Functional Analysis

Prof. P. D. Srivastava

Department of Mathematics

Indian Institute of Technology, Kharagpur

Module No. # 01

Lecture No. # 32

Hahn Banach Theorem for

Complex V.S and Normed Spaces

Lecture, we have discussed the Hahn Banach theorem, that is, the X in linear from the linear function, extension for the linear function in case of a real vector space,.

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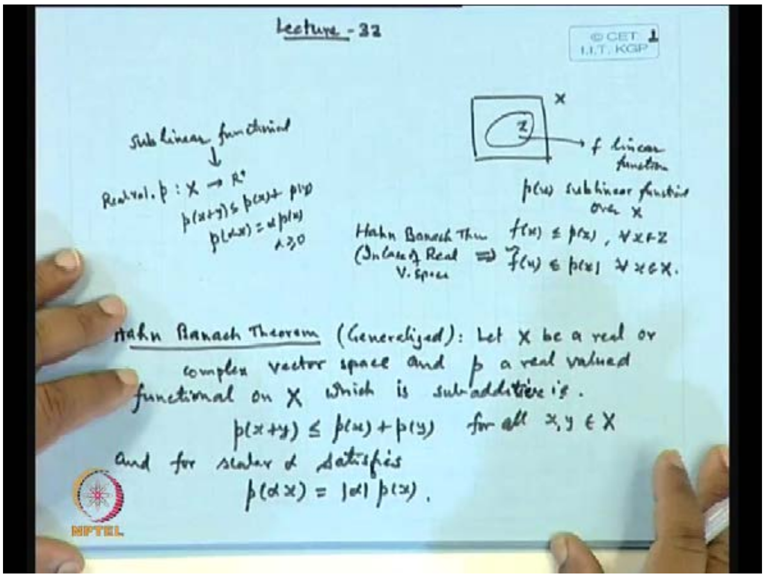

That is, if x be a vector space and z be a vector sub space of this, and f is a linear functional defined over z, and p be a sub linear function over x defined that is point on x. And f is a linear functional which is dominated by p x for x belonging to z, then Hahn-Banach theorem says that, this linear functional can be extended to a linear entire class x or linear function f delta, such that f delta x is also linear and dominated by p x for every x belongs to capital X.

So, this is what we have the Hahn Banach theorem, in case of real vector space. Sub linear functional we mean, a functional which is sub additive, sub linear functional means, a real valued functional p on a vector space real valued functional of p on the vector space x, which satisfied the condition of condition that it is sub additive and positive homogeneous that is, this is equal to p of alpha x is alpha of p x, when alpha is greater than or equal to 0 for all (( )) and x belongs to n.

So, this is sub linear function, it means in case of Hahn Banach theorem, we can extend a linear functional from a vector sub space to the entire class to a bigger vector space x, retaining the same property of linearity and (( )).

Now, today we will discuss a generalized form of the Hahn Banach theorem. We should not restrict only for the real vector space, even if a vector space x is a complex valued vector space, it is still the extension of the linear functional on the entire class is possible, retaining the properties of the functional f on xi whatever this thing.

So, we get Hahn Banach theorem in a generalized form, this we call it a Hahn Banach theorem, the generalized form or you can say it is a general result for real or complex vector space. Let X be a real or complex vector space and p a real valued functional on X, which is sub additive. Sub additive that is additive additive t I v e sub additive

That is p of x plus y is less than equal to p x plus p y for all x comma y belongs to capital X and for and for every scalar alpha, this may be real or complex because we are breaking a complex vector value, vector space satisfies p of alpha x is mode of alpha into p x, p x where again x belongs to furthermore, where the x is element of capital X.

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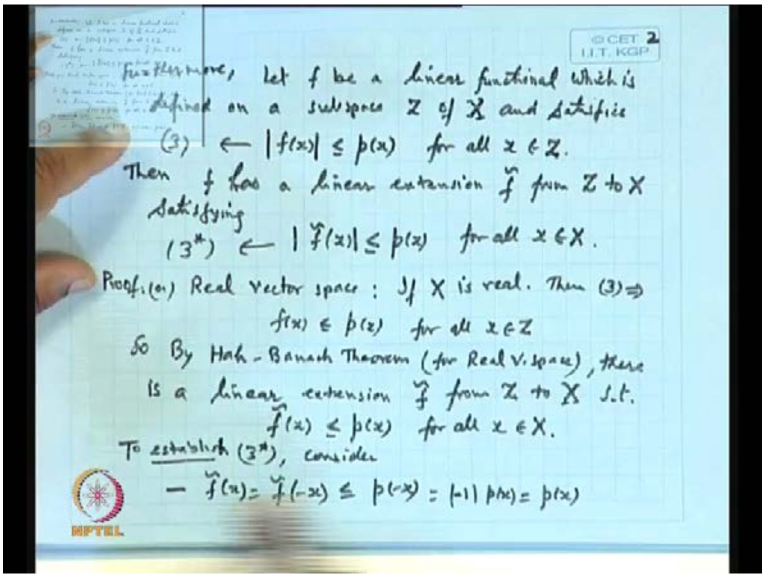

Now, furthermore let f be a linear functional, which is defined on on a subspace Z of X, X is a vector space satisfying and satisfies condition mode of f x is less than equal to p x mode of f x is less than equal to p x for all x belonging to Z.

Then this generalized form of the Hahn Banach theorem says that, the f can be, then f has a linear extension f delta from Z to entire X satisfying the condition criteria condition, f delta x mode of this is less than equal to p x for all x belonging to capital X.

Now, let it be for our purpose, let it be this equation p x plus y, let it be denoted by 1, when the proof we will mention the condition equations let it be this 2, and this part is 3, and this condition let it be 3 star. So, we prove this, any particular case it comes as a real. So, proof: So, let us prove first this for the real vector space, because under the real vector space this condition remains hold good or not.

So, real vector space, for this real vector space let us see first. Suppose, f is real if sorry if x is real, thus then 3 implies that f x will be less than equal to p x, because f is real. So, either it will have positive value or negative value, but absolute value is less than x. So, f x will be less than equal to p x for all x belonging to Z.

Now, p is a sub linear functional because when you take alpha to be positive, mode alpha is replaced by alpha, it is sub additive positive homogeneity x is real, and f is a linear functional, satisfy dominated or measured by p x.

So, we can apply the Hahn Banach theorem in case of the real vector space. So, by Hahn Banach theorem. So, by Hahn Banach theorem for the real vector space, this f can be extended to the entire class capital X retaining the same property of f delta x is less than equal to p x. So, by Hahn Banach theorem, there is a linear extension f delta from Z to X z to x such that f delta x is measured by p x, for all x belonging to capital X. So, this part is clear, now to show that this mode f delta x is less than p x we take this now minus sign.

So, now consider to establish the 3 to establish 3 star, what we do is, let us consider minus of f delta x, f delta x is a linear extension because of this Hahn Banach theorem. So, minus sign will come inside and we get f delta of minus x, but f delta x is less than equal to p x for all x. So, this is less than equal to p of minus x, p satisfies the condition of the homogeneity that mode of alpha is p x, this is this condition, p alpha is mode alpha p x. So, we get from here is minus 1 mode of this p x which is equal to p x.

So, what we see here is that, f delta x is less than p x minus f delta x is also less than p x therefore, it will implies mode of f delta x is less than equal to p x for all x belonging to capital X, and this completes the proof for in case of the real vector space clear.

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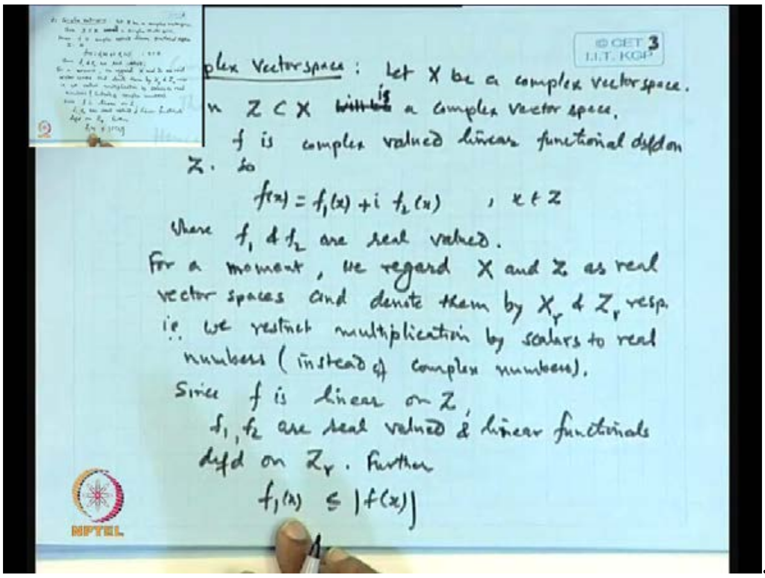

Now, let us come to the second part, where it is for a complex vector space for complex vector space. So, let us take the X to be a complex vector space. Let X be a suppose x is a complex vector space, then Z which is a subspace of X, will also be a complex vector space. will be then z is also will be a complex vector space or then Z is also a complex vector space or you can choose Z is a complex vector space, will be all this, that is better is a complex it can be real, but we are taking this complex.

Hence, f is a complex now, functional f defined on (( )) hence because z is complex vector space. So, functional when we are defining linear functional, it requires the alpha, beta and this entire thing, and we assume that f is also a complex value. Then f is a complex valued linear functional complex valued linear functional define on z defined on Z. So, we can write it, f x as the real part and imaginary part of f, that is f 1 x plus i times f 2 x, where f 1 and f 2 are the real and imaginary are where f 1 f 2 are real imaginary part of f. But f 1 and f 2 both are real valued functions real valued functions, so this one.

Now, we wanted to show this linear this linear functional which is a complex valued linear function can be extended to the entire class X, f delta x and satisfying the conditions of 3 star. So, what we do is, we first look the behavior of f 1 and f 2. So, for this, let us assume for the time being that for the moment that Z and X, which are taking to be the complex value, let it be a real value. So, for a moment, we regard X and Z as real vector space spaces and denote them by X r, r is real and Z r respectively.

What do you mean by this? It means, we are aligned the scalar to run through the real only, that is the meaning is this, that is we are we are restricting or we restrict multiplication by scalar, scalars to real numbers only, instead of complex numbers complex numbers. Now, since f is linear on Z, since f is linear on z f 1 and f 2 are real valued functions, are real valued defined on Z r. are real valued functional and real valued and linear functionals defined on z r

And further, further f 1 x is less than equal to mode of f x, f 1 x is less than equal to mode f x because f x has a real part f 1 and (( )) to 1 it will always be less than real part will always be less than or equal to this absolute value, absolute this modulus value mode of f x. So, real part is less than equal to f x, hence by 3, replace this by the mode f x by f 1 x.

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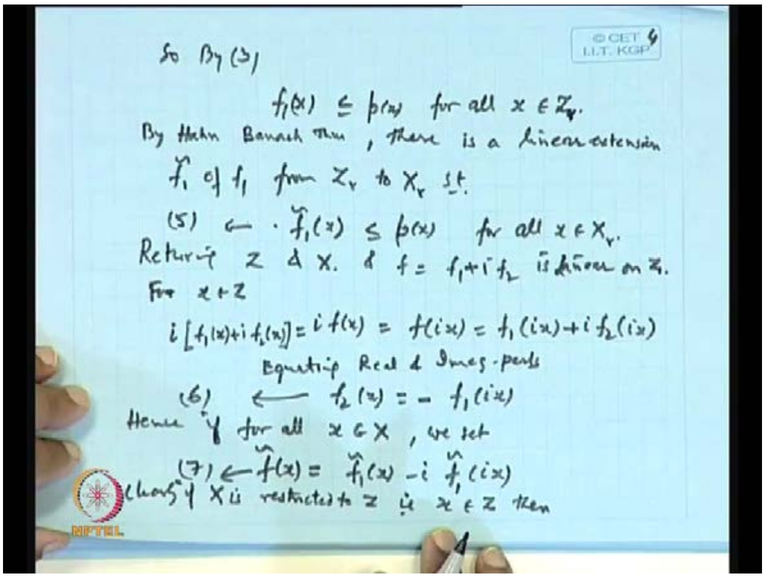

So, by 3, we get that f 1 x is less than equal to p x f 1 x is less than equal to x for all x belonging to Z. Now, f 1 is a linear functional define on f 1 x belongs to r Z r, so this Z r, f 1 is linear on Z r, f 1 is dominated by p x on Z r. So, by Hahn Banach theorem.

So, by Hahn Banach theorem it has an extension linear extension f 1 delta define on the entire class x r. So, Hahn Banach theorem, there is a linear extension f 1 delta of f 1 f 1 delta of f 1 from Z r to X r, such that this f 1 r f 1 star x is dominated by p x for all x belonging to X r. All x belonging to X r So, the real part of the f 1 satisfied that condition of this on a real vector space X on a part of this. So, let us take the (( )).

Now, once f 1 is there, let us take the x 2, now we turn to f 2 now. So, what we do is returning to Z n. So, returning Z and X, means now you take x as an element of Z or as an element of Y. And f, which is the f 1 plus i f 2 is linear on Z, this is linear on Z. So, for every x belonging to Z, i times of f x, this is equal to this (( )) i times of a f 1 x plus i times of f 2 x, but f is linear. So, this will be equal to f of y x again f of x is this. So, we can write f 1 of i x plus i times of f 2 of i x, is it?

Therefore if we equate the real, and equating the real and imaginary part, what we get from here is the f 2 x the real part of this is minus f 1 i x is it not. So, f 1 we have got it f 2, also we get it in terms of f 1. Hence, for all x hence if for all x belonging to x, let us set the functional f delta x as f 1 delta x minus i times of f 1 delta i x. Now, if I set this f star x, when x belongs to (( )) then clearly if X is restricted to Z x is restricted to z that is if x belongs to Z, then then this f delta x.

Let us give the number here again, so this is 5, we have seen up to here 4, what was the 4 here, third star we we have gone is it not? So, what is the 4? the 4 th one was f delta x is, here this is 4, let it be this fourth and then we are taking to be the fifth and sixth, fifth, we can take to be this to be this is fifth, let it be 5 and then let it be sixth is f 2 x and this one is 7, lets mark this thing now we use. So, what will it be if x is restricted to Z that is if I take x, then 6 we can see that f 2 x is nothing, but minus f 1 i x that is f star x is f x on Z. So, then 6 will give because f 2 x is minus f i x and here is f 1 x is also the, so f x this part f x which is equal to f 1 plus i 2 f 2, i 2 f means i 1 f 1. So, here we can write f 2 in place of this.

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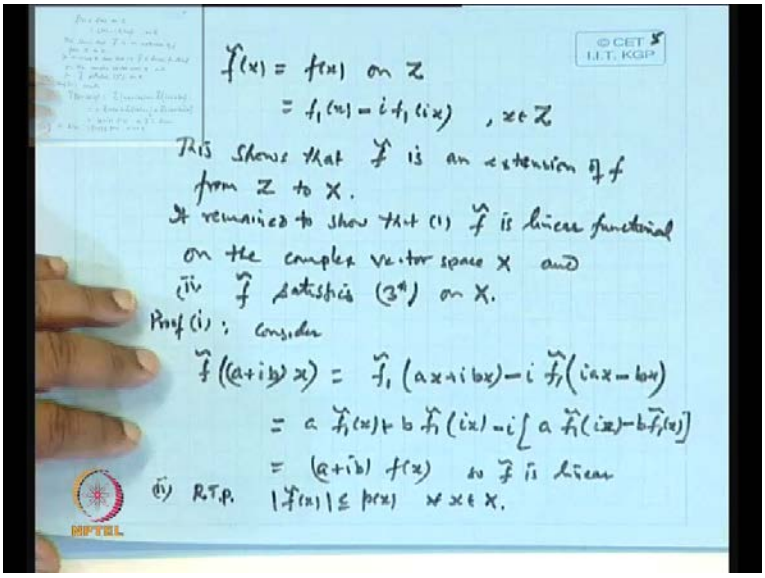

So, whenever the x in Z, then we can say that f is delta x, f delta x equal to f x on Z, is it not? Because, the f x we are introducing in this fashion this is our f x and when f 2 is replaced by this number then basically this seventh and reduced to this point and we get x belongs to Z, then f delta is equal to f x.

Now, we have defined the f delta x to be this for the entire X, and for this one, we are getting this as f 1 x and then minus i times of f 1 of i x, clear I (( )) it means, if I take the extension of this f 2 is minus x. So, the extension of this on entire X is this one clear. So, this clearly shows this shows this shows that f delta is an extension of f f from Z to entire

X. Now, if I prove that f delta is a linear, and then an f delta is linear and satisfy the condition of three star dominated by p x, then our result is proved.

So, now it is it remained remained to show that number 1, f delta is linear functional linear functional on the complex vector space X, and the second part is left is, that f delta satisfies 3 star on X. ok. So, if I prove these two things, then our result is complete, the proof is complete. So, proof of first part, in order to prove this thing, let us consider let us consider this expression f delta of a plus i b x, take any f delta alpha x, if I prove alpha times f delta x, then f delta becomes linear, is it not? Because, addition is also satisfied, f delta a plus b equal to f delta (( )) which can be shown quickly. So, its only proves linearity of this vector space, hence a plus b.

So, now, by definition this is the f 1 delta a x plus i times b x and then minus i times f 1 delta i a, so i of a x is minus minus of b x, f 1 star delta is a linear extension which already you have proved. So, this can be written as a f 1 delta x plus b f 1 delta i x, then minus i a f 1 delta i x minus b f 1 delta x. And then combine this thing, we are getting a plus i b f x that just we can write it (( )) and so on. So, this shows that f delta is linear, to prove the second one, so f delta is linear because f delta define (( )).Now, to prove this thing third, that is required to show is, required to prove that mode of f delta is less than equal to p x for all x belonging to X.

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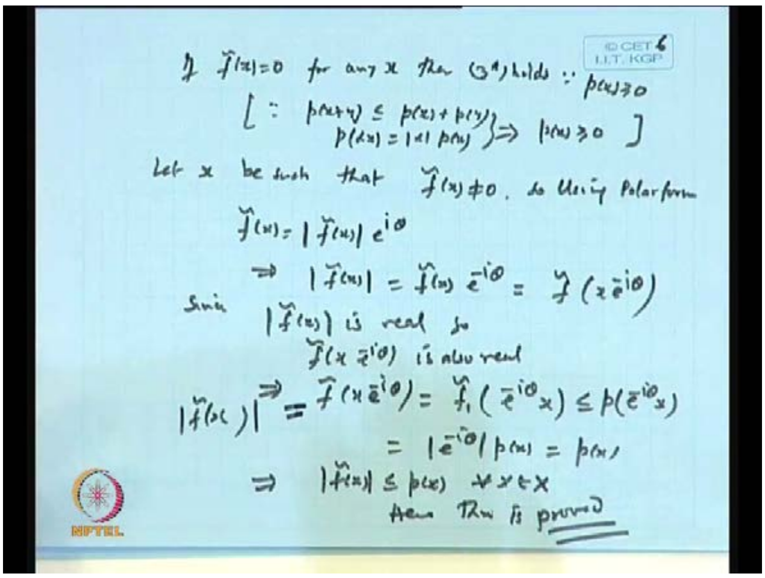

Now, if suppose f delta is 0 f delta x is 0, is 0 then for any x, as such that then third star holds because p x will always be greater than equal to 0, because p x plus y because p x plus y is less than equal to p x plus p y, and p of alpha x p of alpha x is equal to mode alpha into p x. So, if I take here x y 0, x is 0 then we get from here, these two implies p of x will always be greater than equal to 0, ((this is why is 0 let it y is zero)). So, p x will be greater than equal to 0 (( )).So, we get from here. So, let us suppose, let x be such that f delta x is not 0, if delta x is not 0, if it is not 0 then we can write this polar forms.

So, using the polar form, we can write f delta x as modulus f delta x into argument e to the power i theta. Then from here, we can say f delta x is equal to f delta x e to the power minus i theta. Now, e to the power minus i theta can be taken inside and we get f delta of x e to the power minus i theta as f delta is linear. Now, since f modulus of f delta is real therefore, the right hand side should also be real. So, the f delta x e to the power minus i theta is also real.

Therefore, it implies that f delta x e to the power minus i theta should be its real part should be equal to its real part, that is f 1 star of e to the power minus i theta is real, but f 1 is already dominated by p, so p of e to the power minus i theta x. Therefore, this can be written as e to the power minus i theta p x, but this is one, so we get p x. It means that modulus of f delta x, this is equal to what? f delta x is it not? Its real part is this modulus of this. So, this shows modulus of f delta x is dominated by p x for every x belongs (( )) Hence theorem is proved, so this completes the proof of this theorem.

So, we have proved the Hahn Banach theorem in a general case also, when it is real or complex vector space or a complex. Now, if x be a normed vector space, normed space, normed linear space, then also can we apply the Hahn Banach theorem, though it does says anything about the continuity of the functional, it simply say the i f (()) is a linear functional which can be extended from a subspace to the higher, to the entire vector space X. But even if the functional f is a continuous linear functional that is a bounded linear functional, then also one can extend this bounded linear functional from a normed subspace to the entire normed space X, and this we give you further the generalized form of the Hahn Banach theorem in case of the normed subspace. ok

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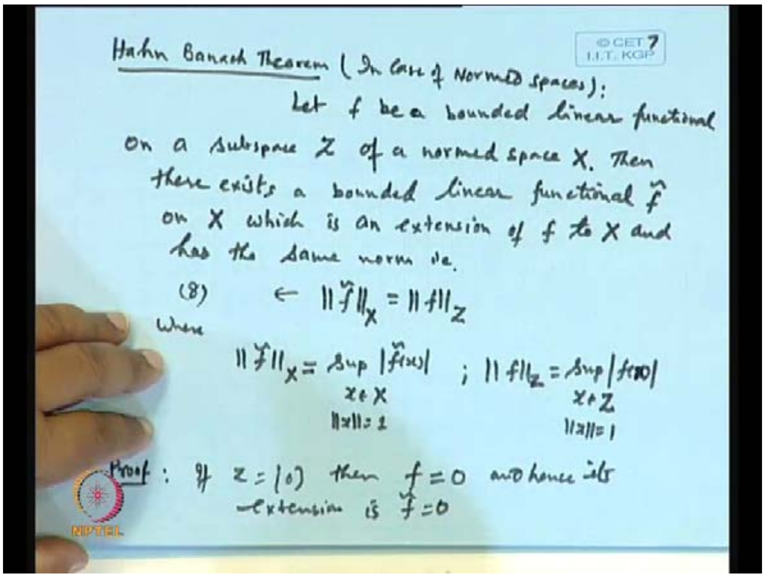

So, let us see the other also normed Hahn Banach theorem, other form of the Hahn Banach theorem in case of normed spaces. Let f be a bounded f bounded linear functional linear functional functional on a subspace Z of a normed space X, then there exist exist a bounded linear functional f delta on the entire space X, on the entire normed space X, which is an extension of f to X, and has the same norm as the norm on f of z has the same norm that is norm of f over the entire X sorry norm of f delta x norm of f delta over the entire X, will be the same as the norm of f on Z.

Where what do you mean by this norm f? where norm of f delta x means, by definition this is the supremum of modulus of f x f delta x where x belonging to capital X and norm of x is 1, this is by definition. And similarly, norm of f z means supremum of mode f x over x of f x sorry f x when x belongs to Z, this is x and norm of x is 1. So, this gives the extension of the bounded linear functional to an entire norm space, from subspace of norm to entire space.

The proof of this theorem is easy, what we do is we try to develop a sub linear functional p in terms of the norm of f over Z, and then once it is a sub linear function is established and f is dominated by the sub linear functional f is linear on Z, then by using the Hahn Banach theorem for real vector space case, one can get the result quickly. Now, let it be this number be eight, so if suppose if z is a single (( )) 0 that is a normed subspace containing only 0 then; obviously, the linear functional will be only f will be 0, and its extension and hence its extension is f delta which is also 0 (( ))So, its nothing to prove much.

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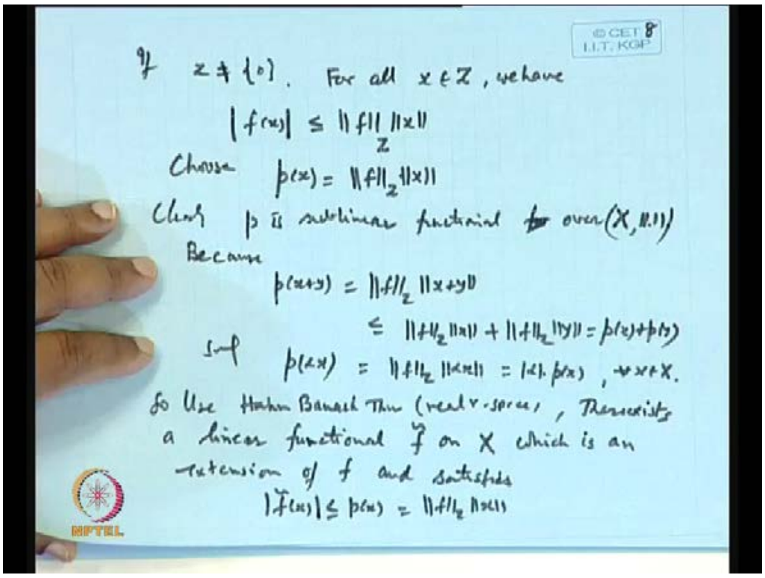

Now, if z is different from single transient 0, now we wanted to make use of the Hahn Banach theorem for the real vector spaces. So, what we do is, let us consider a (( )).So, for any, so for all x belonging to Z, we have mode of f x is by definition, modulus f is a bounded linear function. So, modulus of f x is less than equal to norm of f into norm of x and the norm f is taken over z because x is point in z. So, the right hand side is we can choose p x as norm of f z into norm of x because it is a real valued functional, is a real valued. Now, if p is also sub additive and positive homogeneity, then it will satisfy the clearly p is sub linear, p is sub linear functional.

Why it is sub linear functional? Because if we take this thing, sub linear functional for entire X, for over for all x sub linear functional over entire X x norm. Why why it is sub linear functional over entire norm? Because, because p of x plus y, we can write it as a norm of f z into norm of x plus y. Ok Norm of x plus y because norm is taken to be the same, now this is norm space. So, it is less than equal to norm of f z into norm of x (( )) norm of z into norm of y and this gives the p x plus p y.

Similarly, p of alpha x when x belongs to, remember on x then this is equal to norm of f z norm of alpha x, then this will be equal to mode alpha into p x when x belongs to capital X. So, p is a sub linear functional on this and our f is dominated by the sub linear function of p, so by using the, so use Hahn Banach theorem for real vector space. Ok.

So, what we conclude is, there exist a linear functional f delta. So, there exist a linear, there exist an extension or we conclude that, there exist a linear functional f delta on X, which is an extension of f and satisfied and satisfies the condition mode of f delta x which is less than equal to p x which is basically equal to norm of f z into norm of x, Norm of f z into norm of x now divide by x.

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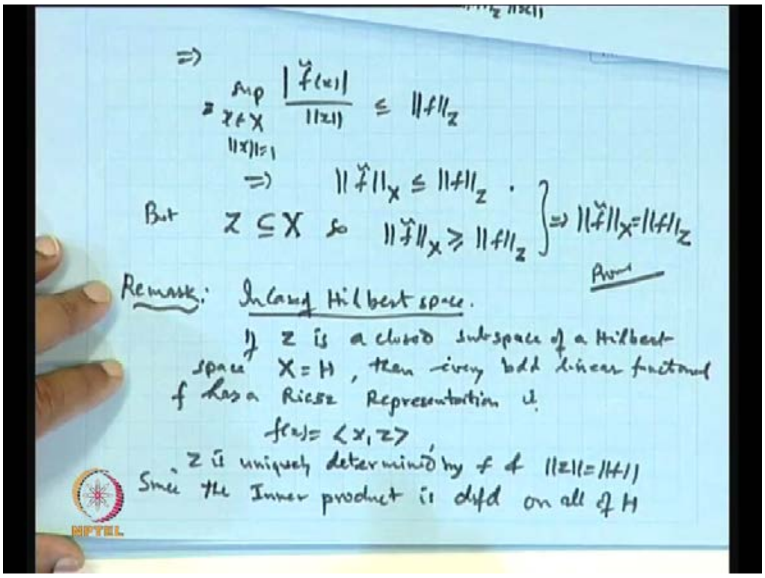

So, we get from here is f delta x over norm x, take the supremum over all x belonging to capital X where norm of x and supremum of x over the norm x is equal to 1, we get this is less than equal to norm of f z. But this 1 is nothing, but norm of f delta which is equal to, so this implies norm of f delta f delta over x is less than equal to norm of f z clear, but z is a vector subspace of x sorry norm subspace means it is a vector subspace of X.

So, norm of f delta x cannot be lower than the norm of f over z, that is because its extension therefore, combine this two, we get norm of f delta X equal to norm of f Z and that proves the (( )) that proves the result. Now, clear. So, we have seen the vector space, we have seen the case in the normed space, now in Hilbert space, the situation is very simple. So, remark in case if the Hilbert space, space that is subspace of Hilbert space, this can be obtained with the help of the Riesz Representation theorem.

So, let us take if Z is a closed subspace of a Hilbert space X which is say equal to H, we are denoting Hilbert space by H, then f has a Riesz Representation then every bounded linear functional functional f has a riesz Representation, this we have already discussed, that is every bounded linear functional, that is f x can be expressed in terms of the inner product x z where z is uniquely determined by f, Z is uniquely determined by f and norm of z equal to norm of f, this is by we are not, this is by Riesz Representation theorem. Ok Since this inner product is defined over the entire H. So, this gives the linear since the inner product is defined on all of H.

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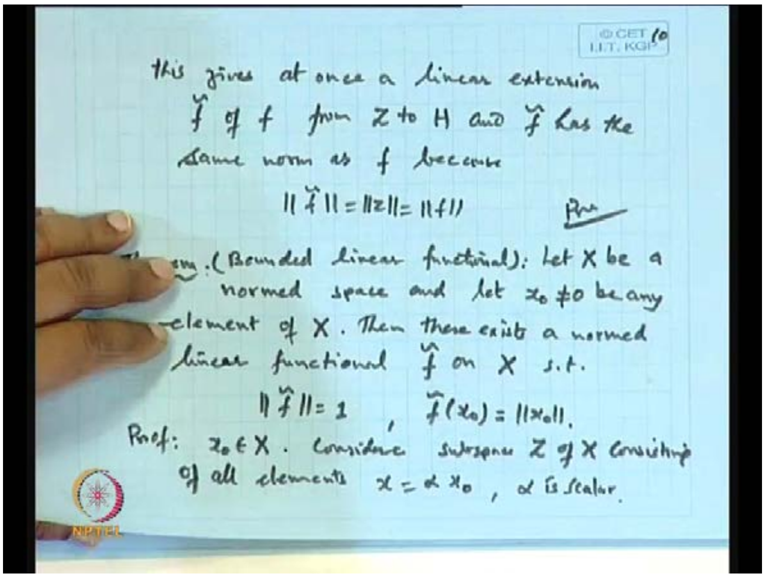

So, this gives gives on all of h. So, this gives at once a linear extension of delta of f, f delta of f from Z to H. And f delta has the same norm same norm as f because norm of f delta is also norm of Z and norm of Z is norm of f, we have already proved with the help of the Riesz Representation theorem, so this completes the proof. So, we can go for the Hilbert space also, now as a consequence of this, one can find out 1 Hahn Banach theorem gives the guarantee that a normed space is enriched by a bounded linear function. That is, we have sufficient number of the bounded linear functional defined on the normed space, so that the theory of the dual space can go very smooth.

And in fact, a result will tell you that if we pick up an element x in a normed space x then corresponding to this, we can always find a bounded linear functional satisfying the (( )) and that result is very interesting one. So, we form a result in the form of theorem, that is a bounded linear functional, we also bounded linear, this result is for bounded linear functional on the normed space.

Let x be let x be a normed space and let x naught is not equal to 0, be any element of X, then there exists a normed linear functional normed linear functional f delta on X such that norm of f delta is 1, and the value of f delta at x naught will give you the value norm of x naught. So, what this shows? This shows that if x be a normed space, and if we picked up any non zero element vector in x, then there will always a bounded linear functional associated with this x naught, which has this property that norm f delta is 1 (( ))

So, the norm space are very much enriched with the bounded linear functional, that is what it says the proof is still there. To prove this, lets let us consider a subspace, now x naught is there, our x naught is about point in z x 0 is a point in X. Let us consider a subspace Z of X consider a subspace z of x consisting of all elements x which is of the form alpha x naught, where alpha is a scalar Now,. So, this form is a subspace clear because it is a generated by all.

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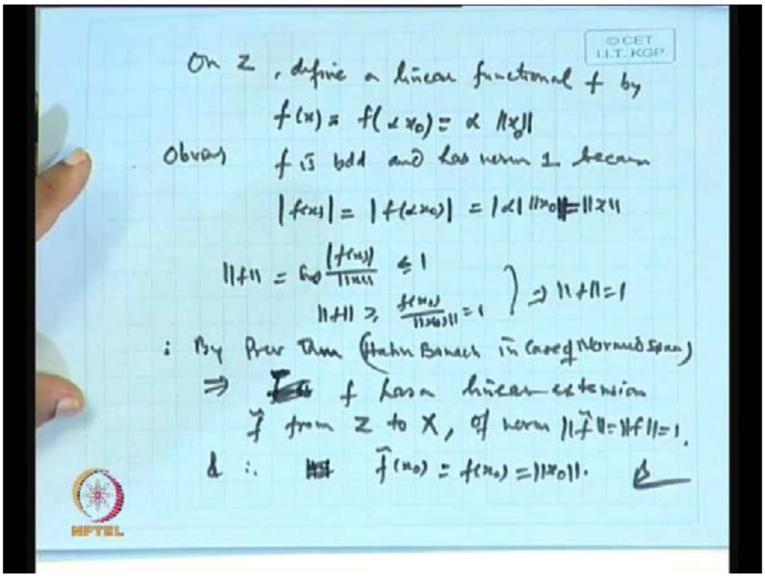

Now on Z, on z define a linear functional f by f of x as x means alpha x naught alpha times norm x naught. Then obviously, f is bounded because the mode of f x is less than equal is bounded mode of f x is, and has the norm 1, why? because what is the mode of f x? mode of f x means this norm, mode of alpha x naught, but alpha x naught is mode alpha into norm of x naught, which is equal to norm of x because this can be written as (()) Now,. So, divide by this, so we get f x over norm x supremum of this is less than equal to 1 which is the norm of f, but norm of f we can choose the point x equal to x naught, but norm of f is greater than equal to f x naught over norm x naught, and that is equal to 1, so this shows norm f is equal to 1, so it has a norm.

Therefore, So, by previous theorem for Hahn Banach theorem, in case of normed space, if we apply this theorem then this implies that f delta, f has a linear extension, is a or f has a linear extension f delta from Z to X of norm, same as norm of f which is 1 and this if I put it again therefore, what is the end? What is the norm of f x norm of x naught? If I take this f f delta x naught, then f delta x naught will be equal to f x naught which is equal to norm of x naught, and this proves the result. Thank you.