Functional Analysis

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Module No. # 01

Lecture No. # 36

Adjoint Operator

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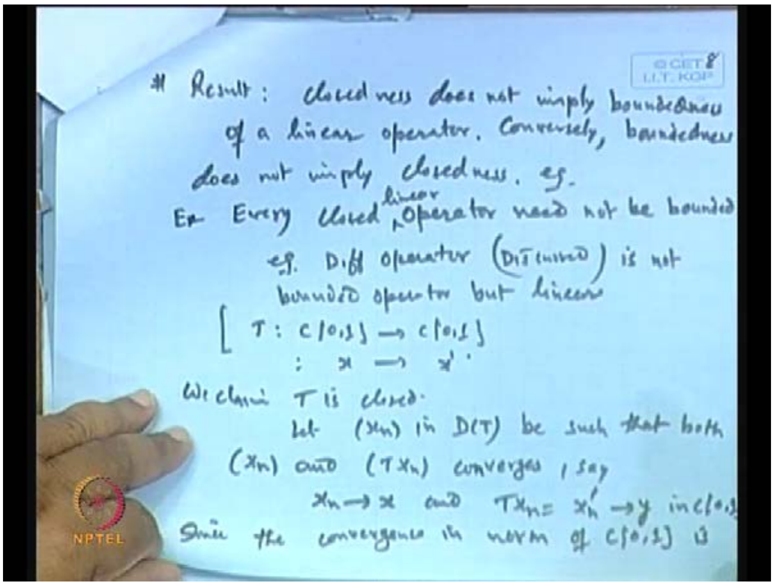

Last lecture, we were discussing this result. The closedness does not implies the boundedness of a linear operator and conversely the boundedness does not imply the closedness.

So, first part of this result we have already shown by an example where the operator is closed, but not bounded and an example we have taken a differential operator from c 0,1 to c 0,1 and in that case, we have shown that operator is not bounded because of a particular function if we take, x n T to be T to the power n, then this is unbounded operator as we have already discussed and this operator; however, it is closed operator. Is it not?

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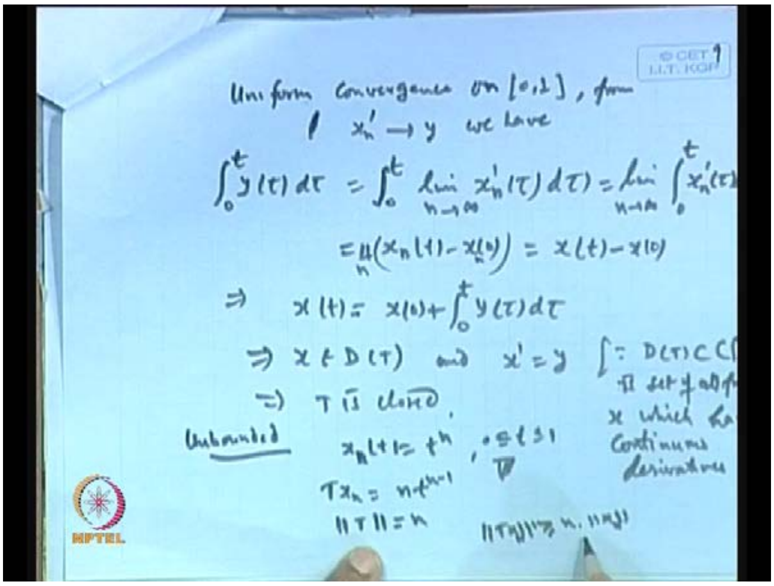

So, that operator is closed we have seen and for unboundedness, here we can say unbounded operator because if I take x n t to be t to the power n, t lying between 0 and 1, then the derivative t x n becomes n t n minus 1 and the norm of t x t is equal to say n, say n.

So, norm of t x, x n is greater than equal to n times norm of x; is it not, but norm of x n is 1. So, we cannot find c. We cannot find c such that is always be less than equal to n; means this condition less than equal to this does not hold good. So, this part we get say an unbounded operator (( )).

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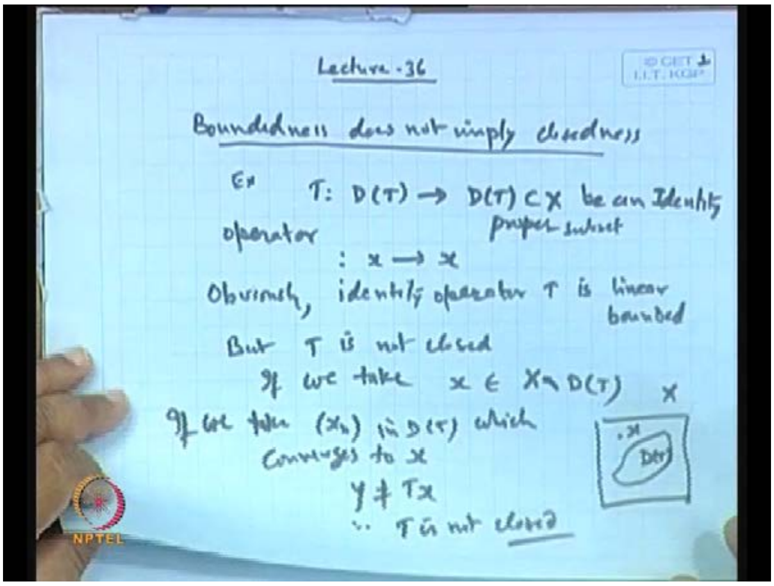

Now, the second part of this that boundedness does not imply closedness, does not implies closedness and the example is, let us take an operator T from D T to Y which is D T to D T which is subset of X and this operator be a identity operator, be an identity operator; that is an operator which transform x to x.

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Identity operator. So, it is a transform x to x so; obviously, this identity operator T is linear, is it not. Then nothing to proof for it linear, as well as is a bounded operator because norm of T x equal to norm of x. So, it is a bounded with the norm 1. So, it is a linear and bounded operator, but T is not closed. Why? This we are choosing as D T is a proper subset, this is a proper subset it means the range set is a proper subset of x.

So, if I pick up, if we pick up, if we take a point x belongs to x minus D T; this is our x and here is D T. I am picking up a point x outside of D T. Now, if it is T is closed, then graph of T must be that is what we get it is x n converges to x, T x n converges to y and then y is equal to T (( )) T x that is what we get it.

So, if we take a sequence x n, then this follows. So, if we take sequence x n in D T, in D T, then if we take a x n in D T which converges to x. It means all the limit point of that this does not belongs to this.

So, if we take a sequence x n, the image is T x n. That T x n will not go to T y, is it not. y is equal to T x will not be there. So, it is not. So, y is not equal to T x is it clear not whereby therefore, this operator T is not closed. So, the operator is bounded, but it is not.

So, it means we cannot have a relation between the closedness and boundedness, but under a certain restriction, we get the relation. So, that restriction we will see in the afterward. In fact, their lemma which shows is like this.

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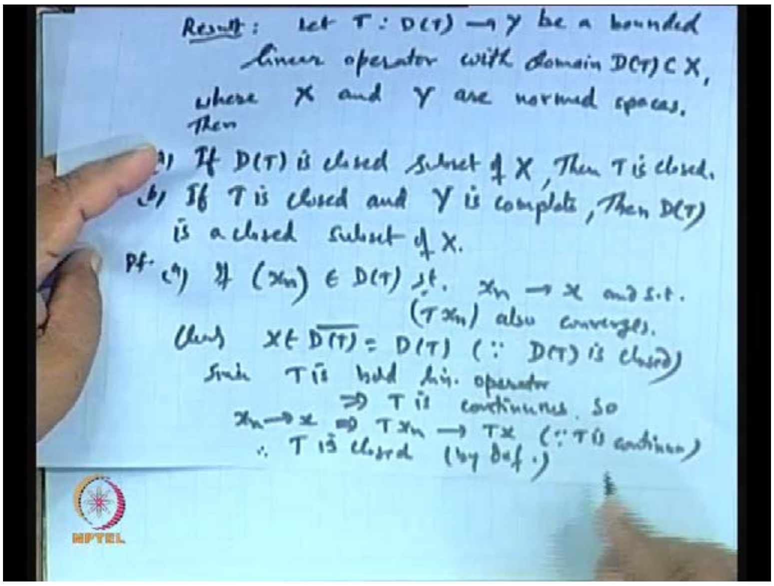

There is one result. The result is, let T from domain D T to y be a bounded linear operator, bounded linear operator with domain D T which is subset of X, where X and Y are normed spaces normed spaces; these are the normed spaces.

Then, if D T is closed, D T is closed subset of x, D T is closed subset of x, then T is closed. T is closed means this condition is D T must be a closed subset of x. All of its limit point must belongs to D T. Then only the T will be closed set. Second result says, if T is closed and Y is complete, Y is complete, then the domain D T is a closed subset of X closed subset of X.

Now, proof of this; it means, this gives the condition under which the T will be a closed operator. The condition is sufficient condition if D T is closed, then T will be and conversely if T is closed and y is complete, then our domain D T will be a closed subset of this and y should be a complete on it; that is proved.

So, let us see the proof for the what we want is the T is closed. So, let us take a point in that closure of that T x closed means, x n converges to x and T if T x n converges the T x n converges to y then y is equal to T x. So, that is what we wanted to show.

So, let if x n is in D T, x if x n belongs to D T and such that x n converges to x and such that T x n also converges. This sequence T x n also converges. That is what; now if T x n converges, then converges this term then y is equal to T x. That is what we wanted to show ok.

Then x, now since D T is closed, x n is a convergence sequence convergence to x. So, x is the limit point of the sequence x n. So, x must be belongs to the closure of D T. So, clearly, x belongs to D T closure which is the same as D T because D T is closed is it not. So, that is a…

Now, T x n converges. Now what is given is T be a bounded linear operator. So, since T is a bounded linear operator and the linearity and the continuity are the same thing. Sorry if a linear operator, then boundedness in continuity are the same.

So, since T is a bounded linear operator therefore, since T is bounded linear operator. So, it will imply, T is continuous; is it not. So, since T is continuous, so, we get from here that it will transfer the convergence sequence to convergence sequence.

So, x n converges to x implies T x n will go to T x because of the continuity because T is continuous. T x n converges is already given. So, it will converge to T x. Now T x is a point in D T. So, image of this will be the in the domain in the range of T. So, there will be a point y where the T x n will go to T.

So, this shows that y, T x belongs to D T. T x is in the range set therefore, T is closed is it or not by definition by definition.

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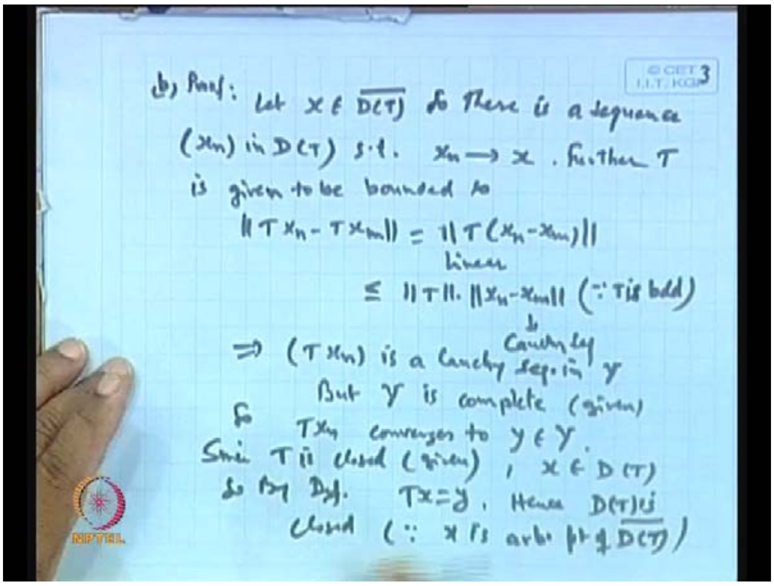

So, this that is proof. Second part, b part proof. What we want to show if T is closed and y is complete, then domain is the close subset of x.

So, let us take all the limits points of the must all the limit point of the sequence in D T must be in D T. So, let x be a point belonging to the closure of T. If I proof that x belongs to D T, then D T will be closed.

Now, since x is belongs to the closure of this, so by definition, there must be a sequence x n in D T which converges to x. So, there exist there is a sequence x n in D T, such that x n converges to x because x is the limit point of this.

Now, further, T is given to be bounded. T is given to be bounded, is it not. So, by definition of the boundedness, norm of T x n minus T x, T x m. This is equal to norm of T x n minus x m because T is linear and again T is bounded. So, this is less than equal to norm of T into norm of x n minus x m because T is bounded is it ok, T is bounded.

So, this will be… Now x n is a sequence which is convergent, is it not. So, what you say is that every convergence sequence is Cauchy. So, this will be a Cauchy sequence. Therefore, this has to be Cauchy. So, this implies the sequence T x n is a Cauchy sequence in y, but y is complete. It is given, this is given.

So, every Cauchy sequence must be convergent. So, T x n; this sequence converges to say y belonging to Y because y is complete.

Now, given that since T is closed, this is given and x is an element of the domain D T. So, by the definition of the closedness, if T x n converges, then the limit point T x must be equal to y.

So, by definition T x must be y. Hence y is closed, hence D T is closed. Hence D T is closed because x belongs to D was arbitrary because x is an arbitrary point of D T closed because if we take any arbitrary point, correspondingly we can always find the y in this such that T x equal to y.

So, this shows the D T is closed x must be in D T is it not. This because of this closedness. So, this completes the proof of this. So, this portion, in fact, we the left out portion of the last lecture.

Now, we will take up the few application of the Hahn Banach theorem, where the Hahn Banach theorems are used. In one of this topic which we require is the adjoint operator. The theory of the adjoint operator and their corresponding result and the properties requires that Hahn Banach theorem very rigorously.

So, let us see the new topic which we wanted to discuss if the adjoint operator in general from one norm space to another norm space and then let us see what is the relation between the adjoin operator and the Hilbert ad joint operator and how this can be represent in terms of the matrix.

Because every linear operator. Every operator defined on a finite dimensional space can be represented by means of a matrix. So, correspondingly the adjoint operator, what we should be the form of the matrix if T is a adjoint operator. So, these things we will discuss here.

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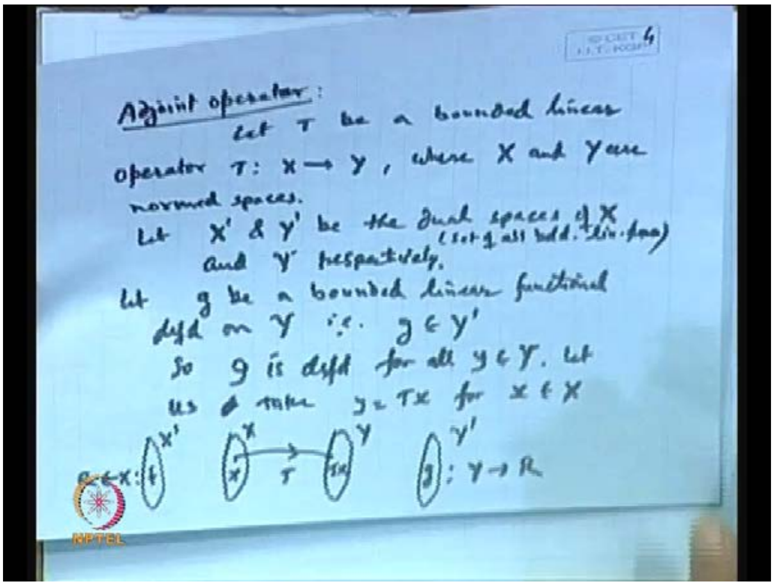

So, let see that adjoint operator. Let T be a bounded linear operator from one norm space X to the another norm space Y, where X and Y are normed spaces. Let us, we wanted to define the adjoint operator of this.

So, let us pick up a bounded linear functional g on Y. Let X dash and Y dash be the dual space of x and y respectively. Dual means set of all bounded linear functional. Dual means set of all bounded linear functional is it not, bounded linear functional. These are the set of all bounded linear functional or the duals.

So, let us pick up a point, let g be a bounded linear functional; g be a bounded linear functional defined on Y. That is, g is a point in Y dash, bounded linear functional. So, since g is a bounded linear functional on y, it means so, g will be defined, g is defined for all y belonging to capital Y because it is a linear functional defined on y. So, it is a bounded linear.

Let us take, let us define or let us take y is equal to T x for some x belonging to capital x because this is our x, this is y. So, if we take any x here, T is a bounded linear operator from x to y.

So, it will transfer the image of x under T as T x. y dash is this; set of all bounded linear functional defined on y is it not. There is a y dash. It means this is our g, g is an element of this domain of g will be T x, domain of g will be T x. That is g is a y to R. g will be y to R. Domain will be y that is R T x. So, g is defined for all y which is T x say.

So, now let us take, start with x. Then corresponding to x we wanted to define a bounded linear functional on x dash. Take x because domain of this will be from x to R. So, fix x, then we wanted to define f of x in terms of g. So, how to define this is, we are taking f once you take x, then you are taking T x here and T x is g of x is a real number.

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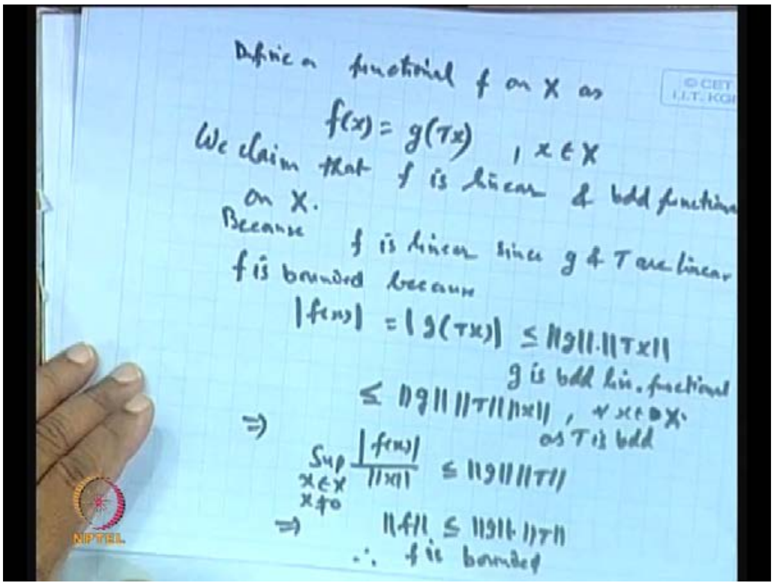

So, we can define, let us define a functional f on X as f of x equal to g of T x because it is a functional means this is a real valued.

So, what I am taking is x belongs to capital X. We are taking the f of x a values, but this value is defined in terms of g and T because T x is a point in y g is a functional, a bounded linear functional defined on y. So, g of T x will be a real part. So, I am defining f of x equal to g x. So, it is a functional. We claim that f is linear, f is linear and bounded.

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No, we are taking g as a bounded linear functional in Y. We wanted to relate this g with a bounded linear functional on x. So, that is why we are wanted to introduce a functional f which is a bounded linear functional on x. So, corresponding to g, we can get the f and vice versa also. That is the ratio. So, that way we can defined the operator from y dash to x dash and which is nothing but a adjoint operator. So, that is what.

So, this f which we have defined, we claim this is a linear and bounded functional on x. Why because f is linear since g and T are linear. g is a bounded linear functional on y. T is a bounded linear operator from x to y. So, composition of the two linear functional will be linear. So, f will be linear. f is bounded because mod of f x; this is nothing but the mod of g T x, but g is giving to be a bounded linear functional. So, by definition, this is less than equal to norm of g into norm of T x, as g is bounded linear functional on this.

Further, T is a bounded linear operator. So, it will be equal to norm g norm T into norm of x, as T is bounded. So, we get the mod of f x is less than equal to normal. This is true for all x belonging to D T belonging to x, sorry belonging to...

So, we can from here, we can be say mod of f x over norm x take the supremum over all x belonging to capital X x is not equal to 0 is less than equal to norm g norm t. But this is equal to norm f. So, this implies that norm f is less than equal to norm g into norm T. So, this f is bounded.

So, this f is bounded. f is therefore, f is bounded is it. So, we have shown that f is bounded. It means if we start with a bounded linear operator from Y, then 1 can associate an bounded linear sorry bounded linear functional in y on y, then we can take the associated bounded linear functional on x. Now this leads the concept of the adjoint operator.

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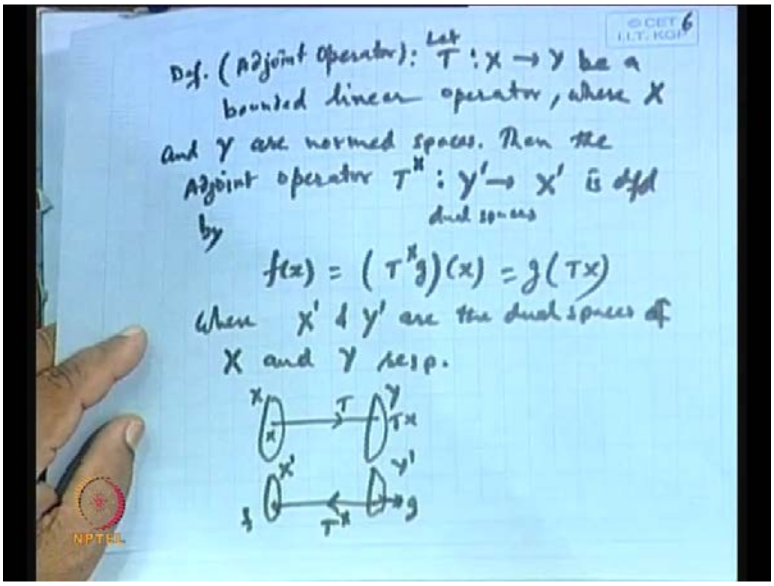

So, now let us defined the adjoint operator is it clear or not. So, define adjoint operator. Let T from X to Y, T from X to Y be a bounded linear operator. Let be a bounded linear operator, let T be a bounded linear operator where X and Y are normed spaces. X and Y are normed spaces.

Then the adjoint operator T cross, the adjoint operator denoted as T cross denoted by T cross is an operator from y dash to x dash. There do not the dual spaces, dual spaces x to y is defined by f of x T cross g x equal to g of T x, where x and y are the dual spaces of X and Y respectively.

So, the adjoint operator T cross defined from Y dash to X dash as T cross g, g is an element of y dash, T cross g of x image of this is the g of T x clear. It means x you are choosing from capital X. So, this is a functional this is a real valued thing. So, this we are taking as f of x is it not. So, we are defining where x dash and y dash are the dual spaces of x and y respectively.

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Small f small g are the bounded linear functional. So, what we are doing is we are just picking up the x. This is our x this one is y, then what we are doing is this is our y dash and here is x dash. So, we are taking x here, then it is giving to be the T x to T x here, then this is from y to y dash is g. This is our g is an element belonging to g. So, whose domain is y So, g of T x we are taking to be the x dash. This is our f.

So, relation between g and f is given by this. Then this operator T cross which relates g to f is it not relates g to f is nothing but the adjoint operator. So, an adjoint operator is an operator from y dash to x dash such that T cross g of x is equal to g of T x is equal to f x. Is it clear. So, corresponding to each g we can find f here and vice versa also 1 (( )).

Now, this operator T cross which you are quoted is also a bounded linear operator and the norm of T cross and norm of T will be the same because if they differ, then it will no use. So, let us see the next result, norm of the adjoint operator.

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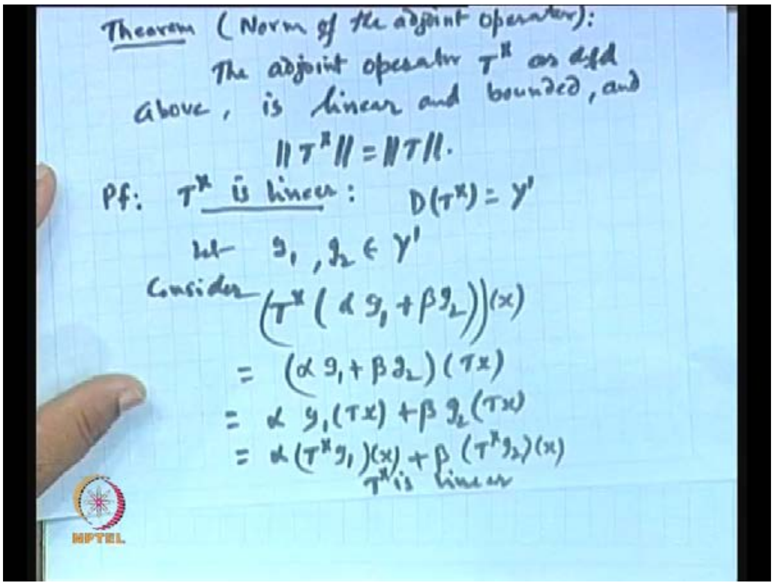

The adjoint operator T cross as defined above is linear and bounded and have the same norm as the norm of T. That is the one we wanted to show, the proof of this, The T cross is linear let us see first, how does…

T cross the domain of T cross is g sorry is y dash is it not. Norm T cross is defined on from y dash to here. So, domain of this T cross is y dash. So, let us pick up the point g1 and g2 in y dash. Let g1 and g2 are the two bounded linear functional defined on y belongs to y dash.

Consider T cross alpha g1 plus beta g2. If I prove this is alpha time T cross g1 beta time T cross g2. Now this is a composition of the two functional. So, image of this under x. Now by definition, T cross g x this is a T cross g x is g of T x. So, by definition, this will be equal to alpha g1 beta g2 x beta of T x g of T x.

Now, separate out. So, alpha g1 T x plus beta g2 T x is it not. Now this will be equal to what; alpha T cross g1 x because this can be written like this plus beta T cross g2 x. So, x can be taken common and we are getting this. So, T is linear is it not therefore, T is linear.

So, T is linear sorry T cross is linear not T. T cross is linear because T cross is. Now we want T cross to be bounded, T cross to be bounded.

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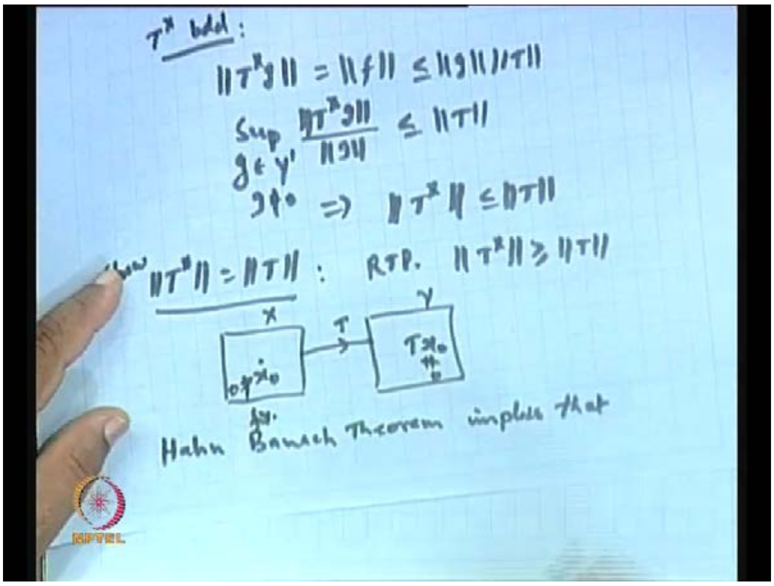

Now, let us take this T cross g. Now T cross g is nothing but what f, is it not? T cross g is nothing but f and f we have proved one result that is, norm of f, yes this will be equal to norm of f equal to norm of g into T.

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This will be f T x and then this is the norm of f equal to norm of.

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So, norm of f, yes this is over there. This one is there. So, norm of f is less than equal to norm of g.

So, using this thing, we can write this is less than equal to norm of g into norm of T. Now divide by norm of g. So, what we get from here is supremum of T cross g over norm of g and g is belongs to what y dash and g is not equal to 0 and that will be less than equal to norm T. Therefore, norm of T cross is less than equal to norm of T. So, this shows that T cross is a bounded operator norm T is here.

Now, we wanted to show that norm of T cross is the same as norm of T. This I wanted to show. It means required to prove is norm of T cross is greater than equal to norm of T because its already shown. Now this we will show.

Now, here we will make use of the Hahn Banach theorem. This is our x, this is say x and then this is y. T is a bounded linear operator from x. So, let us fix up the x0 here. Choose x0 fixed, then corresponding this we get the T x0. If x0 is not equal to 0, then T x0 will also not be equal to 0.

Now, if we pick up a point x0 in x, then there will be a bounded linear functional f such that f of x0 will be norm of x0 and norm of x will be 1 by Hahn Banach theorem. So, let us take x0 fixed. It means you are choosing the T x0 as an element in y.

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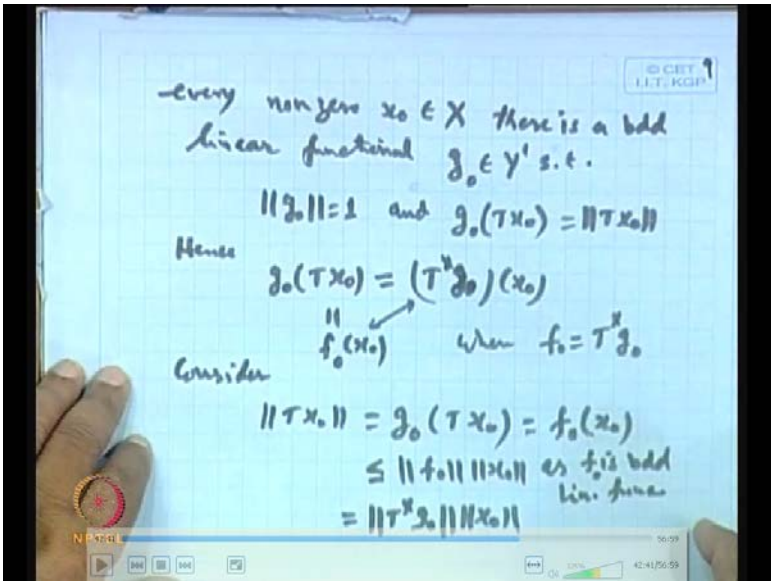

So, by the Hahn banach theorem, so, what we get is, Hahn banach theorem implies that by the Hahn banach theorem, we can say that for every nonzero x0 belonging to capital X, there is a bounded linear functional g0 in y dash such that norm of g0 is 1 and the value of this T x0 under g0 will be norm of T x0 is it ok or not because basically this x0 will give the T x0. So, now, applying the Hahn banach theorem for T x0, so, picked up a T x0 in y, nonzero T x0, there will be a bounded linear functional on y whose norm will be 1 and image of this T x0 under g0 will be the norm of T x0. This is by the Hahn Banach theorem. So, we get this 1.

Hence therefore, g0 T x0, by definition g of T x0 means a T cross g0 x0 is it not, the way in which we have defined, then this will be equal to by definition of the operator, this g0 T x0, but g 0 T x 0 and is nothing but f 0 x0 which is the same as f0, f0 x0. Now consider, so, writing this where f0 means T cross g0. This I am taking as f 0.

Now, consider norm of T x0. By definition, norm of T x0 is g0 T x0. So, this is g0 T x0 and g0 T x0 is nothing but the f0 x0, because of this.

And then f0 is given what, it is a bounded linear functional. So, it is less than equal to norm of f0 into norm of x0 as f is bounded linear. Bounded f0. f0 is bounded linear functional. So, we are getting.

But f0 is nothing but T cross g0. So, this is equal to norm of T cross g0 norm of x0, but T cross g0; that can be written as because T cross is bounded already we have shown.

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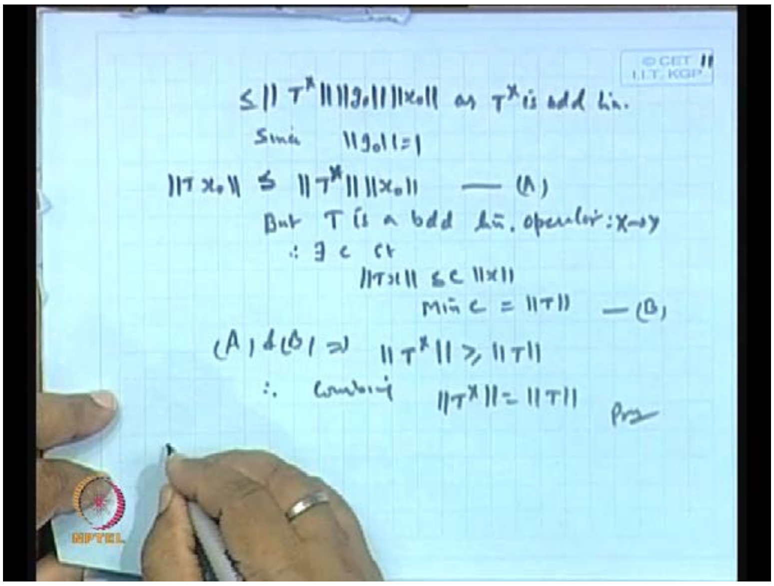

So, this will be less than equal to norm of T cross g0 into norm of T cross norm of this into norm of this.

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I will take another norm of T cross norm of T cross, this is less than equal to norm of g0 into norm of x0 as T cross is bounded. Already proved earlier, T cross is a bounded linear operator linear functional is it not. So, T cross is bounded. So, we are getting this.

Now, from here, we can say, since norm of g0 is 1, so this is equal to norm of T cross sorry it is cross into norm of x0 and this right hand side was T of a norm of T x0. This is less than equal to let it be a.

But we know, but T is a bounded linear operator, linear operator from X to Y. Therefore, there exist a c therefore, there exist a c such that norm of T x is less than equal to c times norm x and the minimum value of c is norm T.

Now, if we look the a and b simultaneously, then norm of T x0 is less than equal to this. Norm of T x is less than equal when minimum value is norm T. It means what will be the relation between T cross and T, the T cross cannot be less than norm of T because norm of T is the minimum value of c.

So, a and b implies that norm of T cross will always exceed by norm of T. Therefore, combining these two, we get combining earliest, we get norm of T cross is the same as norm of T.

(( )) for any x.

Which one?

This all relation (( ))

Let's say T x less than equal to c 0.

For any x. For any x.

So, x is particular And a is for a particular x0. a is particular x0, but even that this constant cannot be exceed by the norm of T cannot be less than norm of T. It may be a greater than or equal to norm of T. So, that is why it is going to be this. So, we get from here is that norm of T x0 is less than equal to norm of… Therefore, this proves this.

Now, let us choose the example where this adjoint operator, how the adjoint operator can be represented by means of a matrix. Just like every linear operator T in a finite dimensional space, we can represent by a means of matrix. Here we will show that T cross which is a adjoint operator of T will also be represented by a matrix and in fact, it will be the transpose of the matrix obtained by T. So, let us take an example here.

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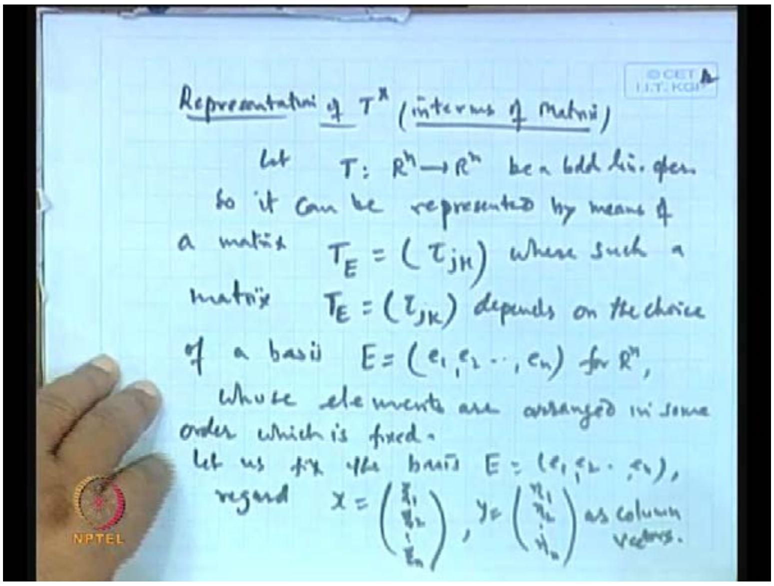

The representation of let us take a new page, representation of T cross in terms of the matrix.. So, let us suppose T, let T be a bounded linear operator from Rn to Rn, be a bounded linear operator from Rn to Rn is it ok? T be a bounded linear operator from Rn to Rn and since this is a finite dimensional. So, it can be represented by means of a matrix where such a matrix by means of a matrix, T E which is said tau of j k which depends where such matrix, T E tau of j k depends on the choice of a basis of a basis E, e1, e2, en for Rn; whose elements are arranged in a definite order, in some order which is fixed in a definite order which is fixed. That is what means the very clear is whenever we want the linear operator, representation of the linear operator in terms of the matrix, then first we have to fix up the base order of the element. Base order of the basis is fixed because a vector space can have a so many basis is it not. So, let us fix up the basis, first element will be e1, second element e2 and e n.

Once you fix up the basis element, then a linear transformation will give you a matrix and that matrix will totally depend on this basis. If we change the basis, the corresponding the transformation which give the another matrix like this.

So, let us fix up the here the basis element. This basis element is chosen with respect to x n y coordinate. So, let us fix the basis E as e1, e2, en as e1, e2, en regarding with regard to x as column vector xi 1, xi 2, xi n and y as eta 1, eta 2, eta n as column vectors.

So, let x and y are the column vectors in there and then with the help of this, you fix up the basis E so that the T E will be fixed. We will have a unique representation here.

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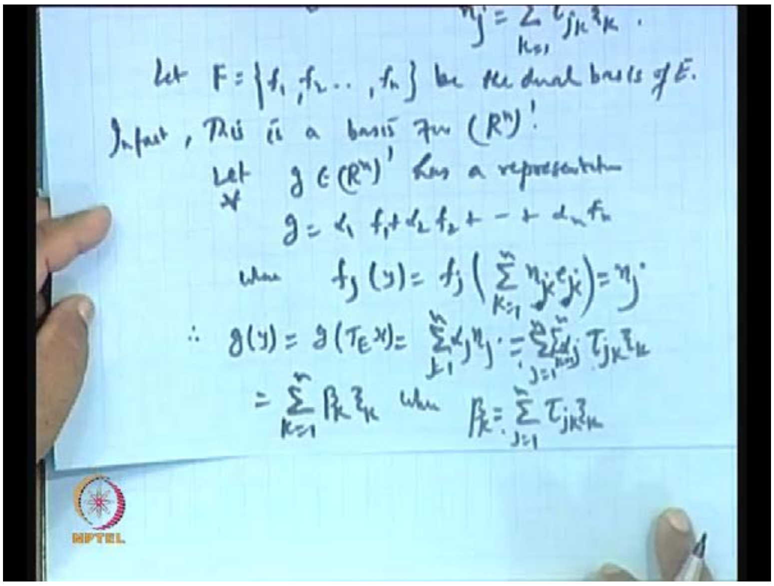

Then, y of y will be equal to T E x and the component form where, what is T E? T E is tau j k. So, where the eta j is the component sigma tau j k xi k; k is 1 to n, is it not.

Now, let us take a set F: f 1, f 2, f n be the dual basis of E, dual basis of E. In fact, this is the basis for Rn dual. Dual basis we have discussed. So, Rn where F f j x k is the chronicle delta e xi is the chronicle delta.

So, it means, let f belongs to g belongs to Rn dual has a representation. Representation is alpha 1 f 1 alpha 2 f 2 alpha n f n every element, in fact, this is true for every g belongs to f we have a representation like this. Then dual basis of this and where what is a there? Where f j y means f j y is what? y is an element in R n is it not. So, R n means it is of the form sigma eta j eta j or eta k ek; k is 1 to n.

And now, this will be f j e k will be 0 when j is different from k and otherwise it is 1. So, basically we are getting eta j. So, we are getting this one. Now from here, we can say. So, this one therefore, we obtained is gy is equal to g of T E x which is equal to sigma j equal to 1 to n alpha j eta j because y is an element which is alpha g eta k, eta j you can write in this form. So, you can write sigma j.

(( ))

which expression?

Summation k is equal to is equal to n. This one, k equal to 1 to n.

(( )).

[fl] This is k.

(( )).

This j alpha j and now tau of j k xi k. The summation, this is the double summation. k is equal to 1 to n. Now this you interchange the summation. So, this we can write it as sigma k equal to 1 to n beta k xi k where beta k is summation j equal to 1 to n tau j k xi k.

Now, this will represent our matrix T cross. It means we should transpose this matrix a and then you are getting T cross. So, this will show that if f x is equal to,

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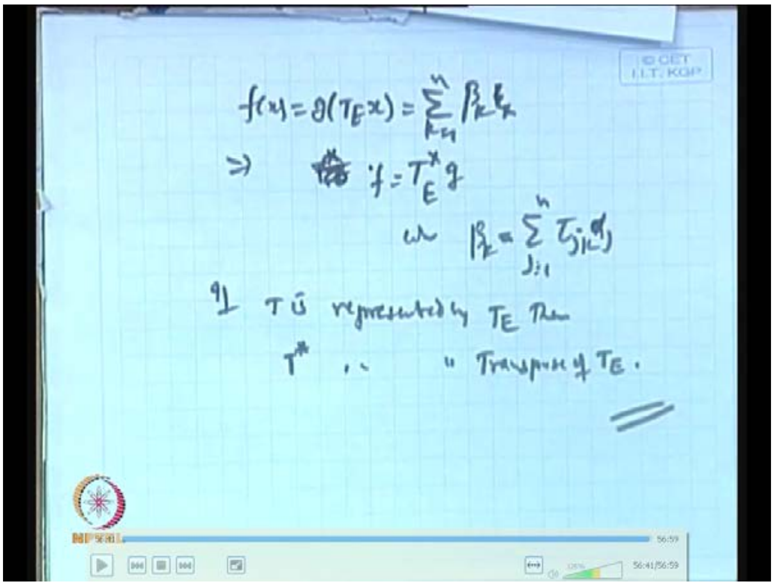

So, f x equal to g of T x which is equal to sigma k equal to 1 to n beta k and xi k. Now, this implies that T cross E, this T cross E, this g equal to f where beta k is the component form is tau j k alpha j and j is 1 to n. So, we say if T is represented by T E, then its adjoint operator T cross is represented by the transpose of the TE, transpose of T E. This

I will explain later on. Let us see. Thank you thank you very much. I will explain this thing in next class.