Functional Analysis

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Module No. # 01

Lecture No. # 34

Open Mapping Theorem

Lecture we have discussed hann banach theorem and uniform boundedness theorem; the third week theorem we will discuss today is the open mapping theorem open mapping theorem, this theorem basically concerns with the open mappings.

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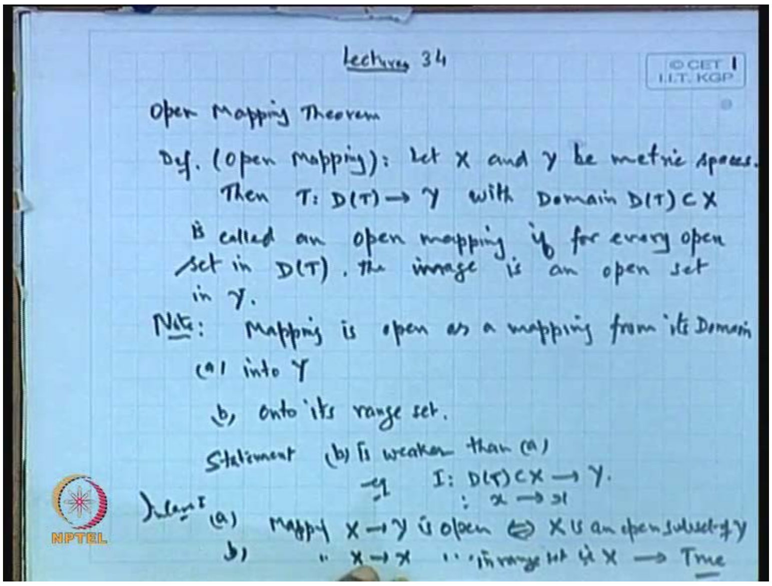

Open mappings are those mappings which send the open sets to open sets that are the image of the open sets, under these open mappings are open; the open mapping theorem gives a certain conditions under which a bounded linear operator becomes an open mapping.

So, we will basically discuss first what are open mappings and then the open mapping theorem what is the statement, and improving the open mapping theorem we require the category theorem - baire's category theorem as a tool for poof.

So, let us see what is our open mapping; let X and Y, definition open mapping, let X and Y be metric spaces, then an operator T from the domain D T to y with domain D T subset contained in x is called is called open mapping is called an open mapping if for every for every open set in domain D T, the image is for every open set in D T; the image image is an open set open set in y in y then such a mapping we call it as open mapping. Now, when we say that image of an open set under this mapping is open set in y, then we can have two types of possibility, one is, when this image when the mapping is mapping the sets from domain into y, and second one when it matches the sets in domain on to it is the adjoint.

So, these two types of possibilities are there, so we should be careful while writing this statement, because both are not equivalent condition one is weaker another one is stronger. So, basically, we say as a note or remark say that when we say the mapping is open mapping mapping is open is open mapping is open as a mapping as a mapping from its domain from its domain into y this is the first possibility, and second is onto its range set range set y.

Now, we claim that first statement b is weaker statement weaker than a, statement b is weaker than a, the reason is, for example, if we take a mapping, say, from identity mapping I from domain D T which is subset of X to say y which is an identity mapping X to X, and in that case if I consider the mapping as our first case when X is into mapping say here then the X has to be open in y.

Then, in the first case we can say when X is mapping say here be there the mapping X to X from which is open, the mapping in case of the first, in case of a mapping from X into y is open if and only if X is open in y, X is an open subsets of y subset of y, while in the second case when we say the mapping X to X is open then automatically it is true into range set, it is an open in the range set, that is X itself, then it is automatically true, because X is only taken to be an open sets; so, this part is weaker, this is weaker, and this one is stronger here, you have to put a condition also that is all.

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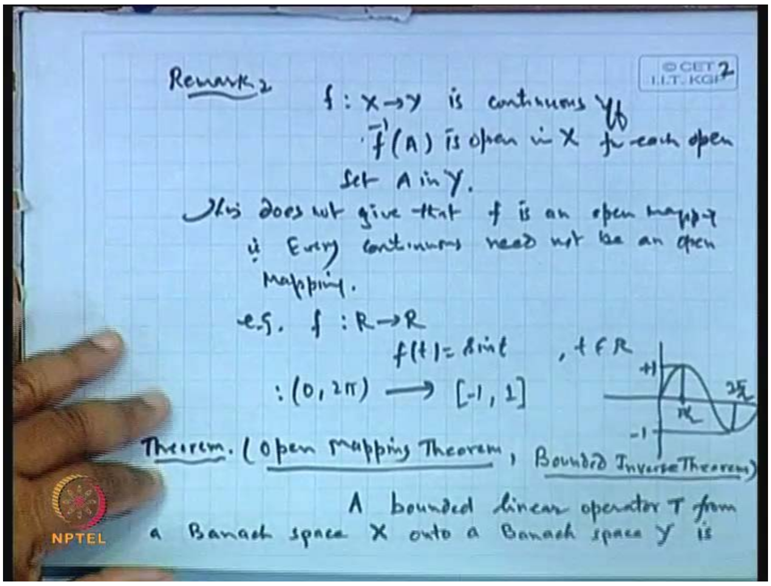

Now, another point which we have already noted that, earlier that a function f is continuous function, a function f from X to Y is continuous if and only if the inverse image of its open set f inverse of a is open in X for each open set a in Y, that is the inverse image of the open set if it remains open then such a function is said to be continuous function and vice versa. Now, it does not mean this does not mean give you that f is an open mapping, that is not necessary that every continuous function is…, that is every continuous function need not be an open mapping.

For example, if we take a continuous function say mapping from R to R, and the function is say f such that f of t is sine t where the t belongs to R, this is a continuous function, well define function. Now, when we say f is a continuous function when we take the image of this 0 to 2 pi under this mapping then this mapping transfer 0 to 2 pi under into a set minus 1 to 1 under the mapping sin t, Because sin pi by to is 1 and sin minus pi by sin pi 0 2 pi sin, this is a sin curve, sin 0 is 0 sin pi sin 2 pi; so, this is pi by 2 here is 3 pi by 2 pi plus pi by 2, where it is a plus sign, 1, and here it is minus 1; it means, this is an open set, but under the continuous function it does not transfer to R open set.

So, every continuous function need not be an open set; therefore, some extra condition is require for a function to be an open mapping, that is why…, so this gives you the next theorem which is known as the open mapping theorem.

So, let us see the theorem now, this theorem which is known as the open mapping theorem; now, as a consequence or when we add some extra condition then we get another theorem which is known as the bounded inverse theorem. So, basically both the theorem can be stated in one way is stated, and we have only we require some extra condition to give the statement for the bounded inverse theorem; so, what is our open mapping theorem, the open mapping theorem says a bounded linear operator T from a banach space X onto a banach space y, y is an is an open mapping.

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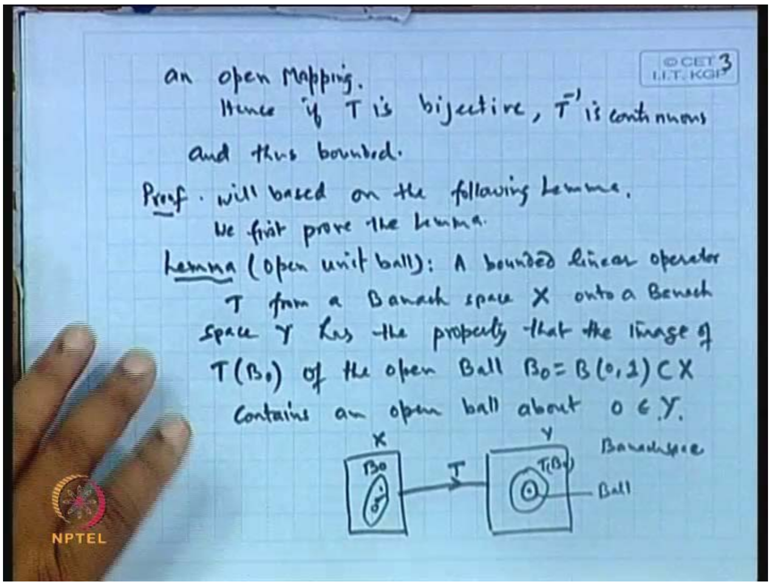

So, this is the statement of the open mapping theorem that a bounded linear operator T from a banach space X onto a banach space y is an open mapping. Now, if we bounded linear operator T from banach space into banach space y is an open mapping onto a banach space y is an open mapping. Now, if we have for extra condition, if T is bijective, T is bijective, then T inverse is continuous T inverse continuous and thus bounded.

So, if we put extra condition T is to be 1 1 bijective means 1 1 1 2, then one can easily show that T inverse exist, then it will remain continuous and bounded and this statement will give you the bounded inverse theorem. So, bounded inverse theorem say that if a bounded linear operator T is given from banach space to banach space onto and it is bijective mapping then inverse operator will also be bounding that is what bounded inverse theorem says.

The proof of this theorem basically based on the foreign lemma, so before going to the proof we will state a lemma, the proof will based on the following lemma, what is the lemma is…, so we first show first we first prove the lemma.

The lemma is which is actually given for the open unit ball, the lemma says a bounded linear operator a bounded linear operator t from a banach space X onto a banach space y has the property that the image of an open ball B naught of the open ball B naught centered at 0 and radius one which is contained in X, under the mapping T contains an open ball open ball about the point 0 in y.

So, what this theorem says is that if suppose this is our X, and here this is say y, both are given to be banach spaces, these are banach spaces, remember this open mapping theorem and the corresponding lemma is valid only when X and y both are taken to be banach not just like a bounded uniform boundedness theorem we have chosen the domain X to be the banach, but y to be a normed space.

In case of the hann banach theorem, we do not require at all any completeness, simply X and y to be a vector spaces and real or complex and we can extend it, but in here in case of the open mapping theorem and as well as the closed graph theorem we will see that both X and Y must be a banach space.

So, what this tool lemma says a suppose T is a bounded linear operator from one banach space to another banach space, and if I take a set B naught open ball B naught centered at 0 and radius 1 then image of this ball under B operator T must contain an ball around the point 0 in y; so, this is the ball centered at 0 and radius y a ball about ball 0. Now, if this lemma, let us assume the lemma is true, we will prove the lemma afterward, let us review assume the lemma is true the proof when goes of the theorem goes like this.

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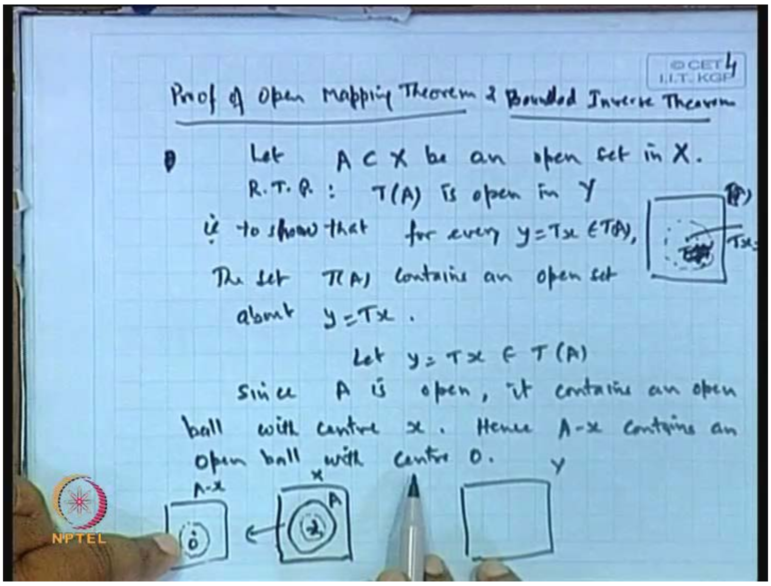

So, first proof of this theorem proof of the open mapping theorem mapping theorem and bounded uniform boundedness theorem, sorry, and this bonded inverse theorem inverse theorem we will assume the lemma is true, and let us see the first proof of this, and we will go for the proof of lemma.

So, what we have, what is required to prove in case of open mapping we want a bounded linear operator T from banach space X onto a banach space y is open mapping, it means if I picked up a open set in X, then the image of this open set X under T must be an open, that is what we wanted to show.

So, let us see for every open set…, so let every open set y has a image let a which is subset of X be an open set in X, now what is required to prove is, required to prove is that image of this a under T is open in y, this we wanted to show. So, how to prove this thing to be open, if this is our T A and we want this to be open, it means, there must be some ball open set centered at T A with a suitable radius which is totally contained in y, if I prove this part then its y becomes open.

So, basically what is required to prove is, the T is open means, that is, that is to show that for every y which is the image of T x an element of T A for every y belongs to a the set T A the set T A has contains an open set in y, open set in open set about the point y in which is T x in that is what we wanted to prove.

That this T A is open, it means, if we want to take any point y whose image say T let it be this T x which is y, and this set is our T A set; then if I find an open set centered at this point which is totally contained in T A then T A will be open, so that is what we want.

Let us take let y which is T x belongs to T A; now, since a is given to be open since a is open, so it contains an open ball an open ball it contains an open ball with centre X the centre X. Hence A minus x contains an open ball contains an ball with centre 0; what is the meaning of this, this is suppose X, and here this is y, now this is the point A is this is our set a in X, it is an open set, so if we picked up any point X here then we can find an open ball around this point which is contained in a; now, if I make a transformation from A to A minus x A to A minus x, it means the centre will x transferred to 0.

So, this ball will transfer to a ball whose centre is 0, that is what he say; if a is open it contains an open ball with centre a then now linear transformation a minus x will transfer this centre x to the origin, so corresponding ball will have the centre 0, so this will have a open ball with centre 0, that is what he say.

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Now, let us let this radius of the ball be r, let the radius of the ball be r; now, consider a set k to be say 1 by r, so r becomes 1 by k, then what will be the k A minus x the k A minus x, this will be if suppose this is our set a, then alpha of a will be something like this, it is magnificated with alpha, it is positive or negative; so, depending on…; so, each point x will give the position will take the position alpha x like this.

So, k of A minus x which is which contains a ball at the centre 0 with a radius r; when you take k of this, then the radius will be k r, and k r means, it is k is 1 by r, so it becomes 0; so, centered at 0 and radius 1 that this space be that, so A minus x contain open ball a then k of A minus x contains the unit open unit ball v centered at 0 and radius 1, that is what…

Now, let us see this lemma, the lemma says that if a T is a bounded linear operator from x to y, then image of an open ball under the operator T whose ball is center 0 and radius 1 will contain an open ball about the point 0. So, this is an open ball here in this set, and T is an operator from here to here, so the image of this will contain an open ball as for the lemma, so by lemma by lemma what we get it now is, implies by lemma it is implied that or it is implied that the T image of this T of k A minus x, that is because T is a linear bounded linear.

So, k can be taken outside, and this T A minus T x this thing contains an open ball contains an open ball about T x is equal to y about T x equal to y, is it not, but y is an contains an open ball about 0 not T x equal, because this no, because this contains…, so about this will contain an open ball about 0 about 0; and so, k is simply a scalar, so T A minus T x this set will contain will contain a ball about 0; therefore, T A will contain, therefore T A will contain a ball about the point T x y is equal to T x, therefore and but y is an arbitrary set point is an arbitrary point, so T A will be open, and that is prove the result. So, we have shown that image of an open set under this is an open y clear. So, if T is a bounded linear operator then every open set will have an open image in y that is the proof of the open mapping theorem.

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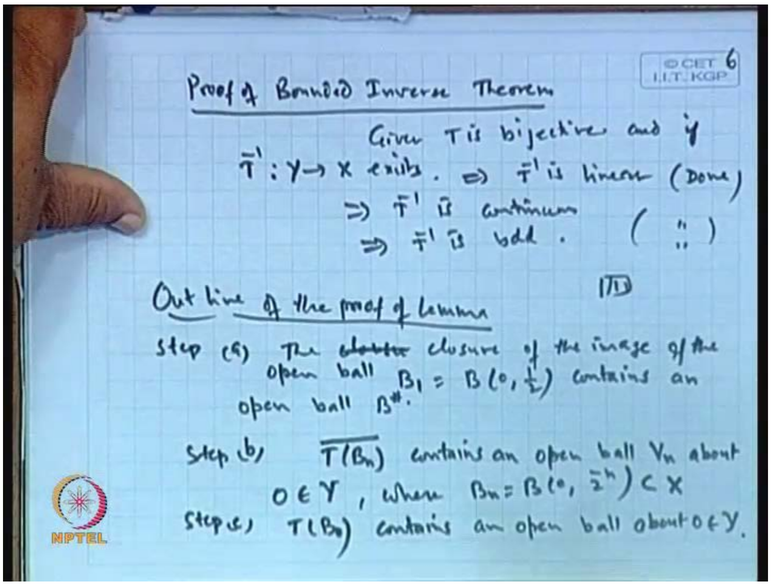

Now, let us prove the boundedness bounded inverse theorem; the proof of the bounded inverse theorem proof of the bounded inverse theorem, here one extra condition is given the T is bijective, that is, 1 1 and 1 2; so, once T is bijective, so inverse will exist; and if inverse exist, and if T inverse from Y to X exist, then it is now there is one result which we proved earlier if T be a bounded linear operator, and T is 1 1 1 and 1 2, inverse exist, then inverse is also linear, so this implies T inverse is linear, this was already done earlier, and again because T inverse is linear and T is continuous. So, T inverse will also be continuous, and once it is continuous then it is bounded.

So, this already we have shown earlier as if and when we dealt with the normed spaces, then we have proved that in case of the linear operators every linear operator continuous and boundedness is the same thing. So, we have proved the T is linear. T inverse is exist when T x equal to 0 implies X is 0, that is one oneness, and when T inverse is their continuity is their T is continuity inverse will be continuous and it is bounded, so this completes the proof.

So, we have proved the open mapping theorem in a, but what is left? Now, the proof of this lemma, now proof of this lemma we can drop or should we add it, I think let us drop it or let us see the outline of the proof let us see the outline of the proof of lemma; let us see, so what we wanted to prove in the lemma is that a bounded linear operator from one banach space to a bounded linear operator T from a banach space X onto a banach space y has the property that image of an open set contains an open ball 0, this we wanted to prove.

So, this we will prove in three steps; first step is, we will prove the closure of the image of the open ball open ball B 1 centered at 0 and radius half contains an open ball open ball B star, so this the first step one. Then step two or step B, we will say the closure of this T B n, that is the image of the ball B n with centre 0 and radius 1 by 2 n contains an open ball v n about the origin belongs y where B n's are the ball centered at 0 with a radius 2 to the power minus n and lying x. And third finally, steps is that image of this ball B naught under T contains an open ball about 0 belongs to y. So, this will be the steps in proving the lemma; let us see the details of this step proof of the step.

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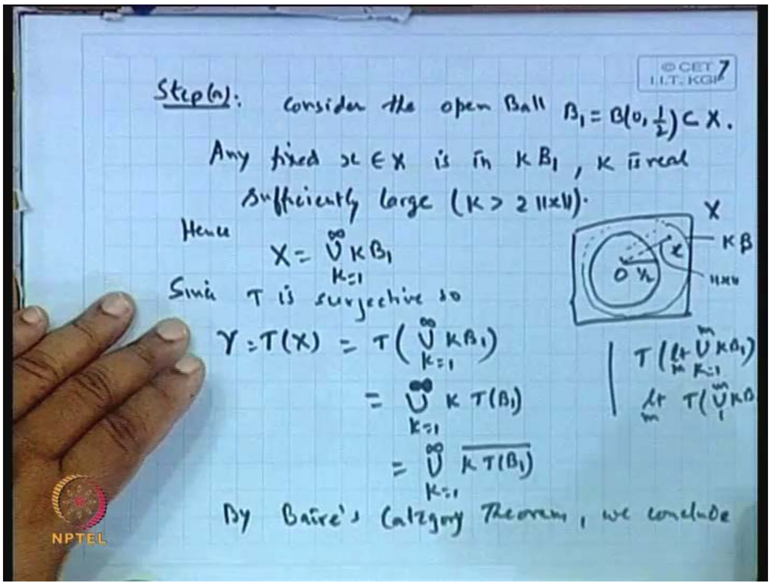

So, first the step one, a to prove this step a let us consider the open ball consider the open ball B 1 centered at 0 with a radius say half which is contained in x; now, if we picked up any point X here, take any point X here, then with a suitable k suitable scalar K one can find out the K belongs to x belongs to K B 1 for a suitable K.

So, any fixed x belonging to X is in K B 1 with where K is real constant in K B 1 with K is a real constant and sufficiently large sufficiently large say K must be greater than two times norm of x, why it is so, because this is our ball centered at 0 with a radius half, and x is this, this is our x.

Suppose, I take X here any fixed X; now, this is the norm of x, this will be the norm of x, suppose distance from the origin, now if we take K times B 1 then each of this will be enhance and we get a ball around this point, this is our K B, so k must be as large, so that the X point lies within this K B; it means, the radius of this ball must include this point norm of x, so what is norm of x K by 2? Norm of x is less than K Y 2. So, if I take the norm of x then a by 2 then multiply by the radius of half norm of x become less than 1, it means, it will conclude, it will include in this point, that is what we want clear.

So, that is….; hence now x is an arbitrary point, so we can pick up an arbitrary point here and corresponding we can find that suitable K, so that we can get in B 1; so, x can be expressed as the countable union of all B K B 1 with K 1 to infinity, is it not, every point x belongs to 1 of the K B 1, where it is called suitable K, and K must be at least this one.

Now, since T is given to be surjective, T is surjective, why because the lemma is…, this lemma was this, what the lemma T is mapping from banach onto banach, so because of the ontoness T is surjective, so T is surjective, so the image of T x which is the whole y is T of image of this union, but T is linear operator, so k can be taken outside, and we are getting this is union of union of k is equal to 1 to infinity, this is infinity k of T B 1 clear, because T is a linear operator, so it can get the summation first k equal to 1 to m, m tends to infinity, T is a this we can write it T of limit union k equal to 1 to m over m and k B 1.

Now, because T is continuous limit can be taken outside and we get T of image 1 to m k B 1 and again T is linear, so this can be taken outside, and we gets the 1 to infinity and like this, so 1 to infinity K T. Now, can we say this is also equal to 1 to infinity K T B 1 closure, why? Why it is equal, because what is the closure, closure means it will include the limit point of this sets also, but when we take the union of all such elements in the sets in case one to infinity it is nothing but the entire y, so it means that also include the limit points.

So, even if you take the closure of this it is not going to change the equality sign, so it will remain the same, so y can be expressed as countable union of the close set hence by baire's category theorem; now, apply the baire's category, category theorem at least one of them will contain the open set by definition.

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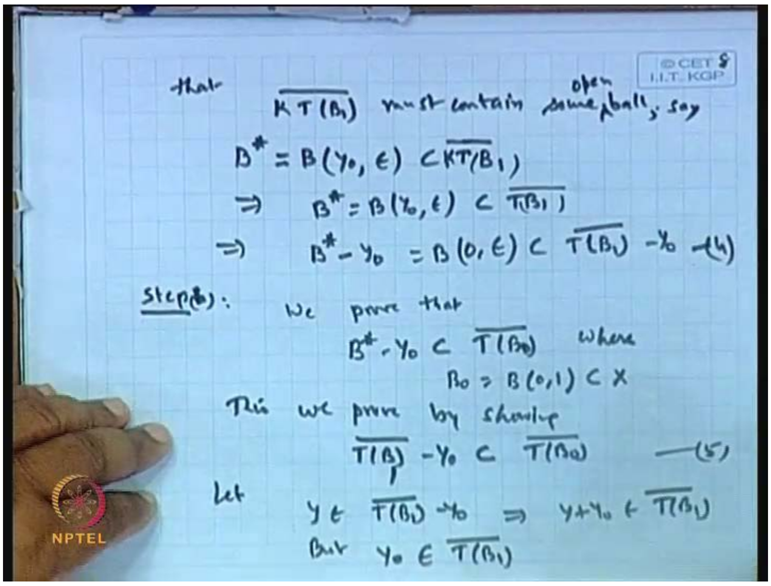

So, we conclude that this set that K T B 1 closure must contain some open ball some open ball, because what is the category theorem says is, if x is complete then it is a of second category; it means, if can be expressed as a countable union of the closed set A k then one of the A k must contain the open ball. So, let us suppose, that this must contain an open ball contains an open ball, say, I am saying this is say B star which is centered at 0 and radius epsilon contained in T B 1 closure since K times T B 1 must contain open ball, so this will also contain the open ball or let it be K.

Then this implies the ball B star which is B y naught epsilon is contained in T B 1 closure T B 1 closure this bracket, I will use because K is simply an constant we transfer the image send the points each point of T B 1 2 K times of that point, that is all. So, if this contains the open ball some open ball, this will also contain the same open ball with centre y naught and radius; now, this follows it follows that if I take a linear again transformation, that is, transferring the set y naught to origin, so B star minus y naught will contain a ball centered at 0 with a radius epsilon in T B 1 closure minus y naught, this is also just by itself.

So, this shows the first step, what is the first step? What the first step was…, this was our first step the closure of the image of an open ball B 1 contains an open ball, so I have in the closure of the image B 1 contains an open ball B star minus y naught that contains in B 1.

Now, second step, proof of second step B, so what we prove that, we prove that B star minus y naught is contained in T B 1, this we wanted to prove, where what is B naught? Where B naught is given where B naught this is given in this theorem, that is, this is our B naught open ball B naught centered at 0 and radius 1 contain in x, this is our B naugh.

So, this we showing in that; now, what we show, this we show, this we prove by showing the T B closure, T B 1 closure minus y naught is contained in T B naught closure T B naught closure. Now, let it be this, let it put the number numbering is may because it will require, so let it put this number 5, and then others number 4 then we have to go through…; what we require is B naught minus this will be 4 like this. So, let us take only 4 3 and others number we will clear; now, let y belongs to T B 1 closure minus y naught, so this implies that y plus y naught belongs to T B 1 closure.

And if we remember that y naught is this one, but y naught is the centre of this point, so y naught belongs to T B 1 closure already, because it is a centre of that point T B 1 y 1, B 1 was the centre, and we have shown that y 1 is the centre, this one, so y 1 is already in the T B 1. So, it’s we get, this element belongs to closure of this y naught belongs to closure of this, so there is one result if a point belongs to the closure, there must be a sequence of the point in T B 1 which will converge to this as well as there is another sequence of the point which converge to this.

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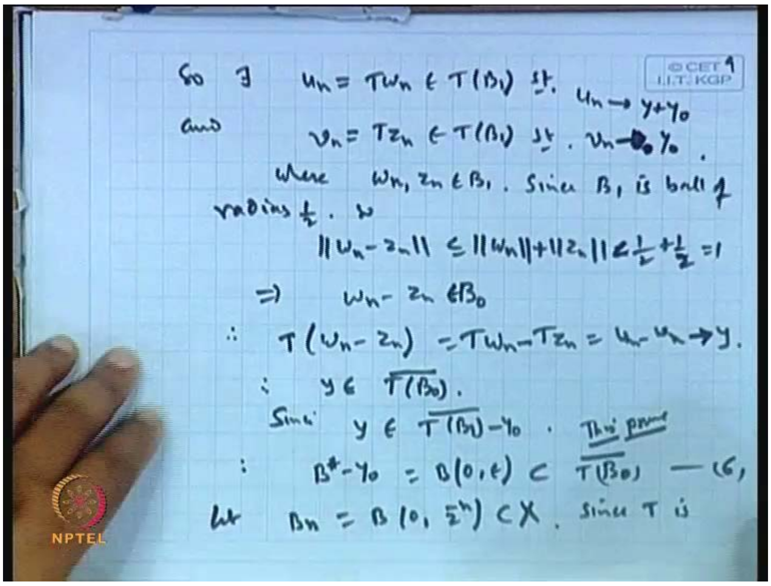

So, there exist, so there exist, the sequences w n there exist u n which is say T w n in T B 1, such that, u n will go to y plus y naught and sequence v n which is T of z n belongs to T B 1 such that v n belongs to y naught by definition of this, v n belongs to y naught, that is all, where w n z n, where w n z n belongs to B 1 .

Now, since B 1 is a ball centered at B 1 with a radius say half has a radius since B 1 is a ball of radius half, so we can say…, so what is the norm of w n minus z n, this is less than equal to norm of w n plus norm of z n, and this is equal to half plus half less than equal to half plus half, and that is 1; it means, w n minus z n, this will be B naught, because B naught is a centre 0 and radius 1. So, this length is less than 1, so it will be the point in 1 and again T of w n minus z n that will be equal to T w n minus T of z n and that will be equal to u n minus v n which tends to y.

So, we are getting the point y; therefore, y belongs to this tends to y, y belongs to closure of T B naught B 0; now, since y is an element of T B 1 closure minus y naught, this shows… so, this proves our step 2, this we wanted to show. So, if y belongs to here y belongs to this, so this proves the that this proves three steps three steps.

Now, we thus say we have B star minus y naught which is equal to B centre 0 with a radius epsilon is subset of T B naught, clear; and let B n is suppose six let B n is centre 0 with a radius minus 2 to the power minus n contained in x.

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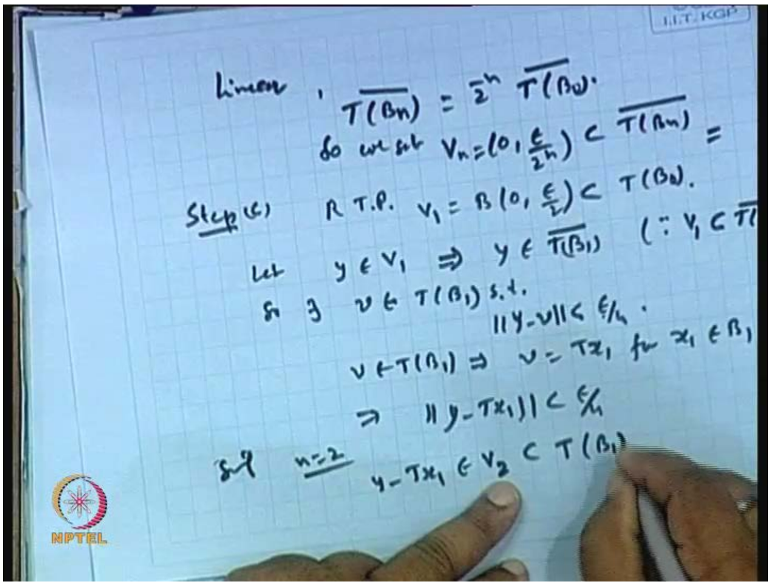

Now, since T is linear T is linear; therefore, image of T B n under this will be 2 to the power of minus n T of B naught closure, and from this we get from this we obtain. So, we get that v n which is 0 epsilon by 2 n, this will be contained in T B n is closure, and this completes the function and step third C again.

Now, so, what we finally prove required to prove is that this ball v 1 centered at B with a radius epsilon by T 2 is contained in T B naught this we wanted to show. So, let us take the point, let us take y as a point in v 1 then y will be, so y will be in T B 1 closure, why because v 1 is a subset of T B 1 earlier that is proved T B 1 hence this belongs to…

So, if y belongs to the closure it means there must be some point available in T B 1 which is close to y, so there is a v belongs to B 1 such that such that norm of y minus v is less than say epsilon by 4 very close to this. So, now, v belongs to…, imply this, but v is a…, so v belongs to T B 1 implies that v is of the form what T x 1 for some x 1 belong B 1 for x 1 belongs to B 1, again T x 1 and y will be very close.

So, we get norm of y minus T x 1 is less than epsilon by 4 substituting this thing, and from here take n is equal 2 again, similarly if we continue this, similarly for n is 2 we get y minus T x 1, again this is an element belongs to v 2 centered at 0 with the radius 1 by 2 square, and then it contained in T B 1 closure, so difference between this again…

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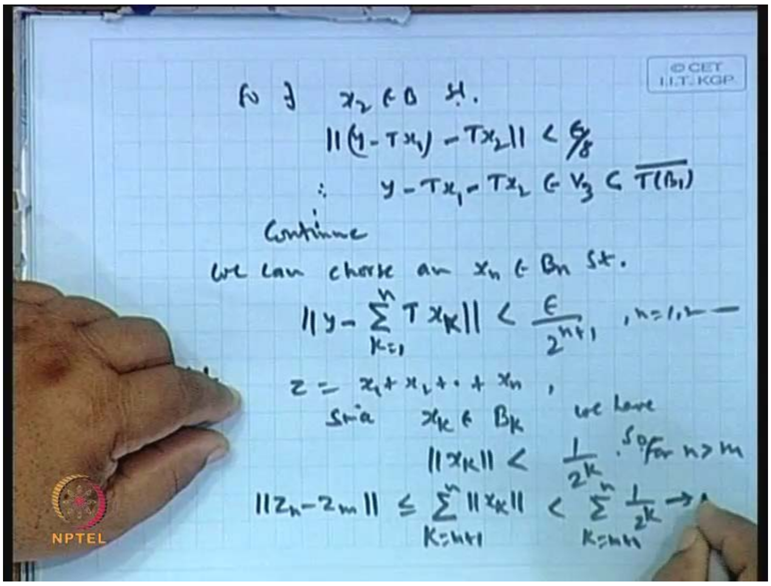

So, there exist an x 2 belongs to B such that norm of y minus T x 1 minus T x 2 is less than epsilon by 8; therefore, y minus T x 1 minus T x 2 belongs to v 3 which is contained in T B 1 closure and so on, so continue this is steps what we get in the last we can choose we can choose an x n belongs to B n such that norm of y minus sigma k is equal to 1 to n t of x k x k norm of this is less than epsilon 2 to the power n plus 1 n is 1 to n and.

Now, let us take z to be the x 1 plus x 2 plus x n since x k since x k belongs to B k, because these are the centre's we have the norm of x k is less than 1 by 2 to the power k, now this implies, so for n greater than m we get norm of z n minus z m this is less than equal to sigma k equal to m plus 1 to n norm of x k, and this will be strictly less than sigma 1 by 2 to the power k, k is n plus 1 to n to be tends to 0 as m tends to infinity as m tends to infinity.

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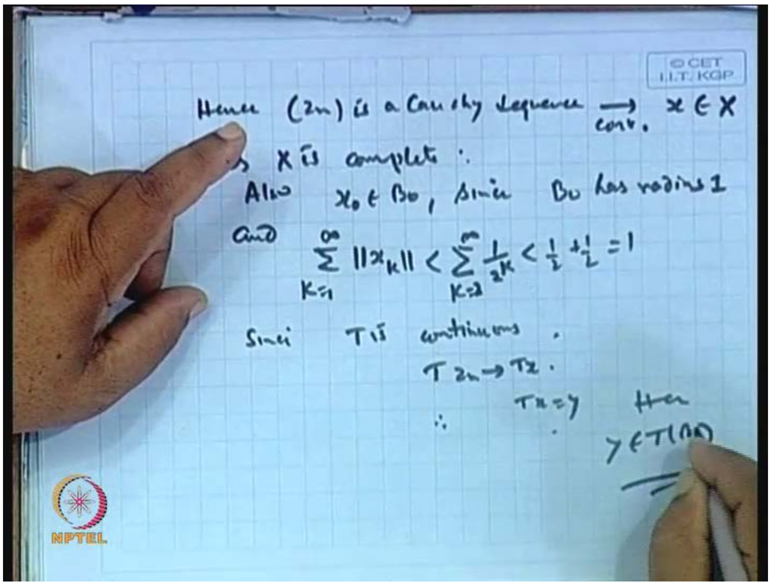

Hence z n is a cauchy sequence cauchy sequence cauchy sequence, but and which converges to converges to x belongs to capital X as capital X is complete capital X is complete, also x naught belongs to B naught, since B naught has radius has radius 1 and sigma of the norm x j x 1 norm of k equal to 1 to infinity norm of x k which is less than sigma k equal to 2 to infinity 1 by 2 to the power k which is less than equal to half plus half which is 1, norm of this is less than k equal to l to infinity, not l to infinity, this is what 1 to n etcetera then we are getting this one, since T is continuous, therefore image of T z n will go to T x and thus T x is equal to y, hence prove the result. Thank you, that is all.