Functional Analysis

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Module No. # 01

Lecture No. # 31

Hahn Banach Theorem for Real Vector Spaces

In last lecture, we have given a brief idea about the four fundamental theorems of norm and Banach spaces namely, Hahn Banach theorem, Uniform Boundedness theorem, Close Craft theorem and Open Mapping theorem and related concepts, which will be used in further, for a further study of these theorems. Today, we will discuss the Hahn Banach theorem.

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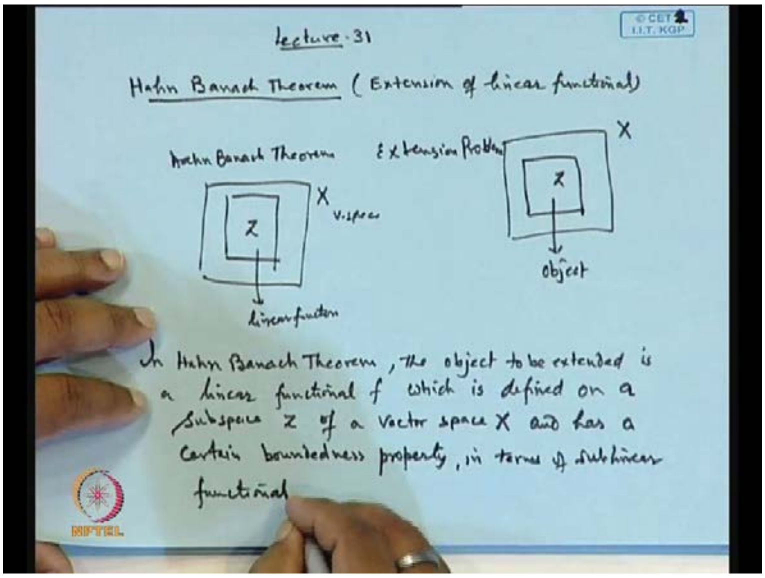

It is basically an extension theorem for a linear functional. So, Hahn Banach theorem it is an extension theorem of linear functional. Now, this theorem gives a guarantee that, there are adequate theory of adequate number of linear functional on the norm space or norm space are enriched with the bounded linear functional, hence the theory of the dual space can be taken up very well and gives you the results, which we need for that.

Now, normally this extension problem is basically, we mean by the extension problem see, suppose z is a subset of X and an object, which is defined on z we wanted to extend this object on the entire X, in such a way that the basic property of these object remains intact then, this problem is known as the extension problem. In case of the Hahn Banach theorem, the extension problem is our linear functional.

So, where we have the linear functional defined on a subspace z of a vector space X, which satisfy the certain property of boundedness, that is, which is, dominated by a certain function p X, which we will defined later on as a sub linear functional and so.

Now, this linear functional which is majorized by a certain function p X, we wanted to extend it to the entire vector space such that, the property of the linearity and majorization remains intact. So, this is what we do in case of the Hahn Banach theorem. So, basically we say in Hahn Banach theorem the object to be extended, extended is a linear functional, is a functional f which is defined on a subspace z of a vector space X of a vector space X and as a as a certain boundedness property boundedness property.

Now, this boundedness property, we will take up in terms of this sub linear functional. So, we will use the first taken as a definition of a sub linear in terms of the sub linear functional, in terms of sub linear sub linear functional, sub linear functional. What is the sub linear functional? It is first introduce the sub linear functional. A sub linear functional p x p x is a real valued functional real valued functional is a real valued functional on a vector space X, vector space X which is sub additive, that is p of x plus y is less than equal to p X plus p y for all x, y belongs to capital X and positive homogeneous homogeneous.

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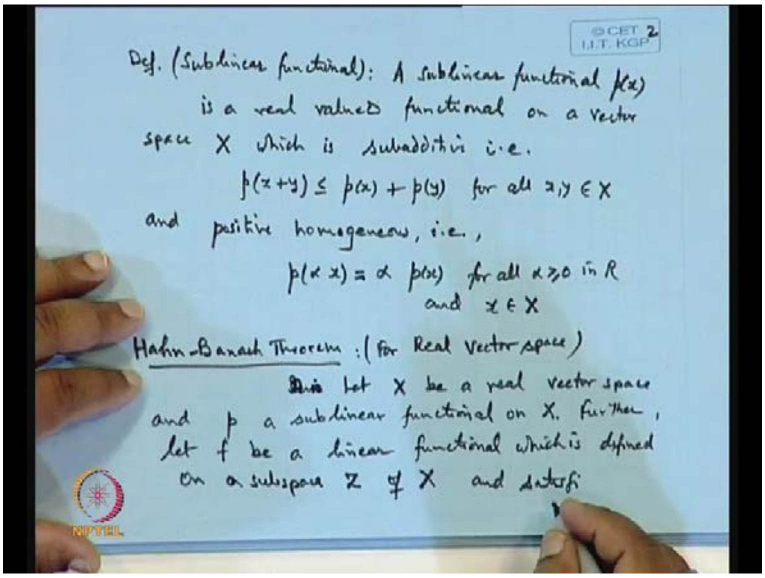

That is the meaning is that, p of alpha x equal to alpha times p x, when alpha is for all alpha greater than equal to 0 in R and x is an elements of X and for x belongs to capital X. So, such a functional p, which is sub additive and positive homogeneous is said to be a sub linear functional.

Now, we wanted this functional f to be extended from z to X, we satisfy the property that f is linear functional and dominated by or majorized by the function p, that is f x is less than equal to p x for all x belonging to z.

Then the Hahn Banach theorem says that, we can extend this functional f to the entire X, we such that, the extended functional f retains the property of linearity as well as majorization, so this is the crux of the theorem. So, we will see the exactly statement of the Hahn Banach theorem is this is first we will do it for a vector space, for real vector space. In fact, the Hahn Banach theorem is first introduced by Hahn in 1927 and then later on a in a modified form a by Banach in 1929.

So, this is discovered it is discovered should I I think is I will the Hahn Banach theorem, it is discovered by H Hahn in 1927 and in a modified form by S Banach in 1929, in the form by Banach in 1925 in that, then this Hahn Banach theorem it discovered real for the real as for real vector space and later on, it is extended to complex vector space by H F H F bohnenblust and sobczyk in 1938.

So, we will discuss this Hahn Banach theorem first for the real vector space which is discovered by Hahn and Banach that is why it is called the Hahn Banach theorem. The statement of the Hahn Banach theorem is, let X be a real vector space let X be a real vector space X be a real vector and p are sub linear functional sub linear functional on X. Further let f be a linear functional let f be a linear functional, which is defined on on a subspace z of X, z of X of X and satisfied and satisfies the majorization property.

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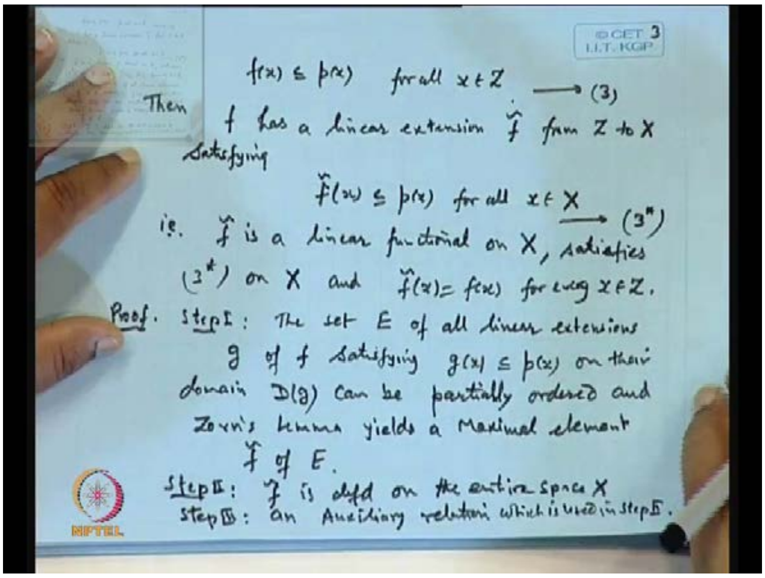

Satisfies, f x is less than equal to p x. Now, let it be this is 1, this property we will use a second and this one is third for all x belonging to z, for all x belonging to z this is third, then what this theorem says, then f has a linear extension, say f delta from z to X, satisfying the property f delta x is less than equal to p x, for all x belonging to capital X.

Let it be 3 star, that is f delta is that is f delta is a linear functional on X and satisfies satisfies 3 star on X and f delta x equal to f x when x belongs to z for every x belonging to z. So, it is an extension for this clear. So, this is a statement for the real. Now, we will proof this theorem in 3 steps in the first step, we will consider a class e of the set of all functional f which are the extension of f, that is set of all linear extension of f on e, satisfying the condition of majorization and then, we will see that a suitable partial order can be introduced which will convert this set e into a partially order set.

Then by applying the Zorn's lemma, we can get a maximal elements and that maximal element we will behave we will see, that it behaved as our desired function, that is it will be linear on the entire x and majorized by p x.

So, in the second step we will prove that, this f delta which you are getting is a desired function on the entire space X and in between establishing this we need certain auxiliary relation, so that will we will show in the last step that is step 3.

So, the proof of this will be divided into 3 steps, step 1 what we do in that, the set e the set e of all linear extension, extensions g of f linear extension g of f, satisfying satisfying the condition g x is less than equal to p x on their domain d g can be partially ordered partially ordered and Zorn's lemma and Zorn's lemma it is a maximum maximal elements f delta of f.

Then second step, what we do is f delta, so obtained is defined on the entire space X and third step, we will use an auxiliary relation auxiliary relation which is used in part b used which is used in step 2.

So, the proof of this will, we will break up into 3 steps, in the first step we will show the existence of f delta and second step we will show this f delta, in fact is the entire defined on the entire x and third step a relation is will be proved which is used for establishing the step 2.

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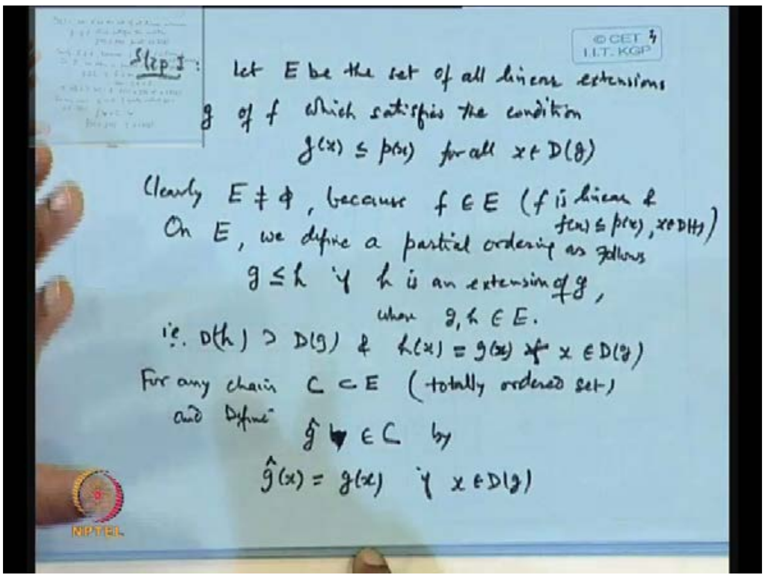

So, suppose E be the set of let e be the set of all linear extension, extensions g of f, f is a already given to a linear functional defined on z, which satisfy the condition, the conditions g x is less than equal to p x for all x, belonging to the domain of g for all X belongs to the domain of g.

Now, this E is clearly a nonempty set, why because because f belongs to E, why? Because f is linear and majorized by this and majorized by this in the domain of f where x belongs to the domain of f, so it is a majorized by this and x belongs to this domain. So, there exist at least one element f belongs to e, so e cannot be (( )) e is different from empty set sorry e is different from e cannot be an empty set.

So, we can now introduce the partial order on e on e, we define we define a partial ordering as follows, g is related to h, if h is an extension of g extension of g, where g and h are where g and h belongs to a, e is the collection of linear functional we are taking the 2 elements of e that is 2 linear functional, we say the g is related to h, if h is an extension of g that is the meaning of this, that h domain of h should cover the domain of g, because it is an extension and whenever the x is belonging to the domain of g then h x must be equal to g x for all X for every x belonging to the domain of g.

Then we say, h is an extension of g now we claim that this is a partially ordered set ordering relation, why? First is, it is a reflexive each g is related to itself, because g is the extension of g itself, domain of g is covers g it is domain of g covers and this condition is satisfy.

So, it is reflexive property satisfy anti symmetric, if g is related to h then this condition holds, if h is related to g then reverse condition hold, that is d h is covers d g's and both are so in fact, both will be equal, so domain of g and domain of h will be equal and they will satisfy.

So, again anti symmetric and transitive also follow, so it is a partially ordered set that is we are able to introduce the partial ordering relation on it, now let us consider a chain in e, so for any chain C in E. Chain means a totally ordered set, totally ordered set means that any 2 element of this class c are comparable, if we pick up any 2 element, either one is extension of the other or vice versa, so let us take any chain for this.

Now, we defined and define and define a new function g hat, has by on this chain g hat defined g hat by g hat define g hat which belongs to c, by g hat x equal to g x g x whenever whenever g hat x equal to g x. If X belongs to x belongs to domain of g x belongs to domain of g, that is in this chain, I am defining the any arbitrary element g dash x you pick up, then it will coincide with 1 of the g x if x belongs to the domain of g.

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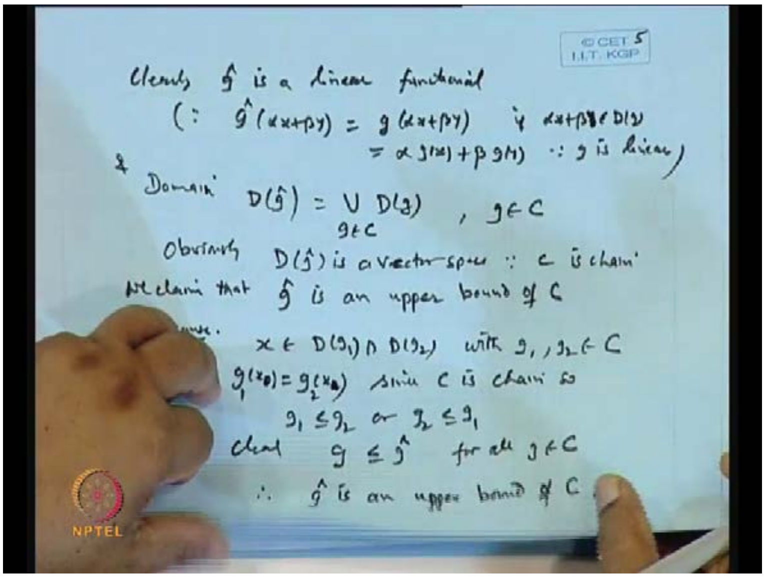

So, whenever the g dash x, x belongs to this then it is same, now this g dash clearly a linear functional, clearly g dash is a linear functional, why it is linear? Because if I take g dash alpha x plus beta y, then say alpha x plus beta y, then by definition this will be equal to g of alpha x plus beta y for, if this point alpha x plus beta y belongs to the domain of g, is it not, because it is a vector space, so if x y belongs to the than the domain and g is a linear. So, it will be taken as g of this g of y, because g is linear g is linear. So, we get this thing that is g dash is a linear functional the domain of g and the domain and the domain of g hat will be the union of all the domain of g, where the g belongs to c.

Because when you take the x, then x may belong to the domain of g 1, x may belongs to the domain of g 2 like this. So, all are for all g dash x is defined as g 1 x g 2 x and so on depending on x. So, domain of g dash will be the union of all the domains of the g hat, where g belongs to c, where g is an elements or linear functional on c. Now, obviously, this is a; obviously, domain of g hat is a vector space why?

Because, domain of g it is a vector space, g is a linear functional defined on a vector space and this is a chain, because c is a chain, that is why, we can get that unions of this will be domain is a vector space. Now, another point which we claim we claim that, that g hat in fact, is a maximal element of c, we claim g hat is an upper bound upper bound of c is an upper bound of c, why? Because because like this.

If we take x is any elements of this, because if suppose x I take in the domain of g 1 intersection domain of g 2, then what happens is? That with g 1 and g 2 belongs to the chain c. Then according to this I g of x 1 equal to g of x 2 because, it is a common element and since, c is a chain c is a chain, so these g 1 and g 2 g 1 oh sorry g 1 g 2, this is not x 2, g 1 x equal to this since c is chain, so they are comparable.

So, either g 1 will be less than equal to zero. So, either g 1 is less than equal to g 2or g 2 will be less than equal to g 1, one of them will hold, but always g will remain that is is less than equal to g hat, because g hat will be according to this g hat define as g x if x belongs to g.

So, g hat will be g 1 x or g hat will be g 2 x therefore, in all the case g hat will be the upper bound for this. So, this shows the g hat will be this is true for all g belongs to c therefore; g hat is an upper bound upper bound of c, so this will be. Now, e we have already established is a partially ordered set and c is a chain, every chain c is an arbitrary chain, now every chain as an upper bound. So, according to the Zorn's lemma it must have its maximal elements because Zorn's lemma's.

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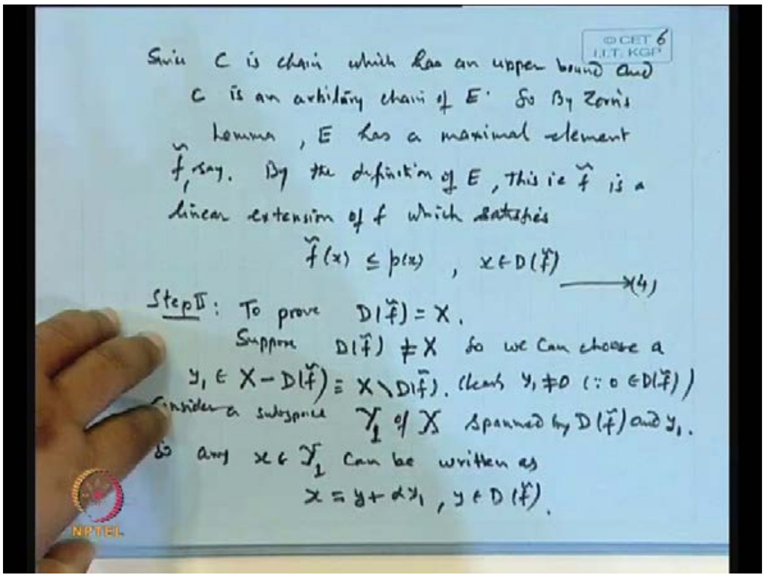

So, we say since, c is a chain in which has which has an upper bound has an upper bound and c is c is an arbitrary chain of E. So, if you take any arbitrary chain c, it will have a upper that is every chain, every totally order set in e has an upper bound. So, by Zorn’s lemma by Zorn’s lemma E has a maximal element maximal elements maximal element f delta say f delta say f delta. So, what do you mean by this maximal element? It means by definition, so by definition by the definition of E, E is the collection of all the linear functional and f delta is an element maximal, so it will also be linear functional and majorized by p.

So, by the definition of E this this is this means that is f f delta is a linear extension of f extension of f, which satisfies which satisfies the property that f delta x is less than equal to p x for x belonging to the domain of f delta and let it be 4. So, we have establish the existence of f delta, now we will show that, this f delta is a, is linear on the entire x defined on the entire x. So, step 2, we now shows that. So, if it is defined on the entire x, it means the domain of d f bar, if it is x then we say it is defined on the entire x.

So, we now show, to prove the domain of f delta is the entire X is all of X suppose, this is not true suppose, d of f delta is not equal to X, that is there exist some point X 1, X which is in x, but not in d f, so there exist. So, we can choose a y 1 a point y 1 in X minus domain of f X minus domain, that is X or we can also write that is equivalent to X difference domain of f delta, domain of f delta, consider which is in x, but in f.

Now, consider a subspace subspace y 1, y 1 of X, y 1 of X X spanned by domain of f delta and y 1, domain f delta that is any element of that is any element of this y 1 can be expressed as the sum of the elements of the spanned d f bar plus the element alpha of y 1 time, so that is 1 can write. So, any x belonging to y 1 y 1 can be written can be written as x is equal to y plus alpha y 1, where the y is an element in the domain of f now 1 more thing which, I have in a.

Now, here when we say this y 1 belongs that clearly, this y 1 is not equal to 0, why it is not equal to 0? Because the domain d f bar contains 0 and y 1 is a point which is not in domain d f bar, because 0 is an element belonging to the domain, domain f bar is a vector space and it contains 0 element, so y 1 will be different from 0.

So, let us take the subspace which is generated by the d f bar and y 1 that is by the spanning of d f bar and the alpha y 1 this one. Now, we claim that this representation. That is is unique first we claim that this representation is unique.

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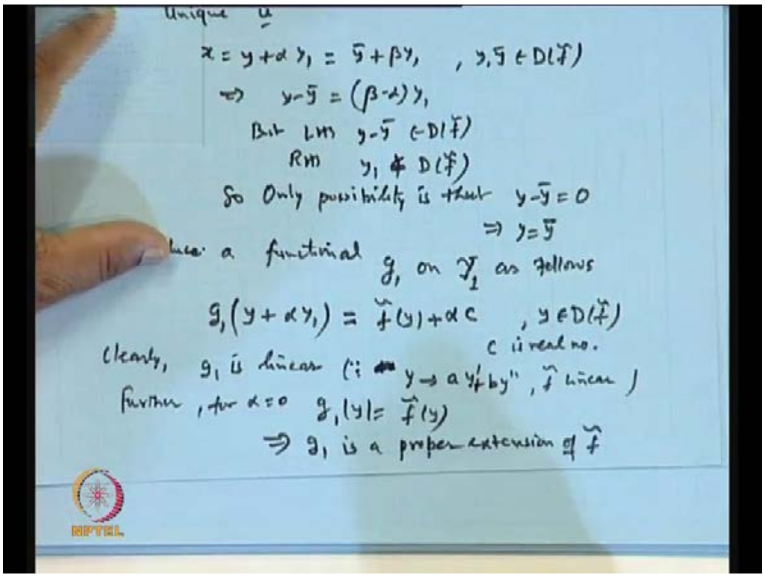

That is if we take suppose 2 representation suppose, that is if X can be written as y plus alpha y 1 and same as say y bar plus beta y 1, where y and y bar these are the element of the domain f bar and alpha beta are the scalar, then this implies that y minus y bar is beta minus alpha y 1. But left hand side y minus y bar is a point in domain f bar and right hand side y 1 is not a point in domain of f bar, sorry it is not a point in, yes it is not a point in because it is in outside of y domain of f bar, but zero is a.

So, the only this result holds, so only possibility is that, this y minus y bar should be 0, that is y is equal to y bar. So, the representation is unique, this shows representation unique. Now, once we have this representation let us introduce the linear functional. So, introduce introduce a functional g 1 on y 1 as follows, we are defining the g 1 on y 1 means the element will be of the form y plus alpha y 1, where y is in d f bar and y 1 is a point not in d f bar, as the f delta y plus alpha into c, where the y belongs to domain of f bar and c is a real number c is real number.

Then clearly g 1 is linear because, if I take y because, if I take the y 1 say alpha let it be some like this y replaced by, say replaced by y, a y dash y 1 y dash plus b y double dash and then pick up the 2 element like this, then we say because of the f bar because, f bar is linear therefore, it will break up into this and get the linearity. So, g 1 is linear, so I need not to show this part.

See, g is a linear, then further because, alpha equal to zero for alpha equal to zero g 1 y coincide with f delta y, it means g 1 is a that is this implies that, g 1 is a proper extension of a is a proper extension of f delta that is an extension such that, that is that is an extension such that, d f bar f delta sorry is a proper subset of is a proper subset of D g 1 D g 1 D g 1.

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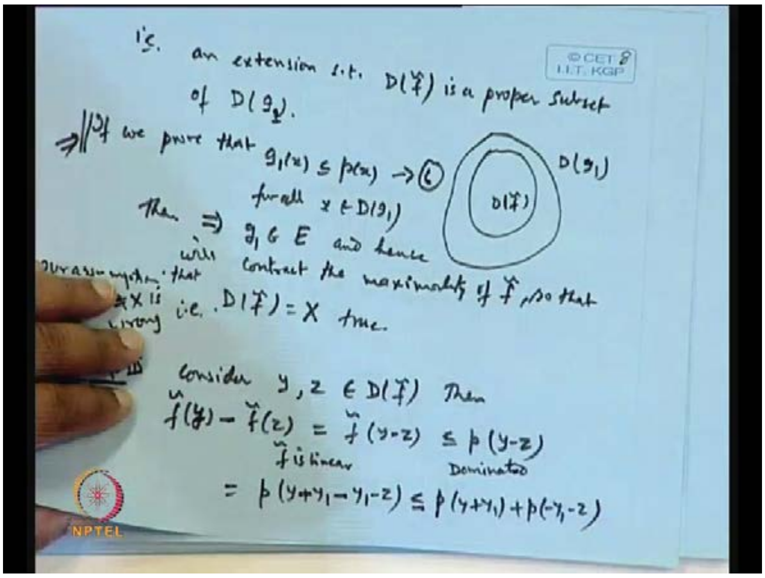

So, this is our domain of f delta and here it is domain of g 1, we say it is a, we have seen that this is a proper subset of this, now if I prove and g 1 if I prove that g 1 belongs to e by showing.

So, we if we prove, if we prove that, this g 1 that g 1 x is dominated by p x for all x belonging to domain of g for all x belonging to the domain of g 1, then this will show this implies, that g 1 is an e because it is an extension, it is an extension dominated by p by c hence and hence will contradict will contradict the maximality maximality contradicts the maximality of f delta.

Because if f delta is maximum maximum element, then a functional which is linear and extension of this cannot be an element of E, here we have shown that g 1 is an extension of f delta, if I prove that g 1 also satisfy this condition, then g 1 is linear dominated by p majorized with p x then; obviously, the g 1 will be the elements in E hence it will contradict the maximality of f delta.

So, this contradiction is this because our will reach, because our long assumption that d f delta is different from x, the contradict the maximality of a f delta, hence so so that d f delta is equal to x and that is what we wanted to show.

That is so the d f delta is X means contradiction, so contradiction of a, so that d f delta cannot be X. So, d f our assumption that, d f delta is subset of X is is not equal to X is wrong, that is d f delta will be equal to X, hence this is true, hence it will give the f delta is a entire defined on the entire x.

So, basically we wanted to this part, we wanted to show basically this portion, this is our and this we will prove in the step three. So, once it is established automatically this gives you the proof of the (( )). So, let us see the third part. So, let it be this equation is 6, I think this 5, which one is 5 equation g alpha, this equation let it be 5, this part is the fifth equation and let it be this is sixth equation.

So, we will make use of the number that is why we putting that. So, now let us prove that g 1 is less than equal to p x. So, what we do is consider the, for any y and z, so consider 2 elements say y and z in d f delta domain of that delta. Then start with this f delta z minus f delta y or f delta y minus f delta z because, f delta is a linear f delta is linear. So, y property of linearity this can be written f delta y minus z and then because, f delta is dominated by the sub linear functional p. So, it can be written as this, because this is the dominated by p, dominated by the way of delta this is by property by assumption, that is given in that dominated by this.

Now, this side we can write it as p, y minus y 1, y plus y 1 minus y 1 minus z and this will be because p is a sub linear functional, so we can get the less than equal to p, y plus y 1 and then p minus y 1 minus z. Now, transfer this toward there.

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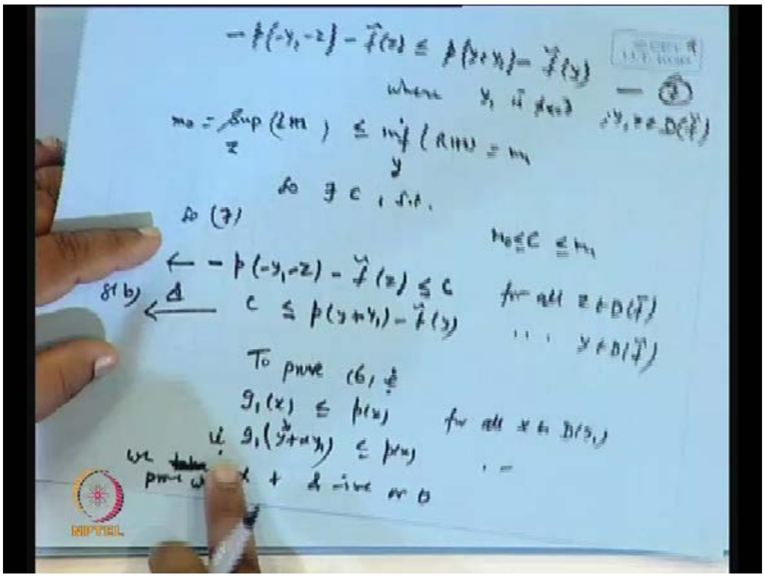

So, what we get it is here is, minus p minus y 1 minus z z minus f delta z is less than equal to p of y plus y 1 minus f delta y minus f delta y. Now, y 1 is fixed element where the y 1 is fixed y and z these are the elements of d f delta.

Now, left hand side is independent of y, right hand side is dependent on independent of z, so if I take the supremum value of this side over z and the infimum value of y over y then the result will continue to hold good. So, we take the supremum of this side supremum of this side left hand side left hand side is less than equal to infimum of the right hand side, where supremum is taken over z, infimum is taken over y.

So, supremum and infimum both are there. Suppose, these values are m naught and this is an m one. So, there exists a real number c, so for as c, so there exist some c such that, c lays between m naught and m 1, this may be equal to or, so there clear. Therefore, this side seven this would be seventh equation, so this 7. So, we get from 7, we get from here that minus p minus y 1 minus z minus f delta z is less than equal to c and and c is less than equal to p, y plus y 1 minus f delta of y. This is this is true for all, all z belongs to d bar d f bar and this is true for all y belongs to d f bar, clear.

Now, let it be this 8 a and this is 8 b this one is 8 a, this 8 b. Now, we wanted to establish 6, we wanted to establish this result 6, g 1 x is less than equal to p x for this now g 1 x is defined by 5 as this value x is y plus x is y plus alpha y one. So, this is x equal to this.

So, we wanted to show 6. So, to prove 6 to prove 6, that is that is g 1 x is less than equal to p x for all x belonging to d g 1 or x we can write it as our alpha, that is x we can use at this one that is g 1, x means according to fifth y plus alpha y 1, this is this will be the x is less than equal to p x that is this we can put it for all x this is it not, for all x d g dash.

So, we want in order to prove this thing, we will take the help of alpha because, alpha depends on it. So, to prove this result we take alpha positive and negative both, we prove rather than, we prove when alpha is taken as positive, alpha is taken negative or zero, so our proving of 6 depends on alpha, so we will choose for this alpha to be positive, alpha to be negative and alpha equal to 0 then this is at holds good.

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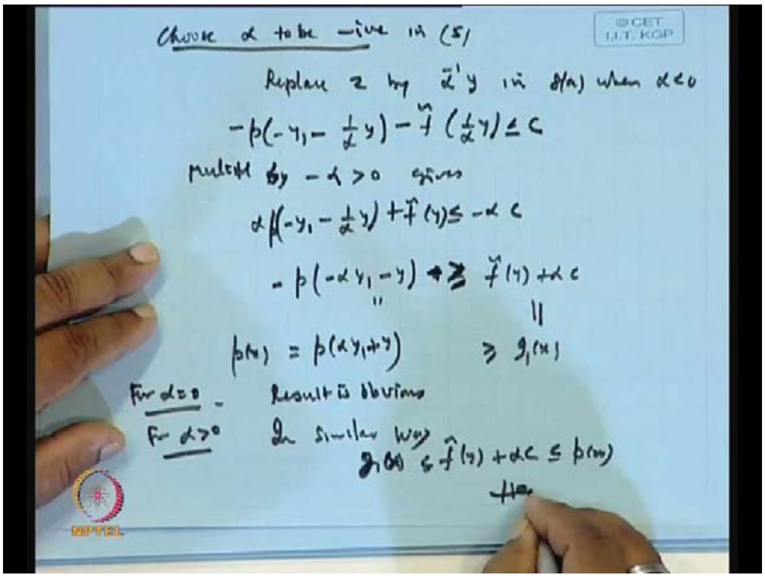

So, let us take this first alpha is positive or negative, so choose alpha to be negative, take alpha to be negative. In fact, then for in 5, what is the alpha is negative in 5 in 5, 5 mean this. So, we wanted to prove this result when alpha is negative, so positive. Let us replace replace replace z by alpha inverse y in 8 a, when alpha is negative. So, we get from here 8 a, 8 a is this problem here this is 8 a.

So, here we can replace z by this, so we what we get is this is minus p minus y 1 minus 1 by alpha y minus f delta 1 by alpha y is less than equal to c now multiply by minus by minus alpha which is positive. This gives alpha of p minus y 1 minus 1 by alpha y plus f delta y is less than equal to minus alpha c or this will be equal to what alpha this is cancel. So, p of p of alpha minus alpha y 1 minus y plus plus sorry is less than equal to f delta y plus alpha c and this will be equal to x let it be put it x. So, this is equal to what p X is it not?

So, p x, this is p x and this one will be equal to what? f delta plus alpha is nothing but, so what you are getting is sorry, this is greater than, this will be transfered here, so it is greater than equal to sorry greater than equal to this side minus alpha is outside and we are getting to be minus alpha p, yes minus alpha is outside, then we are getting minus alpha into this multiplying by this minus alpha. So, here also is minus alpha times of this. So, alpha and then transferring towards that side, we get the minus and this one will be equal to what g x g 1 x, this is greater than equal to what we are getting is here is p alpha y 1 plus y, but this p alpha y 1 plus y is is our p x.

So, we get g 1 x is less than equal to (( )). Now, for alpha equal to zero the result is obvious and for alpha to be positive in a similar way similar way, we can show that g 1 X is less than equal to f delta y plus alpha c, which is less than equal to p x, hence proof this we get.

Thank you, Thanks.