Functional Analysis

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Module No. # 01

Lecture No. # 35

Closed Graph Theorem

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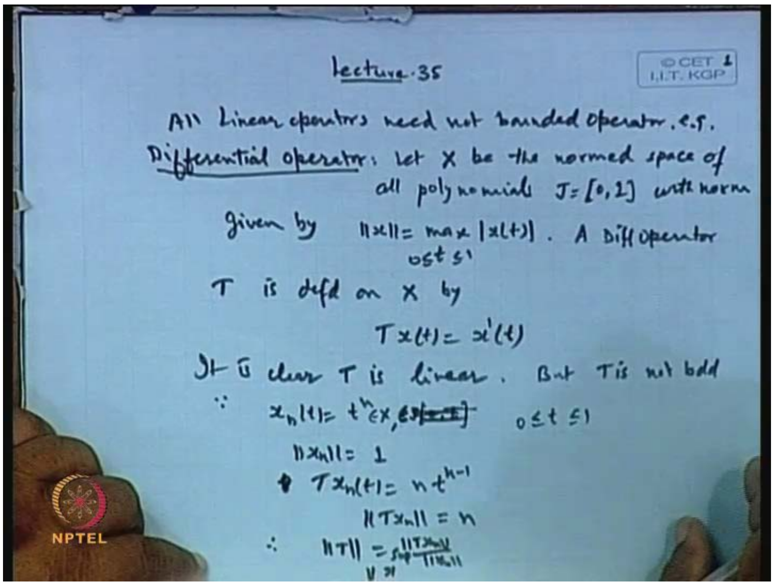

In earlier lecture, we have discussed the linear operators, bounded linear operators and also some unbounded linear non-linear operators also. Now, most of the linear operators, when they are define over certain, and then boundedness will imply the continuity and so on. So, it is a very smooth way dealing with the linear operators. But the entire linear operator need not be bounded. So, all linear operators need not be bounded operator.

For example, if we look the operator say, differential operator, which is a unbounded operator. Why? Because let us suppose, let X be the normed space of all polynomials, polynomials on J closed interval 0 and 1 with norm given by norm of x as the maximum of mod x t, where t ranges from 0 to 1 over J and a operator, a differential operator T is defined on X by T x t equal to x prime t.

Now, if we define the operator T in such a way, then we see that this operator clearly is a linear operator because it replace x by alpha x plus beta y, accordingly the derivative will come here and we get the linear, but t is not bounded.

Why? Because if we look say operator particular elements say in J like x n t if suppose t to the power n, this is an elements belonging to 0 1, J 0 1 is it not. A set of all polynomial now belonging to x, this belongs to x basically where belongs to x, set of all polynomial because this is a polynomial and t lying between 0 and 1.

Now, what is the norm of x n? Maximum of this is 1, but T of x n T of x n T is equal to n t n minus 1 and norm of T of x n is nothing but n because the norm of T to the power will be 1. Therefore, norm of T which is norm of T x n over norm of x n supremum is taken over x such that norm of x n is 1, that will be equal to n.

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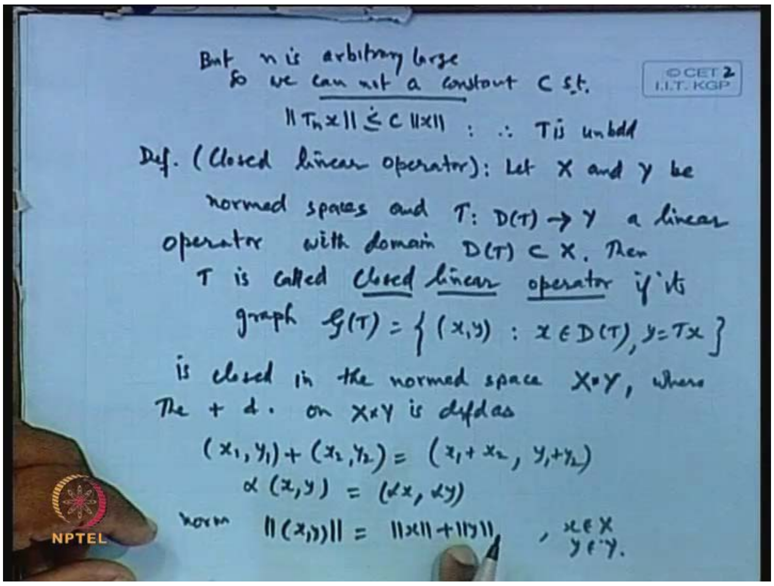

But n is arbitrary, but n is arbitrary so, but n is arbitrary large, one can take. So, we cannot get a constant C such that modulus norm of T n x is less than or equal to C times norm of x. We cannot get a constant C, which satisfy this condition. Therefore, T is unbounded.

So, here we have seen one example which is a linear operator, but is a unbounded operator and you know the differential operator plays a vital role in developing the calculus or the theory of the (( )) calculus, quantum mechanics theory; they use very widely this differential operator, hence many other branches where you require the differential operator.

So, most of this application part requires the unbounded linear operators clear, but it so happens that this all linear operator or unbounded this type linear operator, this linear operator so far we have discussed, though they are unbounded, but they are closed.

So, we can say that, in a state of choosing the linear operator, if we start with a theory of a closed linear operator, if we start with a closed linear operator, then one can get more and more results in compared to only the linear operator. So, here we will discuss today the concept what the closed linear operator is and then under what condition, this closed linear operator becomes outdated.

So, that we will discuss, that will leads to the important theorem which is known as the closed graph theorem. In fact, this gives you a sufficient condition on a closed linear, on a linear operator, closed linear operator to be bounded.

So, let us see first, what is how to define the closed linear operator. Now, when we go for this definition (( )) closed linear operator. Let X and Y be normed spaces be normed spaces and T, a mapping from the domain D T to Y, a linear operator with domain D T contained in the source of set of X, then T is called closed linear operator, if its graph, this we will denote by g T which is the set of all ordered pair (x, y) such that x belongs to the domain of T and y is the image of x under T is closed in the normed space in the normed space X cross Y, where the operation on X cross Y is defined as delta, where the addition and scalar multiplication on X cross Y is defined as: (x 1, y 1) is one element, (x 2, y 2) is another element. The addition is defined as x 1 plus x 2 y 1 plus y 2 and alpha times of x comma y is defined as (alpha x, alpha y) and the norm on this x order pair x t is defined as norm of x plus norm of y, where the x belongs to capital X and y belongs to capital Y. So, if we use this operations, then X cross Y becomes a vector space and under this thing, it becomes a normed space.

Now, this is not only the norm. We can also define the norms in some other way like the maximum of norm x and norm y. This is another way or maybe we can define under root this square plus this square under root of this, that is also a norm. So, there are many ways of defining the norm, but we are choosing the norm as this.

So, what is that closed linear operator is basically a linear operator is said to be closed when its graph is closed. What do you mean by the graph is closed? This graph is closed means that, so, let us discuss this.

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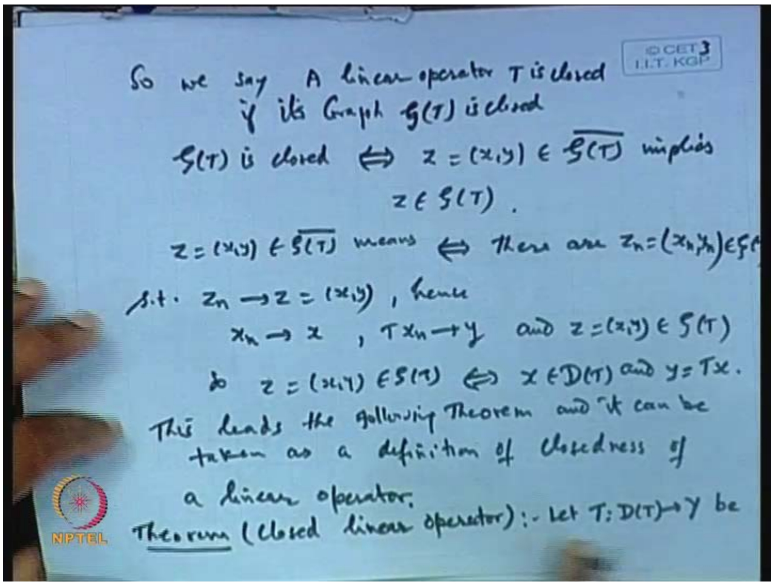

So, we say, a linear operator a linear operator T is closed if and only if this graph is closed. Is it not? T is closed means if and only if its graph sorry if its graph g zeta T is closed. What do you mean by this zeta t is closed. Zeta T is closed means it contains all of its limit points. So, zeta T is closed if and only if all of its limit point belongs to it, if and only if that is if I take a point z in the closure of this set, then it implies z belongs to it implies that z belongs to zeta T. Is it not. z belongs to that is what mean.

Now what is the meaning of this z x y belongs to zeta bar means, that there will be a sequence x n z n in zeta which converges to z under the topology, under that norm. So, this means as that if and only if there are sequences z n x n y n in zeta t belongs to zeta T belongs to zeta T.

Such that such that zeta n this z n will go to g is it not .z n will go to z which is x y. It means the coordinate y is converges is a hence x n will go to x and t of x n will go to y and this z which is x y, must be a point in zeta T. It means this means, so, z which is x y belongs to zeta T if and only if, when it belongs to the zeta T when a point belongs to the graph of T ,when x is lying in the domain of T and y must be the image of that.

So, if and only if x belongs to the domain of T and image y is the image of x. So, that is the meaning when we say a operator is closed if its graph is closed and when graph is closed means basically we are having this.

So, what is the meaning of graph is closed means, that there must be a sequence or if you take any point in the closure of this that point must be the point of this zeta T. It means the x coordinate must be the point of D T and y coordinate should be the image of D T.

Now, this whole discussion leads a very important result which will be used frequently and also as a definition of the closed linear operator. So, we say, this leads the following theorem and it can be taken as a definition of closeness of a linear operator. Now, what is this theorem? The theorem is, see here itself we can start theorem which we call it as a closed linear operator, that is also a definition one can use.

So, let x and let T is a mapping from domain D T to y be a, where D T is a subset of x, be a linear operator.

Where D T is a subset of x, x and y are normed spaces. Normed spaces are normed spaces and that let T be a linear operator D T is our (( )) then T is closed if and only if it has the following property. The property is, if x n converges to x where x n is a sequence in D T and image of x n under T goes to y, then x must be in the domain D T x will be in domain D T and image of x under T must be y.

Now, this definition is a if and only part. Earlier, we have started the definition of the closed linear operator in terms of the graph. What we told a linear operator will be a closed linear operator if the graph is closed. But graph is closed leads to this definition which is if and only part. So, finally, we can say t is closed if we have and this is very easy to apply in order to prove or to just verify whether the operator is a closed linear operator or not, What we do is we simply pick up a sequence x n in the domain d t, whose limit point is x, then find out the images. If that image also converges to y then, we find the relation between y and t if y comes out to be t x and limit point x is in d t then only we say it is a closed one.

Now, this concept is entirely different from the concept of the continuity. What we say is the continuity, if an operator is a continuous operator, then what we see here a sequence x n converges to x will automatically implies the t x n will go to t x .There is not required at all the convergence of t x n to a point y and y is t x because automatically the continuity will implies when it will transfer the convergent sequence to the convergent sequence.

But in case of the closed linear operator it's not so. What we are doing, we are taking we are taking a convergent sequence this is a convergent sequence. But there is no guaranty that t x n will go to y converge, there is no guaranty. If at all it converges, this is necessary that we have to take the condition that this sequence converges, then only we can relate the image point with this x and the relation is that y must be t x and x should be the point in domain d t.

Now, domain d t need not be a closed set, because if it is closed then only the all the limits points belongs to it. So, that also we will see that if it becomes closed then the closed linear operator becomes bounded that is what the closed graph theorem says.

So, these two are entirely different concepts; however, there is a similarity. Slight similarity is what is the similarity is, suppose x n and y n are the two sequences converges to a same limit point ,the images t x n and t y n will also lead to the same image point even if the t is a closed linear operator .That is the similarity so that is problem (( )).

So, we are just defining this again. Now, let us come back to our closed graph theorem under what condition this closed linear operator is bounded. So, theorem which is closed graph theorem.

Now, what is this theorem says, let x and y be banach spaces, and t is a mapping from the domain d t to y, a closed linear operator, where d t is contained in x domain is a subset of subspace (( )) .Then if domain d t is closed in x, then the operator t is bounded. So, what is the closed graph theorem says, that if t is a closed linear operator from one normed space to an one banach space to another banach space remember, the closed linear closed graph theorem open mapping theorem, this requires the banach space to banach.

Uniform boundedness theorem requires the banach to norm, but hahn banach theorem does not require anything, it simply stays a functional an operator from one vector space to another vector space closeness is not required.

But here the closeness is both the x and y is must, similarly in case of the open mapping theorem also. So, x the operator t is bound. So, what is says is this closed linear operator will remain will be bounded when the domain is closed. So, this is the sufficient condition for a closed linear operator t is bounded and as soon as this is bounded because it is also linear. So, every linear operator bounded and continuities are the same. So, basically it becomes the continuous operator.

Let see the proof. So, first we will show that our space which is x cross y because the d is giving to be closed linear operator it means the graph of t graph of t is closed. So, first let us see the graph of t which is a subspace of x cross y is it not subset of x cross y it is a complete space complete normed space we wanted to show this.

Once it is complete normed space then, because x d t is closed graph is close. So, they will become a complete metric space, complete normed space banach space because every close subspace of a complete space is a complete. So, once they are complete limiting point has to be in that. So, that is the idea of the proof.

So, let us we first show that x cross y with norm defined by norm of norm of x comma y is the norm of x plus norm of y .With this norm is complete. This we wanted to show first. So, how to prove this is complete? Let us take a sequence arbitrary Cauchy sequence z n in x cross y and that sequence is converges the limit point belongs to x cross y then our space will be a complete normed space.

So, let us pick up this sequence, let z n, which is of the form say x n, y n; x n belongs to capital x, y n belongs to capital y be a Cauchy sequence be a Cauchy sequence in x cross y arbitrary Cauchy sequence in x cross y.

So, by definition of cauchyness, for any epsilon greater than zero there will exist a n such that the difference between norm x z n minus z m will go to less than epsilon then n m sufficiently large.

So, for the given epsilon greater than 0. So, for given epsilon greater than 0, there is an n integer n such that, the norm of z n minus z m this will remain less than epsilon which is the same as norm of x n minus x m plus norm of y n minus y m is less than epsilon when for all m n greater than capital m is it not.

Now, once this is less than epsilon because these are two non-negative terms are it not. These are two non-negative terms the total sum is less than epsilon. So, individually each term must be less than epsilon. So, this shows norm of x n minus x m is less than epsilon as well as norm of y n minus y m is less than epsilon for all m n greater than n.

Now, this implies the sequence x n is a Cauchy sequence sorry, may be epsilon is hardly matters. I am taking the upper one epsilon is very small quantity, but we cannot say this is epsilon by 2, this is epsilon by 2, no it may be possible this is less than epsilon by 4. And this may be less than 3 epsilon by 4. This total sum has to be less than epsilon that is all. Whether they may be less than epsilon by two each also or may be sum is less than epsilon by 4, other will be less than 3 by epsilon by 4. So, that is why I did not write epsilon n only this is the upper bound for this. So, this sequence x n and y n are Cauchy sequences in the normed space x and y respectively, but x and y are complete are banach space this is given. So, it must converge. So, these sequence these sequences will converge; x n and y n will converge to x and y respectively, where x belongs to capital x and y belongs to capital y clear.

Once we get, this implies that it means this sequence will converge to x and this converge to y. So, our sequence z n this sequence z n will converge to z which is x y is it or not. Why because in this case, in say a star, because in a, let m tends to infinity first. Keep n is constant. So, what we get norm of z n minus z is less than epsilon. So, we get this thing is less than epsilon clear. So, we get from here is this is less than for n for n greater than capital n.

But z n is a Cauchy sequence and which converges to a point z belonging to x cross y because x. So, Cauchy sequence, z n converges to z in x cross y. Therefore, x cross y and it is arbitrary Cauchy sequence is it not, any sequence any Cauchy sequence z n clear. So, x cross y is complete normed space means it’s a banach space.

Now, we will show our thing that is we wanted the operator t to be bounded. So, what we will do is, we will first prove the define an operator t from say graph of t to d t which is a bounded linear operator, then we will we apply the open mapping bounded inverse theorem and bounded inverse theorem will say the inverse operator is bounded and as a consequence we can say t is a bounded. So, that is again.

So, let us assume now since the graph of t is closed, this is given, by assumption this is closed in x cross y and domain of d t why it is closed because t is giving to be a closed linear operator because this is this was our theorem t is a closed linear operator. So, given because t is closed linear operator that is why. And domain of d t is closed in x. This is also our assumption. This is our assumption given if t is closed. So, g t graph of t is closed domain of t is closed in this therefore, graph of closed that is yeah. So, and consider. So, the both…

So, these are complete normed spaces. Why because these are the closed subspace of a complete space this is also close subspace of a complete normed space therefore, zeta t and eta t both are d t both are a complete normed spaces, subspaces you can say is it not subspaces clear.

Now, let us define a mapping p from the graph of zeta t to domain of t which carries the point x t x to x (( )) point in a graph of t and then image this now clearly p is linear and 1 1. Why this is linear let us take because.

(( ))

Domain and graph the points is graph is a ordered pair x comma y y is equal to t x.

So, x comma t x is a point and what a image of this is coming to be a point in d t. So, x now this is linear, why it is linear because if we take the point say x 1 comma t x 1 and then plus x 2 comma t x 2 and under this the image p. So, what you get is this will be added. So, basically you are getting x 1 plus x 2 that is all and this is the same as the sum of these two. Is it not? Is it clear not t is already giving to be a linear t is already given to be linear, is it or not. Then alpha of this x t x. So, by definition this will be equal to p of alpha x alpha t x and this will be equal to alpha x. So, it becomes alpha of p x comma t x. So, it is linear.

And why it is 11? t is 11, it follows very easily. p is 11 sorry. So, let us take p x 1 t x 1 is the same as p x 2 t x 2, where x 1 is different from x 2.

So, this will give p x 1 means x 1 is equal to x 2. It means, if I take the two images is identical, this is only possible when x 1 is equal to x2. So, this implies p is 11.

Then further we claim that p is bounded. p is bounded; bounded means norm of this image is less than equal to constant times the norm of it. So, consider norm of image of this. But what is the p of this will give norm x. Now, this is less than equal to norm x plus norm t x you are adding one more point real number here is it not.

So, this is the… but this is equal to is it not a norm of x comma t x by definition. So, this is norm of it. So, norm of t x is equal to norm of this. It means, the divide by this know norm p becomes one. So, p is bounded because there is a c is one and then p is bounded.

Then further inverse mapping. So, p is 11 linear it is a linear 11. So, inverse exist and bounded that also. So, basically the p inverse will exist. So, p inverse. So, the inverse mapping p inverse will be a mapping from domain d t to, because it is 11 onto and this inverse exist and what is this x will carry it to x t x .

Now, let us come to the bounded inverse theorem. What is the bounded inverse theorem? Yesterday we have discussed it. This theorem says, the operator if the if a bounded linear operator t from one banach space to another banach space is an open mapping and if t inverse exist, t is bi-jective inverse t is bi-ejective, then t inverse is continuous and bounded.

So, what is given is that operator p from banach space to banach space this must be an open mapping. Open mapping means sent the open set to open set and then if t is bi-jective, then inverse is continuous and that is bounded.

So, here everything is because these are all banach spaces. So, we can apply the closed linear operator theorem. So, by say bounded inverse theorem, we say this is inverse operator p inverse is bounded is it not p inverse is bounded.

Bounded means p inverse means bounded means this is the norm of this. So, x t x norm of this is less than equal to some constant b times norm of x.

So, there exist a b such that, this is norm of this. But what is here is let it be b, but norm of t x this can be written as less than equal to norm of t x plus norm of x, which is the same as norm of x comma t x by definition x comma t x is less than equal to b times norm x. So, this will be this will be.

This implies norm of t x is less than equal to b times norm x. Therefore, t is bounded, is it or not. For all x for all x, this is true for all x belongs to the domain d t. So, this implies t is bounded and that is what we wanted to prove the operator t. So, this proves the closed graph theorem is it clear.

Now so, we have discussed this closed linear operators and closed graph theorem and is now. What is the relation now? Can you say if the operator is closed then it will be bounded or if a operator is bounded can you say it is closed or none of the statement holds in general. Is it not just like continuity and boundedness we have, if the operator is linear operator, then continuity and boundedness are the same thing. But here the closeness and boundedness whether they can also be detected or if not then we should have certain examples where it violates.

So, basically here we say none of these statements is true means a closed operator need not be bounded always similarly a bounded operator need not be closed also. So, this is the result which we wanted to discuss, closedness does not imply boundedness does not imply boundedness of a linear operator of a linear operator.

Conversely boundedness does not imply closedness. Let us see this result. So, for example, first we will show that the operator is bound is closed but not bounded. Every closed operator need not be bounded. Every closed linear operator need not be bounded. This we have seen already for example, a differential operator discussed earlier is it not.

If we define the mapping t from the set of all polynomial over the interval zero one and define the t x equal to x prime t, we say that operator is a is not bounded operator is not bounded for example closedness that is why. So, let us take the operator differential operator t is not bounded operator but linear is it not linear bounded. Now let us see, that whatever the difference over that is an operator t from c zero one is it not to c zero one such that the image of this goes to x prime that derivative and lines. So, this operator was not linear was linear but not bounded.

Now, let us see we claim that this operator t that this t is closed. So, how to prove this closeness is, let us take closed means graph t must be close all if it prove that domain of d t is closed is it not x n x belongs to d t and t x belongs is y then it is also closed why because that result a operator t is closed and x n converges to x and the t of x n goes t x n converge to y, x belongs to d t and y is equal to t x then it is closed. So, we will use this (( )).

So, let us take, let x n in domain d t be such that both a sequence x n and the image t x n converges because for the closeness this condition is required is it not t x n must converge then only the y n x can be related to t x converges say, x n converges to x and t x n which is the derivative of x n converges to y.

Now, this convergence is in the in which in c 0 1 is it not c 0 1, the convergence is uniform because it is a maximum of mod x n minus y t is then or integral part. So, since the convergence in the norm of c 0 1 is uniform is uniform convergence is it or not convergence on the interval 0 1. So, from this from x n dash converges to x dash from derivative x n dash convergence to y, we have this integral 0 to 1 y tau d tau is the integral 0 to t sorry 0 to t 0 to t integral 0 to t.

Y t is the limit of y tau is the limit of x n prime tau when n tends to infinity d tau, this is just by definition because t 0 1, the integral the norm can also be defined in terms of the integral.

So, this is and this is nothing but what because the derivative exists. So, we can take the n limit is a continuous function. So, we can take the limit outside we can say limit n tends to infinity and then integral 0 to t x n prime tau d tau d tau.

And then this will give you differentiate integrate it. So, x n tau. So, the value will come to x n t minus x 0 minus x and 0, sorry this limit is what limit is x t I am sorry this limit and then limit of n this will give x t minus x0 is it or not clear (( )). This implies that x t is nothing but x 0 plus integral 0 to t y tau d tau. By this why, I have taken limit because the integration we require that is all. So, this shows that x belongs to d t why and x prime is y.

Why x belongs to d t, because what is the d t d t is, because d t is a subset of c 0 1 set of all functions, this c 0 1 have a continuous derivative d t is the d t is the where d t is the set of all functions x which have which have continuous derivatives continuous.

So, that is why this is a x limit point belongs to d t because it is a continuous norm because this is continuous integral part is continuous. So, x t will be a continuous function of t this is the. In fact, it is differentiable and derivative will come out to be y. By the first principle of (( )) first. So, we get from here this hence this implies t is closed because y is equal to t x. So, this gives. So, it is x 0 is a constant. So, will not defect. So, it is a bounded linear operator is it not this operator is an unbounded linear operator, but it is closed.

Now, the second part, we will discuss next time tomorrow. Thank you because it require certain.