Functional Analysis
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Lecture No. # 33
Baire's Category and Uniform Boundedness Theorems

So, in the last lecture we have discussed the Hahn banach theorem in details and all the form for the vector space for complex vectors in case of the complex vector space, and generalized form for a norm space and soon. The next theorem which we will discuss today will be uniform boundedness theorem; as we have seen or we have told already there are four fundamental theorems, Hahn banach theorem, uniform boundedness theorem,open mapping theorem, and closed graph theorem.

Hanhbanach theorem does not require the completeness, it is simply we takea norm space or we take a metric space, and an extension of the linear functionals are guaranteed over the norm space with the help of norm in Hanh banack theorem. But rest of the three theorems, and that is uniform boundedness theorem, open mapping theorem, and closed graph theorem requires the completeness of this space x. So this also shows that the banach space as a very important role in development of the functional analysis or in the theory of the functional analysis.

A lot of application of these uniform boundedness theorem, open mapping theorem, we can get we can see in the subsequent topics of banach theory or banach spaces.

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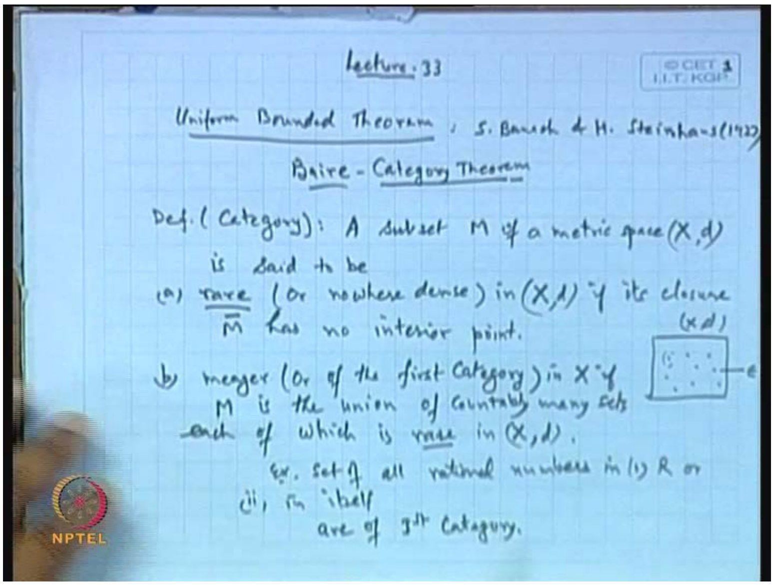

The uniform boundedness theorem is basically, uniform boundedness theorem, this is basically done by Sbanach and H stein Haus Steinhaus in 1927, this is also known as the banachsteinhaus theorem, some author say banach Steinhaus theorem, some book you can find the uniform boundedness theorem.

The uniform boundedness theorem the proof of this requires the category theorem, so before going for this Hanh uniform boundedness theorem we will see discuss the category theorem and the related conceptsof the category.So, basically we are interested in first find in get discussing the baire's category theorem, and then thesubsequently we will proof for uniform boundedness theorem to go for the category theorem where category theorem we require certain definitions or terminology for it.

There are two types of terminology, one is the old one,and another one is the new one, sowe will use the old one inside the bracket, and the new one we have has outside the bracket. So, we first discuss that categorya subset Mof a metric spacecapital X da subset M of a metric space X d is said to be rare is said to be rare or no where dense setor no where dense in X in X, means, X d in X d ifits closure,that is M bar has no interior point.

That is the meaning is, suppose this is oura metric space X d and these are the elements of m, these are the elements of M; now, this set collection of this set is said to be rare or no where dense set if the closure of this set has no integer point, it means, if we draw an open bar around any point then this open bar should not totally contained inside the closure end bar, then such aset M is said to be a rare or no where dense set, no where dense set is a old terminology and rare is a new one.

Similarly, we say ameager,a subset M of a metric space is said to be meager or of the first categoryfirst category in X if Mis the union of is the union of countably many set many sets each of whichis rarein X d rare in X d. So, a set subset M of a metric space said to be meager or of the first category if m is the union of countably many set each of which is rare in X d.

For example, if we choose pickup this setsayset of all rational numberin R or in itself set of all rational numbersrationalnumbers in set of all rational numbers in R or in itselfin R or in itselfset of all rational number in R or in itself, these two sets are of are of first category, why because this is our real line, say real line minus infinity to infinity here is 0, these are the pointsset of all rational numbers.

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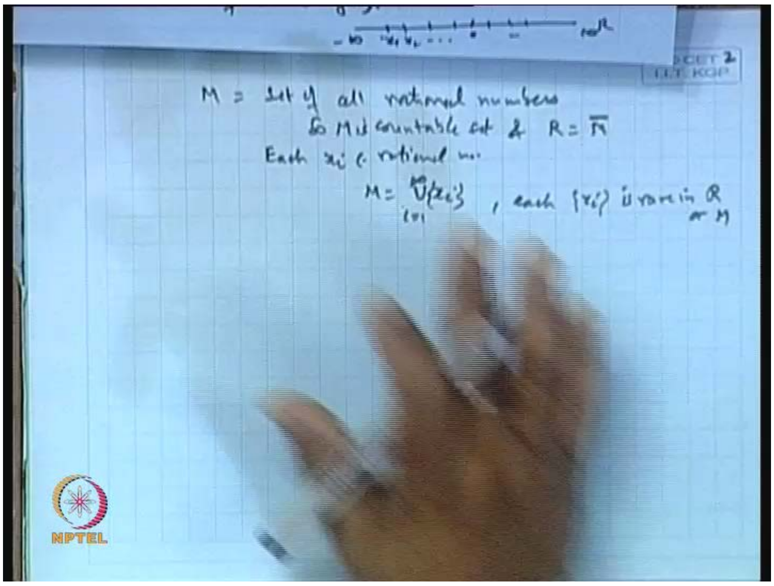

So, M is the set of all rational number,M is the set of all rational numbers;this is set of all rational numbers now you know the set of rational number are countable set, soM is countable, sothese are countable sets; now each one of these if I take each of these say x 1, x 2, x 3, and soon, suppose I take and like this, then each one of thisclosure of this is nothing but x 1, soeach one each x i(s) which is in rational number, it is a rational number, the closure of thisis I mean union of countable many set which is therea countable.

And so, let us said M is a set of pressure, and the set x that is R is the closure of this,R is the closure of this rational in R whose closure is R. So, each of M is the union of the countable set if M is the union, if M is the union of countable set each of which is rare in this, soM can be written as union of xI,this set i is 1 to infinity.

Now, each x i each x i this set is rare in R or M or n m, because the closure of this if we take any open interval close this it is not contain totally in a or totally in R, because in between the two rational theyare the irrational numbers also, sowe cannot choose like this, soboth are of first category, clear.

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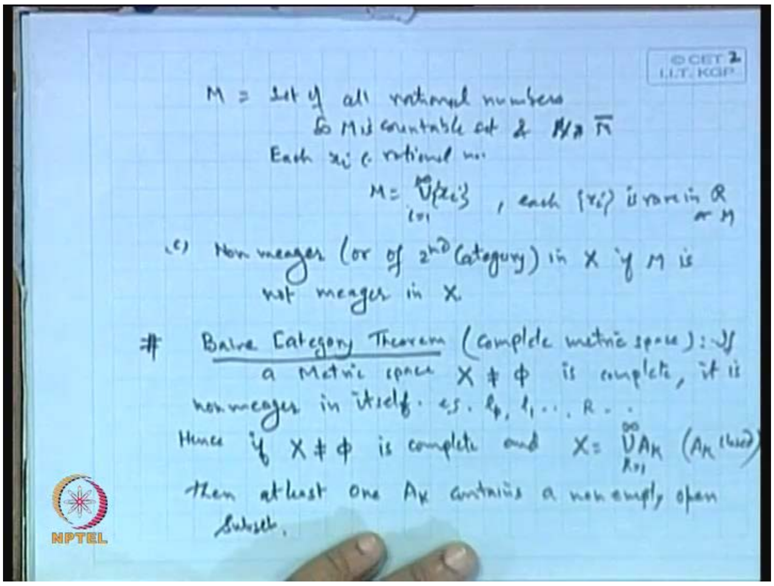

Then this another definition is non meageror of second categoryor of second category in X, a subset Mof a metric space is set to benon meager or of second category in MX if M is not meager in X meager in XM is not meager in X, this is let it be this caseoknot meager in X.

So, if m can be expressed as a countable union of the sets union of this and at least one of them has athat one of them has a non empty open set means interior point then such a set will be a second category or this. So, this type of set is of second, and we willprove that if aspace is complete then it will be non meager in itself or it will be the second category, now this is what we will show in the baire's category theorem. So, let us see the first result the baire's category theorem category theorem, this is for a complete metric space.What this theorem says, if a metric space if ametric space capital X which is non empty is completeand is complete then it is non meagerin itself non meager in itself.

So, baire category's theorem says that every complete metric space which is of course non empty will be of second category that is what. So, all the l p space they are of second category l 1 space, etcetera,for example, l p space l 1 space R and c and soon, these are all second category spaces;the same thing we can sayas a consequence is we can also write this baire category theorem in a more suitable form is that. Hence if X which is a non empty set and is complete, and if X is a union of A k(s) countable union of A k(s) where A k(s) are closed closed, then at least one A k contains a non-emptyopen subset.

So, as a consequence of baire's category theorem we can say that or a more suitable form of baire's category theorem is, if X, if a complete metric space is represented by means of accountable linear of A k(s) where each A k is closed then at least one of the A k must contain a non-empty open set subset, that is what we are (( )); let us see the proof for it.

So, we wanted to show this a every complete metric space is of second category or non meager in itself, suppose this is not true, suppose a metric space which is non-empty and complete, but it is of first category then we should reach a contradiction, that is we cannot be able to write X in the form of this where one of this A k(s) contains non-empty interest; let us see the idea of the proof is (( )), suppose the metric space suppose the metric space X which is non-empty is of first category in itself first category in itself.

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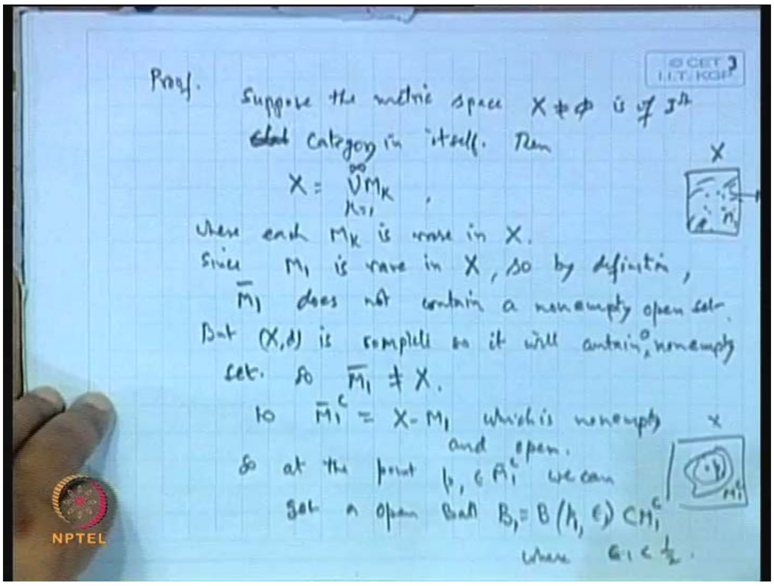

It means,X can be represented asthe countable union of M k(s) countable union of M k,where each M k is rare in X, whereeach M kis rare in X is of first categoryis rare in X, it means, the closure of this it does not contain any interior point,that is what it is.Now, we will show this part as a contradiction, that is, we cannot be able to express X as a countably union X k, where each M kcountable union of M k, where each M k is rare unless x is not complete.

So, when we say X is not empty X is then an is of first category, then we will reach certain contradictionlet us see how. So, the proof is like this, we will construct a Cauchy sequence in M, and since the M is complete, soif Cauchy sequence must converse to it point in M and that point that point if it does not belongs to any one of the M kand that limit point belongs to X because X is complete, soif that point does not belongs to M k then X cannot be represented into this point, sothat is the idea of the proof.

So, let us suppose since M1is rare inX which is a complete metric space, so then… So by definition M1 closure does not contain a non-empty open set, so by definition definition M1 closure does not contain does not contain a non-empty open set, because M1 is here, so this is our X, and this is our M1.

So, because it is rare it means by definition of the rare a set is saidsubset M of a metric space is said to be rare if each of this point does not contain any… If the closure of this does not contain an interior point, so that is why M 1 closure does not contain ainterior point, means, it was not contain any open sets, but X is given to be completeis complete. So, M 1X d is complete, therefore if we take any… And M 1 does not contain.

So, since M1X is complete, soevery Cauchy sequence is convergent, it means, the limit point belongs to it, so it will have an open set which is totally contain inside it; therefore, since X will complete, soit will contain a non-empty of a set, so it will contain non-empty open set contain a non-empty open set that is some point.

Now, sinceour M 1 closure M1 closure does not contain a non-empty open set, but X d is complete contains a non-empty, it means, M 1 closure cannot be X, so M 1 closure is not equal to X, sothere will be some complement for it, so this is our M 1 closure this is the M 1 closure, now this M1 closure M 1 closure will contain the some open set. So, M 1 closure closure of this will be X minus M 1 and of the M 1 bar is not empty, and soM1 which is non-empty which is non-empty obviously because this contain some point X, and this M 1 closure is a subset of X and open.

So, M 1 closure is thissay I am just putting here say this is our M1 closure just this. So, this is a nonempty and open set it means at some point we can find. So, if p 1 belongs to is… So p may… So m at choose the point since it is non-empty; so at the point p 1 which is in M1 closure M1 closure, one can find an open ball which is totally contain inside it, so at the point p 1 which is inM1 closurewe canget a open ball say B1centered at p 1 with a radius say epsilon 1, which is totally contain in the M1 closure, and let us suppose the epsilon one is less than say half.

Now, since this M1 closure is an open set non-empty and open, so we can find out an open ball around this with a radius say epsilon 1, which is totally contain in the M1 closure, so this is 1.

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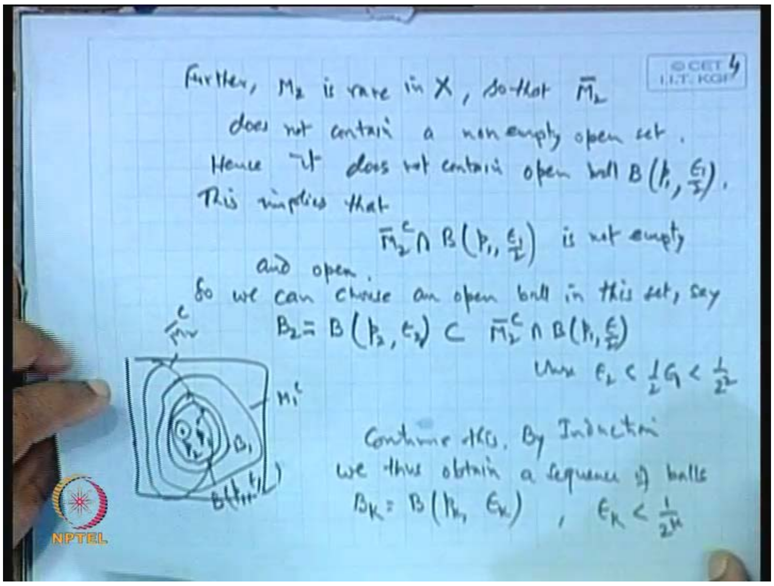

Now, further M2is again rare in XM2 is rare in X, so that closure of thisM2 closure does not contain by definition does not contain any non-empty open set does not contain a non-emptyopen set,clear. So, if it does not contain an open set it cannot contain the open ball this, so hence it does not containit does not contain open ball B say centered at p 1 with a radius say epsilon 1 by 2, because it does not contain any non-empty open set, so it obviously, it will not contain the ball centered at p 1 with a radius less than epsilon by 2, but it is closure will contain the open ball.

So, this impliesthis implies that the closure of this M 2 closure intersection with this B open ball B1 with the epsilon y by 2, the closure of this is not-emptyand open, and it will bean open set and open, because M1 M 2 it does not contain, so closure of this X minus M 2again is an open set and non-empty and since this ball is not contain inside, so intersection part will remain open and non empty.

So, it will have certain… So we may choose… So that we may choose… So we can choose, because it is non-empty an open ball in this set, say B 2which is centered at B 2 with a radius say epsilon 2 which is contained in M 2 bar closure intersectionthe ball p 1 epsilon by 2 and epsilon by epsilon2 is less than half 2 square say epsilon 2 is less than half of epsilon 1, but epsilon 1 is already less than half, sobasically this is less than 1 by 2 square. This is….

So, we can get there open ball in that, it means,thisis ourif we draw the bigger figurethis is say M1 closure, here we are getting p 1, and this ball say B1. Then what we get it is the p 1 with the radius epsilon by 2 we are drawing a ball B2, this is our MB, this centered p 1 and radius epsilon by 2, so b this centered p 1 radius epsilon by 2; now, this ball n intersection withM2 closure M2 closure will be non-empty, so there will be a point p 2.

And we can draw the ball around this point p t with a radius less than 1 by 2 square, then it is totally contain inside it that is fine. So, continue this process, socontinue this; so,by induction we can sayby induction we thusobtained a sequence of ballssay B k(s) with centered B k and radius say epsilon k, where each epsilon k, where epsilon k is less than half to the power 2 k, such that, such that the intersection with M k is non-empty and it is an open set, andthe next ball B k plus 1 will contain inside the B k in side this B k that is.

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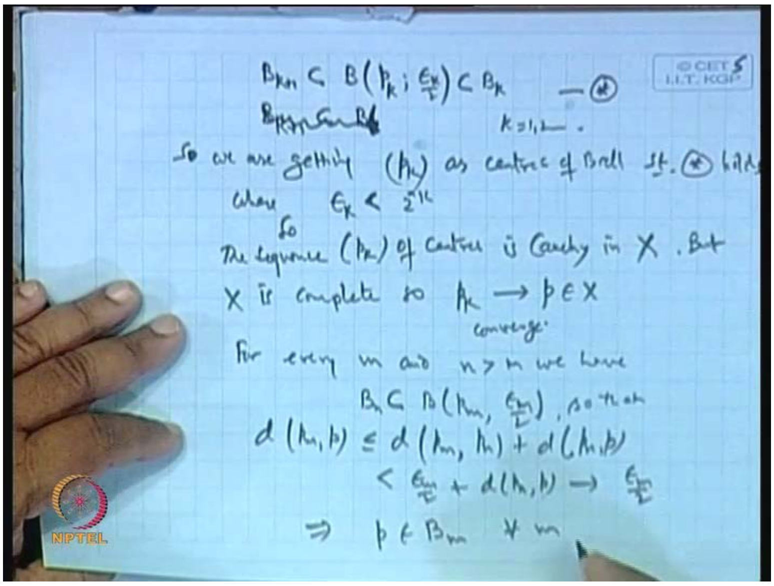

In fact, it will contain between B p k B k plus 1 is contained in ball bcentered at p k with a radius epsilon k by 2 which is contained in B k. So, the B k plus 1 is contained in this, that is what M k is this k is 1, 2, 3; if by induction we are getting slowly the ballsinside these closures, and B1, B2, B n are the sequences of the balls which has a center p 1, p 2, p n.

So, we are getting the sequence p k, now since sowe are gettingsequence p k sequencep k as the centers of the ballballscenters of balls such that this holds, such that, this say star holds; so, we can say that isand where the X whereepsilon k is less than 2 to the power minus k, sothis is going slowly to 0, it means,this sequence of the center, sothe sequence p k of centers of centers is a Cauchy sequence Cauchy sequence in X, but X is complete X is complete, sothis Cauchy sequence will converge, sothis sequence p k will converge to say p belongs to Xwill converge to pbecause it is complete.

Therefore, this point p will be thus limiting point of all p k(s) limited point of; now, the distance between p k and p under this will also go to 0 as m tends to infinity, that is p will be the limit point of this, that is what we wanted, because the reason is this assumption for every M for every M and n greater than M we have we have B n is contained in B with centered p m and radius say epsilon M by 1, so that the distance between p m and p will remain less than or equal to say epsilon mby 2, why? Because this is less than equal to p m p n plus d of by triangle inequality p n p; now, this is less than epsilonm by 2, because of p m is the center and p n belongs to it, and this distance p n p it goes to 0 because of this, sothis total will go to epsilon m by 2; it means, the p is the centered,and p m is a point belonging less than epsilon m by 2.

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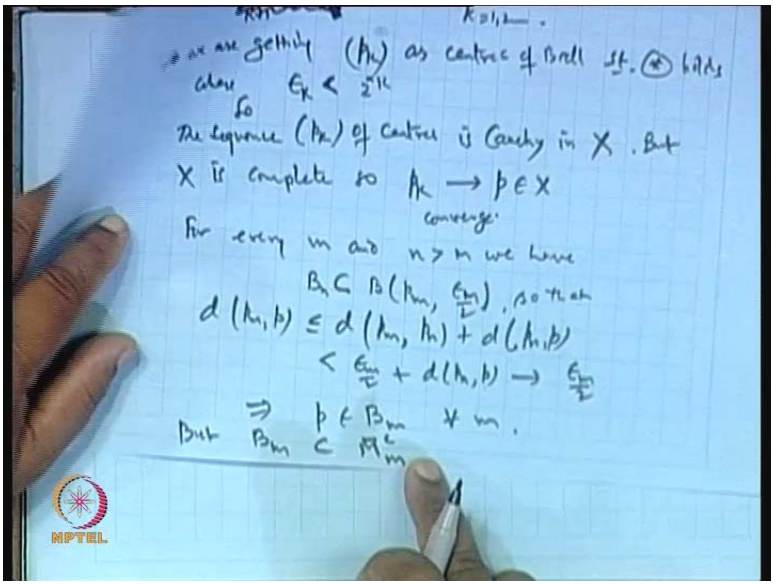

So, it belongs to the class p belongs to theballB m and this is true for every m, but B m is contained in the count, but B m is contained in the complement part of the closure of

Mm, so what he says if p belongs to B m and B m is in this, sop will be the point in the complement part of Mm closure, it means, it cannot belongs to the Mm for any m.

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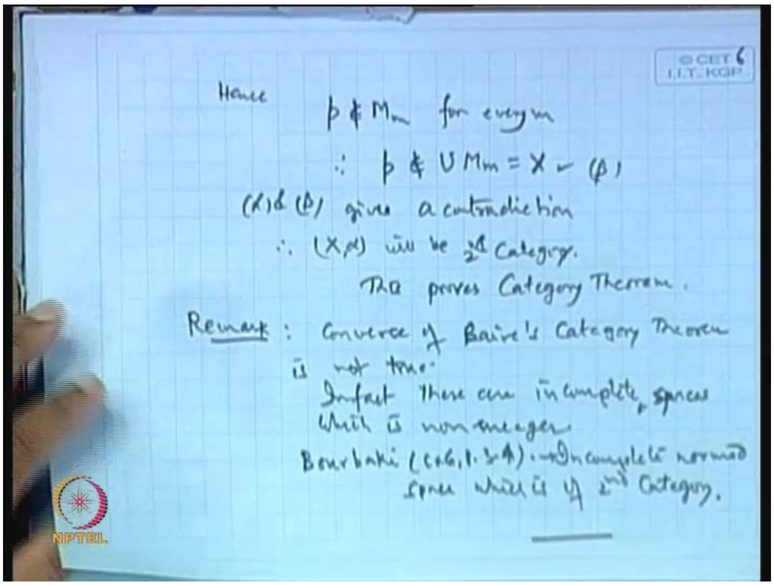

So, this implies, hence p does not belongs to Mm for every m for every m; therefore, p is not the point in the union of Mm, but as our assumption is this is Xby our assumption thereforean X is a complete metric space X belongs to this class here p belongs to this class where the p does not belongs to this, sothese two will leads say alpha and beta will give a contradiction contradiction.

Then this contradiction is this, because our wrong assumption that the set is of first is not of second category;therefore, therefore, the space M which we have provedah discussed earlier assumed this one that is we assumed, say, this is the metric space is of first category is wrong; therefore, the space XX d will be of second category will be ofsecond category,and this proves theproves category theorem, sothis one.

So, everycomplete metric space is of second category by this is not mean the incomplete metric is not is also is not of second category; the converse of the baire's category theorem is not true, that is, if a set is of second category can you say it will always be a metric space, no,it is not true always,it is not true.

In fact, there are incomplete metric spaces incomplete metric space incomplete non spaces. In fact, incomplete metric space which is non meager which is non meager, and this is shown by bourbaki, and his work its 734, he has chosen the incomplete norm space, where he as shown incomplete norm space which is of second category, this is a detailed exercise. So, you conclude that, so that is all. Now, once we complete this category theorem then base is developed to prove our uniform boundedness theorem.

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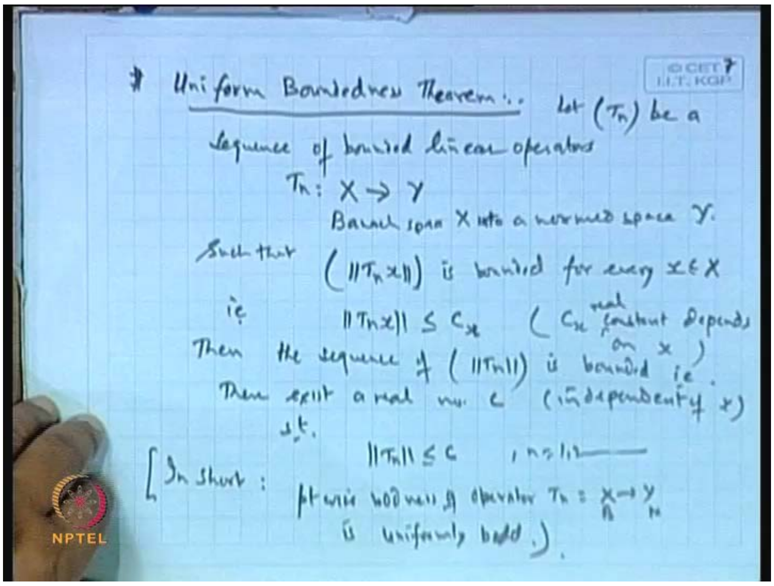

So, let us see the next result is the uniform boundedness theorem boundedness theorem, this is also known as the banach steinhaus theorem; what this theorem says is, let T n, let the sequence T n be a sequence of be a sequence of bounded linear operators bounded linear operators from banach space X to a norm space Y from a banach space X into a norm spaceYX to be banach and Y need not be a banach, such that the norm the sequence of the norm T n x is boundedfor every x belonging to x; it means, corresponding to X we can find a constant C such that norm of T n x remains less than equal to C, that is,that is norm of T n x is less than equal to C suffixes, suffixes means, it depends on this which is a real number,this is a constant real constant depends on xdepends on x is a real number.

So, a sequence of a bounded linear operators are given from one from a banach space to a norm space which is point wise bounded, that is, sequence of the norm operators are point wise bounded; then what thistheorem says is then the sequence of the normof normT n is bounded that is, there is a C there exist a real number C independent of xsuch that norm of T n isless than equal to Cand n is for all n 123.

So, what this theorem says is, in short we can saythe result says that point wise converges point wise boundedness implies the uniform boundedness, that is, in case of a the point wise boundedness of the operator T n from X to Y,X is banach, and this is norm is uniformly bounded bounded that is what the result says point wise boundedness implies the uniform boundedness, if x is a complete norm space in y. So, let us see the proof for it.

The proof which will dependon the category theorem - bairs's category - we will make use of baire's category theorem to prove this results, sowhat we do is, we will first find out the close set, and then we say the X isunion of the close set, and if X is complete already then baire's category theorem it will have a certain open subsets some subset which is non-empty and contains thesequence means open, so that is the one.

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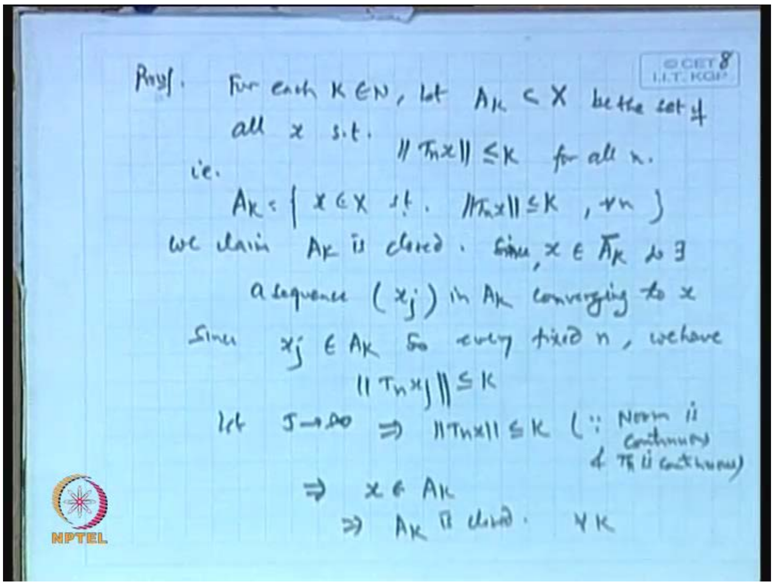

So, let us see for each Knatural number belongs to N, let us consider A k which is contained in X and be the set of all set of all x such that such that norm of T n xis norm of T n x is less than equal to K for each n for all n, means, A k is the set of those X belonging to capital X, such that, under remains of this is bounded by Kfor eacht 1 x is less than equal to k t 2X is less than equal to T n x n.

Now, we claim that A k is closed, soall the limit point belongs to this. So, let us take x belongs consider x belongs to since the reason I am giving sinceif we take x belongs to the A k closure,then by definition there will be since… So, there exist a sequence there exist a sequence x j in A k converging to converging to this x, because A k closure, this is the limit point I am choosing, sothere will be a sequence in A kx j in A k which tends to x under the same norm or this x.

It means, for x j since x j belongs to this, since x j belongs to A k, so far every fixed n fixed n we have the norm of T n x j is less than equal to k by the property of A k; if x and j belongs to this then T n x will be less than equal to k for each n, solet us fix the n and get it once you fix n. Now, let j tends to infinity, let j tends to infinity, so let j tends to infinity.

Now, this implies norm of T n x is less than equal to k, because norm is a continues functionand T n(s) are giving to be a operators. So, when limit you take limit will come inside and it will get the limit of x j which is equal to j, so it will… So, this shows a norm is continues and T n is also continues, because it is a bounded linear operators T n is is also continues, so because of these thing we can get;therefore, this implies that x belongs to A k, so A k is closed and for each k this k.

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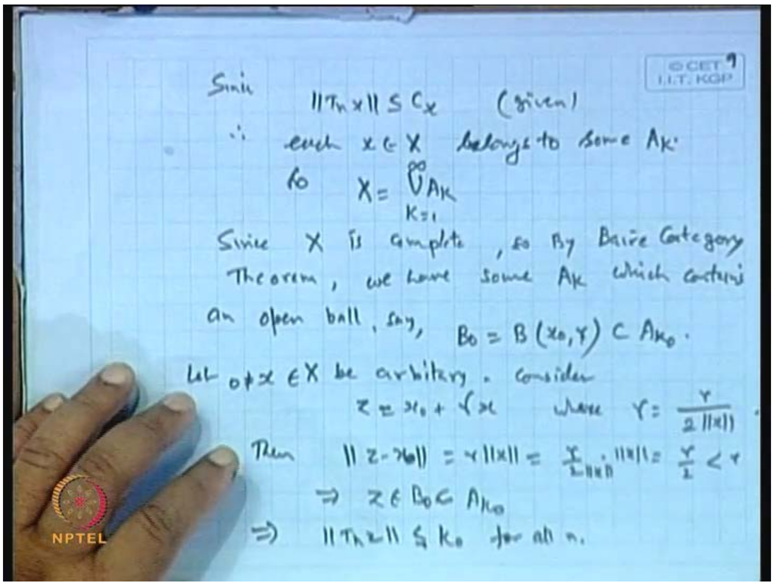

Now, once A k is closed then we can write down this, take any x belongs to capital X it will belongs to one of the A k(s); therefore, sincenorm of T n x is less than equal to C x is given; therefore each of its therefore, each x which is in x will satisfy this condition, it means it, will belongs to any one of the A k's, so belongs tobelongs to some A kbecause of this property.

So, X can be expressed as a union of this, soX can be written as a union of A k,k is 1to infinity countable union of this clear in this order. So, we have X, we have banach space, we have proved that this X can be expressed as a countable union of A k where A k(s) satisfies this condition which is a close set. Now, since apply the baire's category theorem, if X is complete then by the baire's category theorem this representation means one of the A k will contain the open ball centered at x naught and radius suitable radius, sosince X iscomplete.

So, by baire's theoremcategory theoremwe have some A k(s)A k which containsan open ball,say,B naught which centered at x naught and radius say r which is contained in A k naught;some of these A k(s) say A k naught contains an open ball B naught with radius a naught.

Now, let us take a next arbitrarypointarbitrary and different from 0different from 0. Now, consider the point z which is x naught plus gamma of x, where gamma is a constant term gammais say r over 2 norm of x, because x is already given, sowe can find out the norm of x, sothis becomes a real quantity, soz which isx naught plus gamma x naught. Now, this has a property; consider now if I take z minus x naught norm of this, then this becomesgamma into norm of x, but gamma is r by 2 norm of x into norm of x, sothis is equal to r by 2, which is less than r, it means, that a ball centered at x naught with a radius r contains z, sothis belongs to b naught, soz belongs toB naughtby definition.

And further because thezbelongs to Bnaught and what is B naught? B naught isthis open ball which contained in A k naught, which is contained in A k naught, so z will satisfy the condition that norm of this T n x naught is less than equal to k naught T n z; so, this implies the norm of T n z is less than or equal to k naught for all n for all n, that is what (( )).

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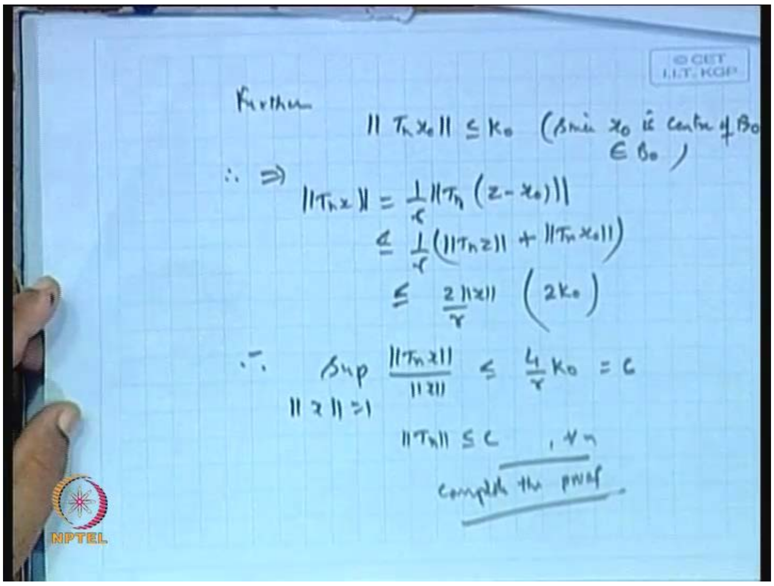

Furthernorm of T n x naught this is also less than equal to k naught, why? Since x naught is the center of B naught x naught is the centerof B naught and belongs to B naught, so B naught therefore again by the same property B naught is a subset of A k naught, so we can get this set.

Therefore,we can get from here, this implies that norm of T n x which is equal to 1 by gamma norm of T n x, means, z minus x naught; now, this will be equal to less than equal to one by gamma norm of T n z plus norm of T n x naught within bracket, now T n z naught gamma is this gamma is r by 21 by r, means, it is 2 norm x by r, and thisis less than or equal to because each one is less than k naught, sothis is 2 k naught.

So, what we get is that 2 k naught, and this will be4 by r 4 by r into norm of x into k naught, sothis will be equal to this one; now this therefore, supremum of this divided by norm x, when x belongs to thex and norm of x is one, then this supremum is bounded by 4 by r k naught which is say some constant c there, but this supremum is norm, so norm of T n is less than equal to c for norm nand this completes the proof, sowe can get this, is it clear now. So, we have proved this uniform boundedness. We can directly use this uniform boundedness theorem as an application one can solve many problems for this.

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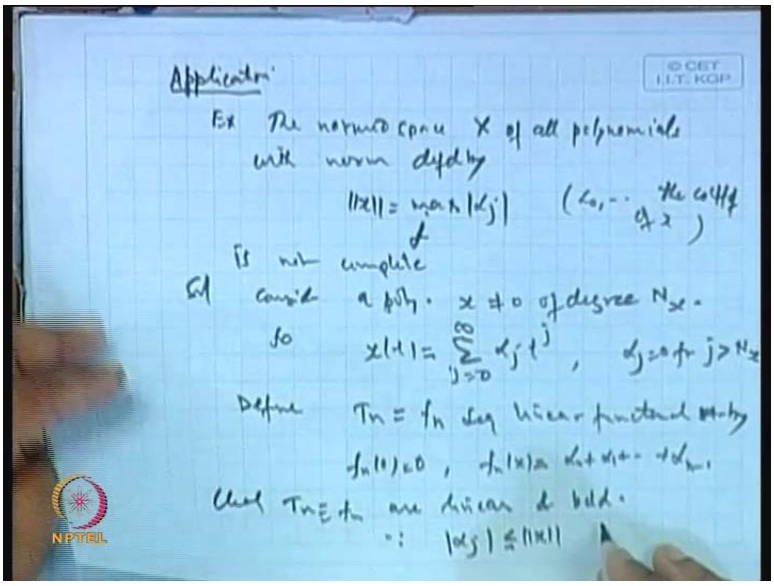

So, first let us see the one application where this uniform boundedness theorem has used. So, let us take the one exercise, the norm space X of all polynomials with norm defined by norm of x is maximum of mod alpha j over j, where alpha naught, alpha 1, alpha 2, alpha are the coefficients of coefficients of x is not complete.

So, what we do is, we will generate a bounded linear functional on x which is point wise bounded, but not uniform bounded, so this by baire's category theorem we can say it is not complete,that is all. So, let us consideran x a polynomial x which is not 0 of degree N or N x, sowe can write x in the form of sigma j equal to 0 to infinity alpha j t j where the alpha j 0 for all j greater than N x, then (( )).

And defineT n as our f n sequence of linear functionalsuch that by this by thisf n 0 is 0 and f n x is some of this alpha 1, alpha 2,alpha n minus 1. So, that we can see from here clearlyT n which is say f n are linear and are linear and bounded, why bounded? Because mod of alpha j is less than equal to norm x, because the maximum is there;therefore, this norm each one f n will be a bounded function.

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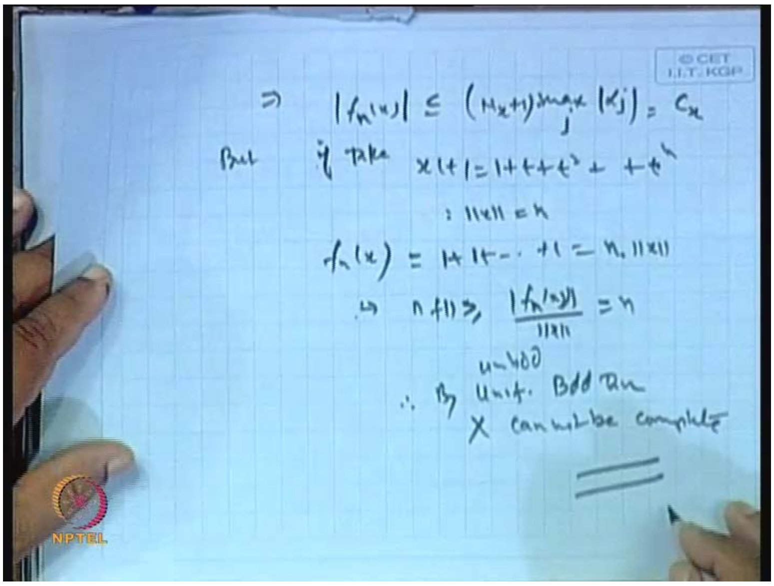

So, mod of f n x, so mod of f n x, this will remain less than equal to some of this N x into maximum of mod alpha j by j, and this is nothing but a C x, soit is point wise bounded but norm of f n is not bounded because but f n x, but if we take x t to be 1 plus t plus t square plus t to the power say n, then norm of x becomes n, and f n x this x becomes1 plus 1 plus1 up to n, son into norm x;therefore, norm of f is equal to n norm of f is greater than equal to mod f n x over norm x, and mod f n x is n, sothis will be n into norm x which we can write it is it not, sothis is equal to n, soit is unbounded.

Therefore, by uniform boundedness theorem uniform boundedness theorem x cannot be complete,and that is what we wanted to show, because it is point wise bounded, but it is not uniform bounded; therefore, this x cannot be complete, because if it is complete it must be bounded. Thank you, thanks.