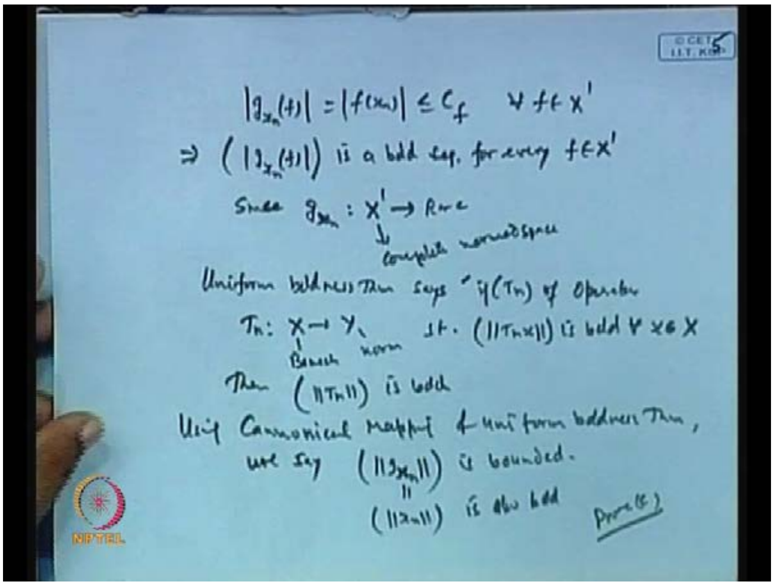

Mod of g of x n f is equal to mod of f of x n, but mod of x is less than equal to C f. So, this is less than equal to C f for every f belongs to the dual.

It means that this sequence mod of g x n f is a bounded sequence for every f belongs to the dual.

So, this operator g of x n is a bounded operator point y (( )) it is point y (( )) bounded because g is defined on x dash, and g of x n f is less than equal to c for every f and c depends on f.

So, it is a point y bounded theorem. Since g x n is defined from x dash, this is defined from X dash to Rnx.

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It is an element of x double dash, but basically the domain will be g n of f will be the point in real. So, it is defined on the real or c.

Now, this will be a complete normed space, whether x is complete or not, dual space will always be complete. So, a sequence of the operator is defined on a Banach space x. This is x to y.

This one is R n c. So, this will be uniform boundedness theorem. The uniform boundedness theorem says that if T n be a sequence of operators from x to y where x is a Banach space, y may or may not be a Banach space, just norm space such that norm of T n x. This is a bounded sequence for every x belongs to x.

Then, the sequence of the norm is bounded. Now g n is defined on x dash which is a complete norm space, Banach space and g is bounded point y.

So, according to the Banach uniform boundedness theorem, we say norm of g n will be bounded. So, from here, using the canonical mapping concept and uniform boundedness theorem, we say that norm of this sequence g of x n is bounded.

What is the canonical mapping? When g of x n is there, then, norm of g x n is equal to norm of x n, which is the same as norm of x n by cannonical mappings because the operator which we have defined is also bounded.

Hence the theorem is proved. So, this proves c because we want the norm of x n to be bounded. So, this is convergent.

Now, in a general norm space, we have seen the concept of the weak convergence and strong convergence. Why do not we have this concept in a real or complex, when x is reduced to the real set of earlier number or set of complex number or in general a finite dimensional space?

The thing is in case of finite dimensional space the weak convergence and strong convergence are equivalent concepts. It means that strong will imply weak and weak will imply strong.

But if x is not a finite dimensional space, then these two concepts differ. Strong always implies the weak, but weak may not imply the strong convergence.

We will see that in case of the finite dimensional space, the strong convergence and weak convergence are identical. So, that is the next target of (( ))

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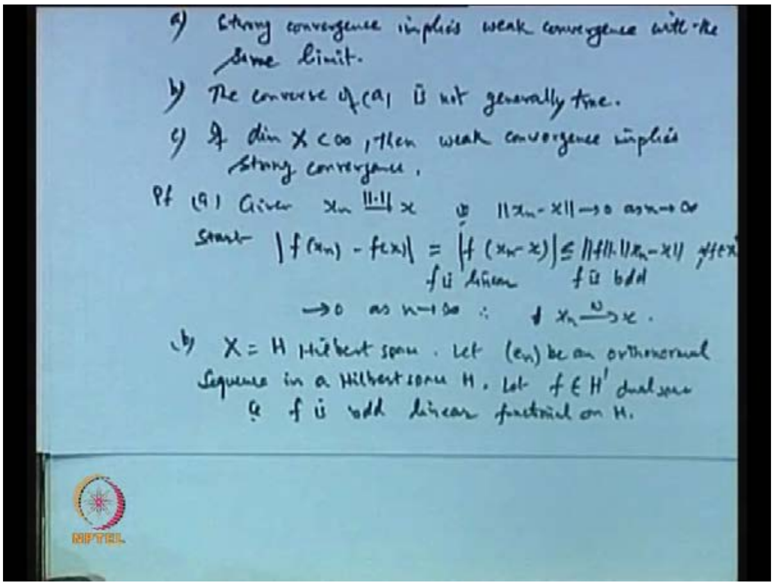

We have the theorem. Let x n be a sequence in a normed space x norm space norm, then the following result hold. Strong convergence implies weak convergence with the same limit. The converse of a is not generally true.

And c part is, if the dimension of x is finite, then, weak convergence implies strong convergence. Let us see the proof.

Strong convergence always implies the weak convergence. So, it is given x n converges to x strongly means under this norm, that is norm of x n minus x goes to 0 as n tends to infinity.

We want the weak convergence. So, start with this mod of f x n minus f x (( ))

Now, f is a bounded linear functional. So, this can be written in this form, further f is bounded. So, we can say this is less than equal to norm f norm x n minus x as f is bounded. This is true for every f belongs to the dual.

Now, x n goes to x is given. So, it will not tend to 0 as n tends to infinity, because this is a bounded and this will go to 0.

So, f x n converges to n. Therefore, f x n converges to x weakly, because this is true for every x.

Now, part b, the converse of this is not true in general. It means in a general normed space, weak convergence need not imply the strong convergence. So, we have to take a counter example where the sequence converges weakly, but it is not strongly. I take x to be a Hilbert space. Hilbert space is also a normed space, but every Hilbert space is not a normed space. We can introduce the norm is an inner product norm of x is the inner product x x under root. So, we can find out the (( )).

So, let us take the Hilbert space and let e n be an orthonormal sequence in a Hilbert space H. Now, it is given that weakly convergent. Let f is an element belonging to the dual, that is f is a bounded linear functional on a Hilbert space H. So, it is representation by Riesz theorem.

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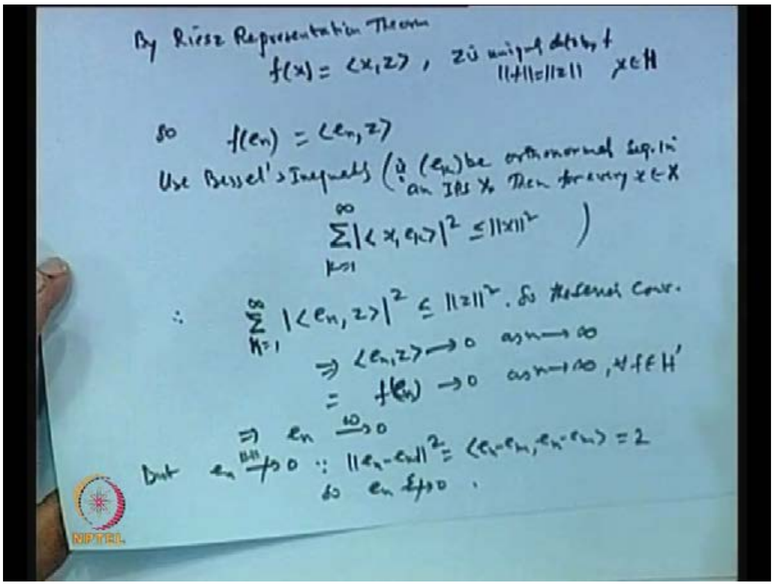

By Riesz Representation Theorem, every bounded linear functional f can be represented in terms of the inner product.

By- bounded linear functional, x can be represented in terms of the inner product x z where z is uniquely determined by f and norm of f equal to norm of z. x belongs to H. H and f is this by Riesz Representation Theorem.

So, we get f of e n f of e n will be inner product of e n z.

Now, use the Bessel's Inequality. What the Bessel's Inequality says is that sigma of the inner product x n y n mod square is less than equal to sigma norm of x into norm of y. That is what is Bessel's Inequality.

Let e k be a orthonormal sequence inner product space, then for every x belongs to x, that is if e k be orthonormal sequence in an inner product space x, then for every x belonging to capital X, the sigma of inner product modulus of inner product x e k whole square k is equal to 1 to infinity is bounded by norm of x square. This is what inner product at the Bessel's Inequality says.

Apply the Bessel's Inequality here. What we get from there is, n is equal to 1 to infinity modulus of inner product of e and z. This whole square is dominated by norm of z square, but this is finite. So, this series converges.

Therefore, the n th term must go to 0. So, inner product of e and z must go to 0 as n tends to infinity, but what is this, f of e n goes to 0 as n tends to infinity.

So, e n converges to 0 weakly because this is 2 for every f belongs to h dash dual of this. So, e n converges to 0 weakly, but e n does not converge strongly to 0.

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(( )) This is because norm of e n minus e m whole square, this is inner product e n minus e n e n minus e m and this will be 2.

So, it is not Cauchy. Therefore, it will not converge. So, e n does not go to 0. (( ))

Therefore, every weak convergent need not imply the strong convergence.

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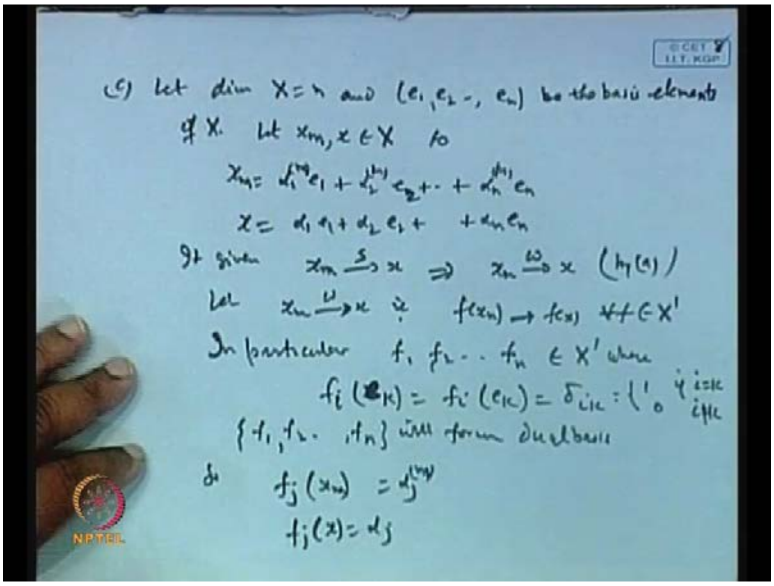

Now, third part: the proof of this, if dimension of this is finite, then weak convergence and strong convergence are the same.

So, let the dimension of X be n and e 1, e 2, e n be the basis elements for x.

Let x n, n comma x are the elements of x. x n can be expressed in a linear combination of e 1 e 2 n.

There exist scalars alpha 1 n, alpha 2 n, alpha n n such that the linear combination of this is x n.

Let the dimension of this be m.

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Now x can also be written as alpha 1 e 1 alpha 2 e 2 and alpha n e n in terms of the basis elements.

Now, it is given x n x m converges to x strongly, then; obviously, x m will converge to x weakly. It is by result, every strong convergence imply weakly.

So, now, let x m converges to x weakly. We want this converge to this. So, that is given as f of x n goes to f x for every f belongs to the dual of it.

For every f belongs to dual, now in particular, the f 1, f 2 or f n's, these are also the elements of x dash where f i e k is the chronical delta k which is 1 if I is equal to k, otherwise 0. f 1, f 2, f n will form dual basis. So, these are the elements of x dash.So, in particular, this f 1, f 2, f n will also be dual. So, image of this x n under f j will be equal to alpha j n x m.

As you apply the f j, f j e j will be 1 and rest will be zero. So, alpha j m and this is true for all and what is the f j x is alpha j.

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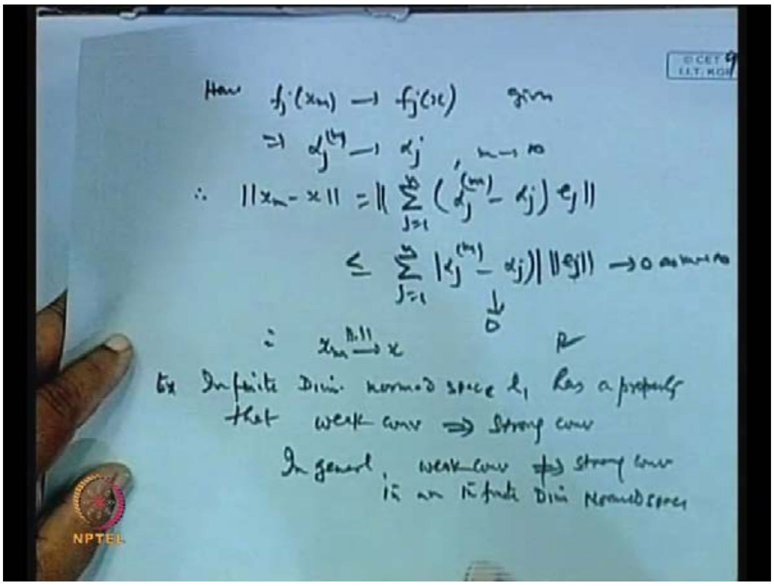

Now, it is given that f j x m converges to f j x. This is given.

(( )) This implies alpha j m goes to alpha j as m tends to infinity. Therefore, norm of x m minus x, consider this which is equal to norm of sigma j equal to 1 to n alpha j m minus alpha j e j.

Now, this will remain less than equal to sigma alpha j mod of this into norm of e j.

Now, sigma j equal to 1 norm of e j is finite and this part goes to 0. So, this will go to 0. This is finite. So, entire thing will go to 0 as m tends to infinity.

Therefore, x m will converge to x under this norm strongly. So, x m converges x weakly implies x m converges to x strongly and that is proved

So, in case of the finite dimensional, this is 2, but it does not mean that infinite dimensional that we convergent never implies. (( )) They are all examples of an infinite dimensional normed space l1 has a property that weak convergence also implies strong convergence. .

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There are other spaces also parallel to l 1, where this is true, but in general, it is not true. In general infinite dimension, weak convergence need not imply strong convergence in an infinite dimensional normed space.

But, there are certain spaces even in finite dimensional where it comes, but we require the proof of this theorem. So, just this is an example that I have given in this.

Now, research is going on to find out the criteria, the sufficient condition when the sequence converges weakly under that norm because always the infinite dimension space is not necessary the weak convergence implies a strong, but we can impose a restriction on the sequences so that the sequence will imply the weak convergence.

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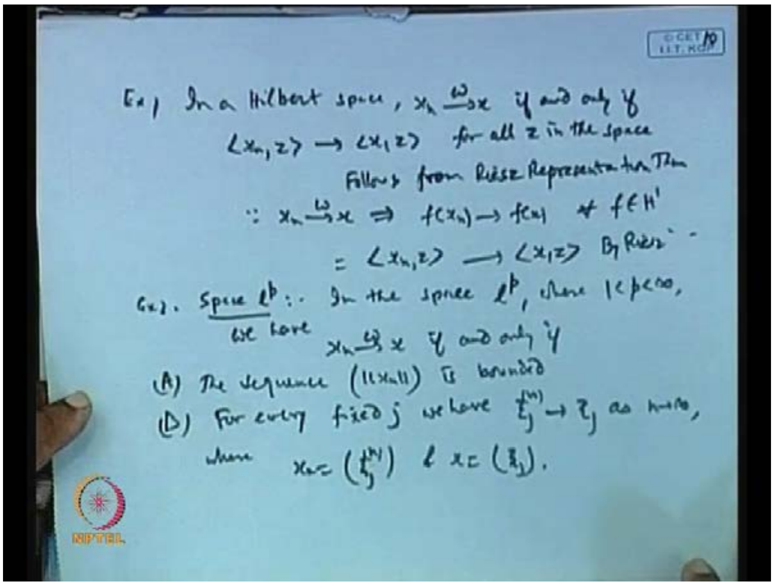

What is the criteria for weak convergence? (( )) I will tell these results. The examples: in a Hilbert space x n converges to x weakly if and only if inner product x and z goes to x z for all z in the space.

I think this follows immediately by Riesz representation theorem because f of x n is x n converges to because x n converges to x weakly means f of x n goes to f x for every f belongs to the dual.

But f of x n by Riesz theorem, this is the inner product of x n z. This will go to the x z by Riesz.

Therefore, in case of this space l p, the criteria is that in the space l p where 1 is less than p, less than infinity, x n converges to x weakly if and only if and only if the sequence norm of x n is bounded. And second part is, for every fixed j, we have x i j n goes to x i j as n tends to infinity, where x n is a sequence x i j n and x is a sequence x i j.

V is for every fixed j, x i j n converges to x i j, it means if f is a point in l p dash dual of l p dash is l q.

So, (( )) belongs to the dual of this then f of x n will go to f x by means of this v convergence. Now, this will converge weakly if the coordinate y is convergence of x n is also clear.

X j n, because j you fix it, means x i 1 n will go to x i 1 x i 2 n will go to x i 2 n. So, if x n converges to x coordinate y as well as the norm x n is bounded, then, the sequence will converge weakly.

We are not going for detail in the proof of this, but these are the results. So, this type of the (( )) continues, means, you take the space, find out the sufficient condition because these are basically the necessary and sufficient in both conditions. Sometimes, we are unable to get both types. So, we get only the sufficient part.

It is required condition when the sequence converges weakly and that. (( )). Thank you very much. .

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