**Functional Analysis**

**Prof. P. D. Srivastava**

**Department of Mathematics**

**Indian Institute of Technology, Kharagpur**

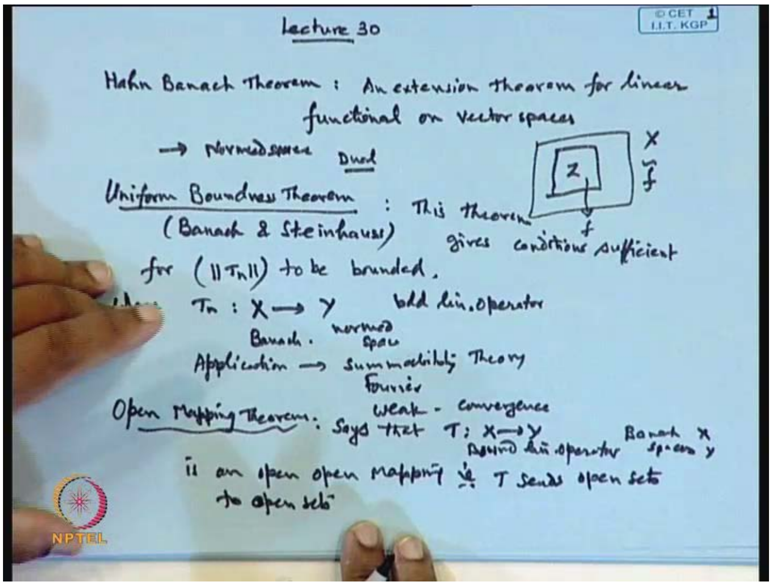
**Module No. # 01**

**Lecture No. # 30**

**Partially Ordered Set and Zorn's Lemma**

Discuss the Norm space, Banach space, the definitions and some fundamental results to go in deep to get a get more deeper results we require some basic fundamental results theorems. There are four fundamentals theorem, which are very much useful in developing the theory of norm as well as a Banach spaces.

(Refer Slide Time: 00:47)




These theorems are Hahn Banach theorem, Hahn Banach theorem which is basically an extension theorem an extension theorem an extension theorem for linear functional for linear functional on vector spaces. In fact, what we do is, we are interested to extend the linear functional from a sub vector space to the larger class of the vector, if we have say z is a vector subspace of x and f be a linear functional defined on z, which satisfy certain property or certain conditions are satisfied by the function f may be, dominated by another function some some linear functional or something by f.

Then the Hahn Banach theorem gives you a guarantee that, it can be extended to the entire class x without losing the property, that is the extended functional f delta will also remain linear as well as it will return the same condition, which the function f x over z, so this is the Hahn Banach theorem we will go in detail. Then, another result is uniform boundedness theorem, this theorem is given by Banach and Steinhause, Banach and Steinhause hause is also known as the Banach Steinhause theorem.

And this theorem gives the condition this theorem this theorem gives conditions sufficient conditions sufficient for for the sequence of norm T n to be bounded to be bounded, where the T n are the bounded linear operators from a Banach space x to a normed space Banach space into a normed space say, y, this is a Banach space and this one is a normed space and this is a bounded linear operators (Refer Slide Time: 04:04).

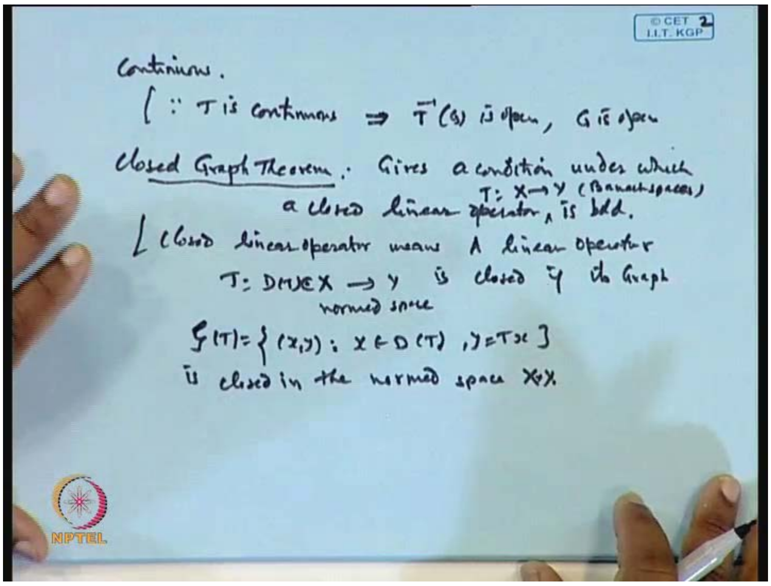
So, this result basically gives you the sufficient condition, when the norm of T n to be bounded. In fact, what is to be shown in this case is, if T n be a sequence of the bounded linear operator from a Banach space x to a normed space y, which is quite wise bounded convergent; then a sufficient condition is obtained by this theorem, in the uniform boundedness theorem by the Banach and Stein; so, that, the norm of sequence T n becomes bounded, so under what sufficient condition thus T n becomes uniformly bounded, so that is why it is also known as the uniform boundedness theorem.

And this theorem has a tremendous application, this applications one can see in the summability theory, in the Fourier series, summability theory in Fourier analysis or Fourier series. We can go for this application then, weak convergence and strong convergence and this also we can see the uniform boundedness theorem has a tremendous application.

Similarly, Hahn Banach theorem gives you the guarantee that, normed spaces are sufficiently interest with the linear functional, so that, one can get the concept of the dual way. So, third one is that, open mapping theorem, open mapping theorem this theorem states that, bounded linear operator from Banach space to Banach space is an open mapping, this theorem says that, bounded linear operator T from x to y. A bounded linear operator from the Banach space Banach space x and y, from Banach space x to a Banach space y is an open mapping, that is it will send the open sets to open set, that is T sends open sets to open sets.

So, when the T sends the open set to open set means, if T is bijective then, T inverse will be continuous. So, that is if hence, T is bijective then T inverse is continuous, it is continuous why, because we know this map a mapping T, because the T is continuous if and only if inverse image of this, it implies inverse open open set G is open, where G is open. So, if the T sends the open set to open set it means, if T is a one to one mapping then, the inverse operator T inverse will also remain continuous. So, in case of this open mapping theorem we can take.

(Refer Slide Time: 07:26)




Now, fourth one, we have the closed graph theorem, closed graph theorem that says that, this gives a condition under which gives a condition under which a closed linear operator is bounded is bounded.

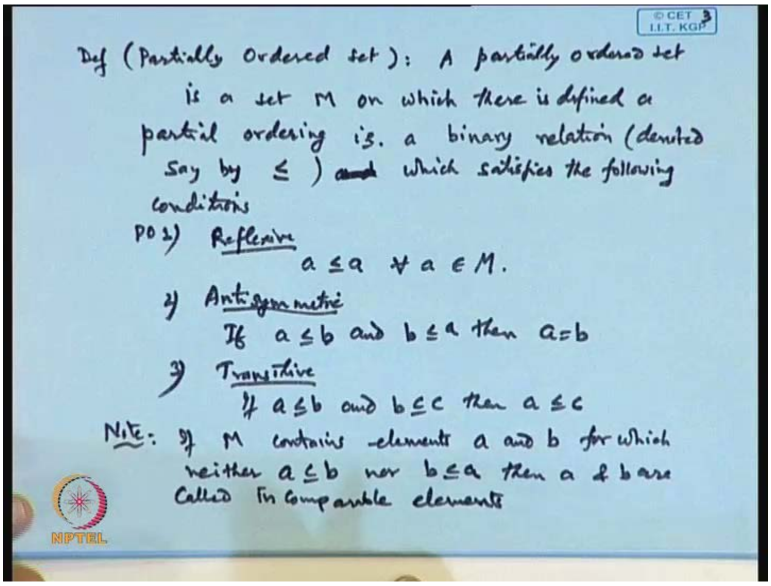
Now, when we say is a closed linear operator, closed linear operator means an operator in which the graph of the operator, graph is closed. So, we define the closed linear operator, as the operator T is called the closed linear, if the graph of T is closed. So, operator is closed linear operator, closed linear means linear means a T a linear operator means a linear operator T from the domain D t, which is a subset of x to y, these are normed spaces is closed, if its graph that is G of T, gait of T the set of x y such that, x belongs to the domain of T and y is T x, this graph is closed in the normed space x cross y, x cross y x y are normed space and T be linear operator domain D is called the… and when we say, x and y both are under which a closed linear operator T from x to y, both are Banach spaces both are Banach spaces.

An example, Banach space then, T from x to y or D T of subset of x to y, if it is a closed linear operator, then the operator T is bounded, that is what closed graph theorems, is it closed? It is a closed operator if and only if, it is closed and the operator is bounded. So, we have this four theorems and this theorem has a tremendous application in the development of the functional analysis; and many results can be obtained quickly by application by using these one of these theorem which we require.

Now, we will go in detail and see, the one by one on Hahn Banach theorem, uniform boundedness theorem then, open mapping theorem and closed graph. First the Hahn Banach theorem, we will see the proof and then, what are the how the Hahn Banach theorem can be applied, can be used to get the other results quickly, rather than for its classical way of proving the results.

So, in order to start with the Hahn Banach theorem or in order to develop the theory or proof of the Hahn Banach theorem, we require the concept of the Zorn's Lemma and the related partially order set maximal, minimum and so on. So first let us see, what is the partially ordered set and then, we go for this Zorn's Lemma and finally the proof for this, Hahn Banach theorem.

(Refer Slide Time: 12:31)

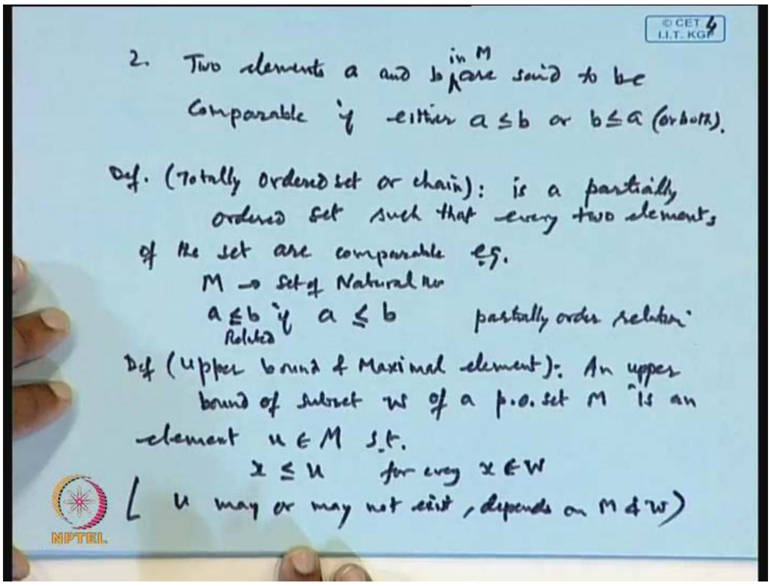


So, partially ordered set, partially ordered set a partially ordered set a partially ordered set a partially ordered set is a set M on which, there is defined a partial ordering, a partially ordering that is, what is the partial ordering? That is a relation, a binary relation a binary relation, which denoted say by this sign, is not less than, it is a notation for using the relation; and satisfy a binary relation, which satisfies the following which satisfies the following conditions, conditions first condition is, we will use the PO 1, Partially Ordered relation for condition 1, it is a reflexive that is reflexive condition that is, a is related to a, for every a belongs to M.

Then, the second condition is, the anti symmetric, anti symmetric that is, if a is related to b and b is related to a then, a must be equal to b that is anti symmetric. And third is, transitive transitive property, if if a is related to b, b and b is related to c then, a must be related to c, so this if these three conditions are satisfied then, we say the corresponding relation is a partial ordered, order relation of partial ordering.

So, a set M together with the partial ordering, which is called the partially ordered set, why partially? Partially emphasize that our set M, may contain some element a b, in which neither a is related to b nor b is related to a with the given relation, so partially implies that, emphasize that, partially emphasize that M may contain element a and b for which neither a is related to b nor b is related to c and such an elements are called incomparable element, if M contains if M contains elements a and b, a and b for which neither a is related to b nor b is related a then, a and b are set called incomparable elements incomparable elements elements. And in contrast, if a and b are the two elements, they are called comparable, if they satisfy either a related to b or b is related to a.

(Refer Slide Time: 17:31)




Second comparable, if two elements a and b are said to be in b in M in M are said to be comparable, if if either a is less than equal to b, a is related to b or b is related to a or may be both. Let us see why, why it is both? Suppose this depends on the relation, in which the relation is defined, if I defined the relation a is related to b as a is equal to b, then both the relations satisfied. Say, over the set of integer if I define the relation a is related to b, if a is equal to b then, it will be a this condition, comparable condition is satisfied.

Now, if there may be elements like this where, none of them is true; either a is need need not be related to b or b is need not be related to a, then two elements are said to be incomparable elements, this type of means we cannot say, always the two elements are related. Say any two elements are there suppose, I say the relation a is related to b, if a divides b, then it is not necessary always the two element if we pick up the one will be divisor of the other, it is not or b is the divisor of a, it may not be possible always clear. So, this will this is what we are getting. For example, 7 and 13, 7 neither divides 13 nor 13 divides 7 like that, so it depends on the relation.

And the correspondingly, the set M will contain comparable or incomparable elements then, we go for the totally ordered set, totally ordered set or chain, a totally ordered set or chain is a partially partially ordered set partially ordered set such that, such that every two element of the sets elements of the sets of the sets set are comparable are comparable.

For example, a partial ordered set such that, every two elements of the sets are comparable for example, if I take set M as a set of integer of natural number say, suppose, natural number and define the relation, a is related to b, if a is strictly less than b, then in this partial a is less than less than or equal to b then, a is related to b, this is related to b, if a is less than equal to b, then this is a partially ordered relation, partially ordered relation because it is reflexive; every element is equal to itself we can say every element is related to a itself, if a is related to b and b is related to a, then a must b means, if a is less than equal to b, b is less than equal to a, a must be equal to b it is transitive. If a is less than equal to b, b is less than equal to c, then a must be less than equal to c, so it is a relation.

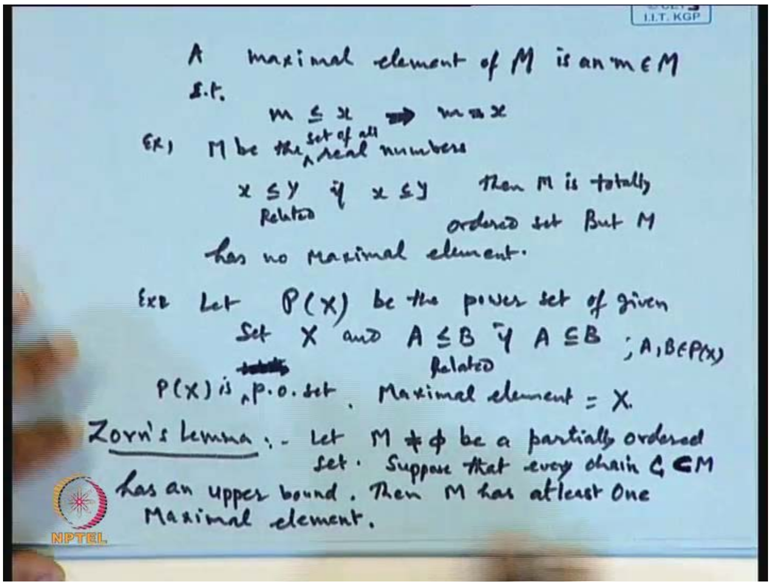
Then it is also a totally set ordered set, because if I pick up any two arbitrary elements of this set means, two natural number then, one can always decide whether one is less than equal to other or one is greater than equal to other, means either a is less than equal to b or b is less than equal to a, will always hold good, it means a is related to b or b is related to one of the condition will definitely hold. So, this will be a totally ordered set.

Definition then, there is concept of upper bound and maximal element upper bound and maximal element, so we may say maximal element, an upper bound of a subset of a subset w upper bound of a subset w of a partially ordered set M partially ordered set m PO set, partially ordered set M is an element u is an element u belonging to M such that, x is less than related to u, x is less than equal to for every u for every x belongs to w.

Depending on M and w, u may or may not exist. Now, this is u may or may not exist depends on M as well as w, what type of the M and what type of w you are choosing accordingly, the max elements upper bound may or may not exist.

For example, if I take this x is strictly less than u say, suppose or x less than equal to u in an open interval or in the real line completely, then upper bound will not exist for that, because it may be the infinity like that, so this will may or may not exist for that. Then a maximal element a maximal element of M element of capital M, M is a partially ordered set is an m belonging to capital M such that, such that m is less than equal to x implies m must be equal to x m must be equal to x.

(Refer Slide Time: 24:56)




Then such an element is called the maximal element again maximal element not exist, because if I take the real line example. Say, let M be the real set of all real numbers be the set of all real numbers and if I define the relation as x is related to y, if x is less than equal to y, then M is totally ordered set, totally ordered set but M has no maximal element. What M has no maximal element?

Because, definition of maximal element says, if m is less than equal to x, then m must be equal to x. Suppose, I take any arbitrary number as a maximal element and then, if this is less than equal to x, it never implies it may not imply then, m equal to, because a number higher than this is possible on the real line, so that is why we should not maximal element will not exist.

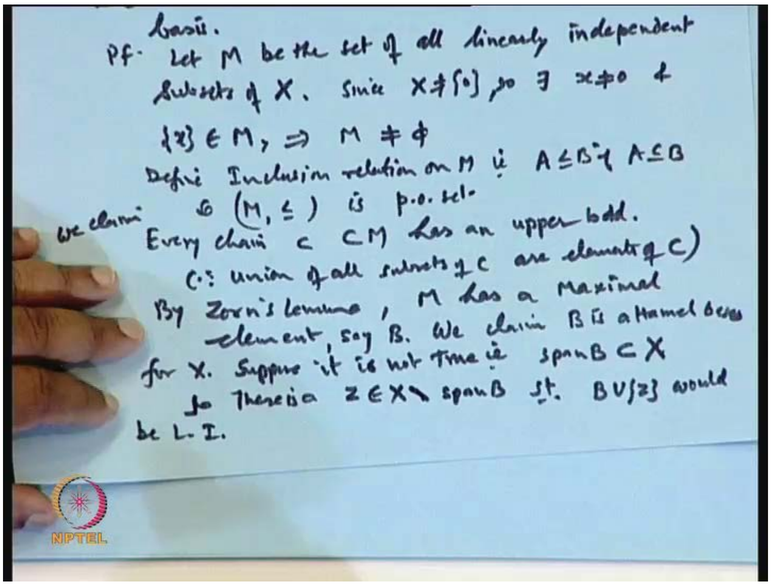
However, the maximal in some case, the maximal element may also exist. For example, this is a power set, let p x be the power set, power set means set of all subsets of given x set x of given set x, so x is given set set of all possible subsets of x means collection of all the subsets of x will give the power set of x. And let us introduce the relation and let A is related to B, if A is the subset of B, this is related to be, where A and B both are elements of the power set x, then obviously, the p x is a partially ordered set and in fact, it will be a totally partially ordered set, it is a partially ordered set the only maximal element of this is partially ordered take any two elements A B either B no, this is partially ordered set, sorry this is partially ordered set and its maximal elements the maximal element will be maximal element will be our entire x, so it has a value. We cannot say it is totally ordered partial, because if we take a two element, one is the singleton set one, other singleton set is two and we cannot say A is a subset of B or B is a subset of A. So, that is why it is only a partially ordered set, is related to if A is a subset of B ok. If A is related to B means, A is a subset sorry, A is related to B means A is a subset, this is the condition is given. So, if we take the two elements A and B one should be the subset of the other. So, x will be the length that is all clear. So, this is my x one element.

Then, let us see this properties are again used and to to this Zorn's Lemma basically. So, Zorn's Lemma what he says is, we will simply states the Zorn's Lemma because, it is used in proving the Hahn Banach theorem. The Zorn's Lemma says, let M which is a nonempty set, be a partially ordered nonempty set, be a partially ordered set let M be a partially ordered set. Suppose that every chain suppose, that every chain every chain c, I am doing to c, which is contained in M every chain c which is contained in M has an has an upper bound, this is c contained sign has an upper bound, this is c upper bound then, M has has at least one maximal element at least one maximal elements.

So, what is Zorn's Lemma says is, if M be a partially ordered set and which has a property that, if I take any chain means totally ordered set and chain is a totally ordered set that is, any two elements are always comparable. So, if we choose a collection of the sets, which is totally ordered set, then this chain has an upper bound, that is every chain there is an element x such that, x is less than such that all the elements are less than equal to x or related to x, then such a less than equal to x then, then M will have at least one maximal element. That if every chain has an upper bound then, there will be a guarantee that it will have an upper maximal element.

In fact, the Zorn's Lemma an axiom of choice they are the set, the axiom of choice, what he says is that, if suppose e is a set and we are considering a choice function from the power set of e to e, then pick up an element e from it, then the c of e will be an element of B, this is the choice function and the axiom follows from this, Zorn's Lemma and vice-versa. So, So, this axiom of choice and Zorn's Lemma they can be treated as equivalent things, a equivalent axioms. We are not interested in the choice of axiom just we will see the Zorn's Lemma, so this is what we get.

(Refer Slide Time: 33:00)




Now, Zorn's Lemma itself has an application in proving the results of the function analysis, the first result is which can be shown with the help of the Zorn's Lemma is we know that, every vector space which is a different from a singleton set 0 has a Hamel base this we know and it can be put with very quickly with the help of the Zorn's Lemma.

Every vector space capital X which is different from a singleton set 0 has a Hamel basis. has a Hamel H a m e l Hamel Hamel basis ok. Proof, Hamel basis we know is x be a vector space, b is a subset of x which is linearly dependent span of b is our x, then we say b is a basis for x, which called a algebraic basis or the Hamel basis for that.

So, let us see the proof further, let us suppose M be the set of all linearly let it means, every vectors has a Hamel basis so, there must be a linearly independent set, which is buzz the x maximal element. Let M be the set of all linearly independent subsets of x consider, M be a linearly independent subsets of x then, we will see first that M is a a we can define a inclusion define on partial ordering only. So, since x is different from 0 it means, there must be a non zero element.

So, it has so there exist an element x which is different from 0 and in capital X and this singleton set x will be an elements of M, the M is the set of all linearly independent, this singleton set which is different from 0 will be linearly independent set, so it will be the elements of M. So that, so this implies that M is nonempty. So, we are having a nonempty set of linearly independent subset of x.

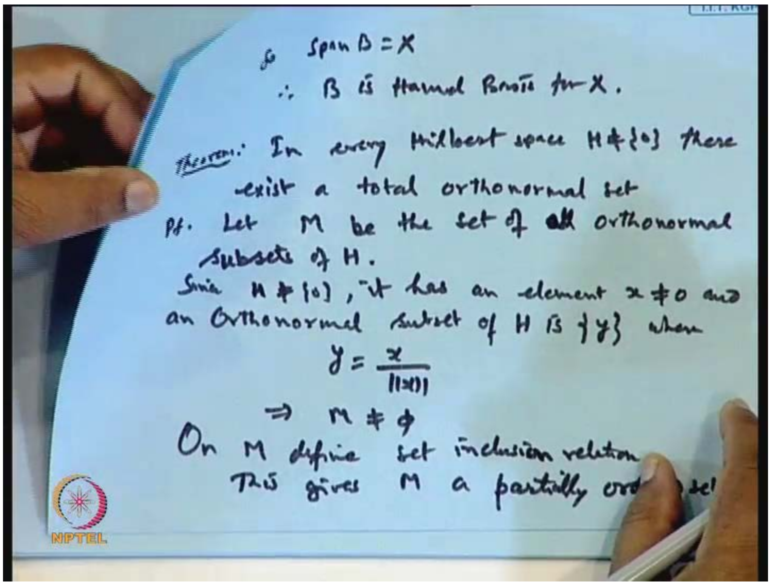
Let us define, a inclusion relation so also subsets are there. So, define inclusion relation on M inclusion relation on M that is two sets are there; A is related to B, that is A is related to B if A is a subset of B this is our inclusion relation. So, this transports the M to a partially ordered set, so M with this relation is a partially ordered set.

Now, we claim that, this is also a totally ordered set, we claim that it is also a total that is if I take every chain in this has an upper bound. So, every chain c which is a subset of M, that is if I take a partially ordered set, totally ordered partially ordered set in M this is possible, because if we take the if collection of the subsets A is subset of B, B is subset of C like this, then take the union, so that it will becomes a chain. So, if we take a chain in c, one chain in c then, we claim that every chain we claim that every chain c, which is subset of this has an upper bound. Because, again this is possible, because the union of all subsets of x which are the elements of c, because the union of all subsets of c all subset of c are the elements of C, so this is ok.

So, every chain has an upper bound, so by Zorn's Lemma Zorn's Lemma M has a maximal element, maximal elements say B. Now, we claim that this B will be a Hamel basis point, we claim that B is a Hamel basis basis for x this is our claim, the Hamel base for x that is the span of B will be this. Let us suppose, it is not true, so suppose it is not true suppose it is not true it means the span of B that is span of B is not equal to X will be a proper subset of X that is what we have.

So, it means there must be some z, so there is there is a z belongs to X minus span of this, but minus span of B minus span of B is it not, minus span of B such that, the B union z B union Z would be linearly independent set, independent set containing B as proper subset, proper subset which contradicts the which contradicts that which contradicts that B is a maximal element, B is maximal element. So, this contradiction shows that our assumption is wrong the span of B will definitely equal to X.

(Refer Slide Time: 40:22)




So, span of B is X therefore, the B will be a Hamel basis this proves the similar type of results is also valid, in case of inner product space. So, this we know in case of the Hilbert space inner product space that can also be proved with the help of Zorn's Lemma. The result is in every Hilbert space H, which is different from singleton set 0, there exist there exist a total orthonormal set. total orthonormal set

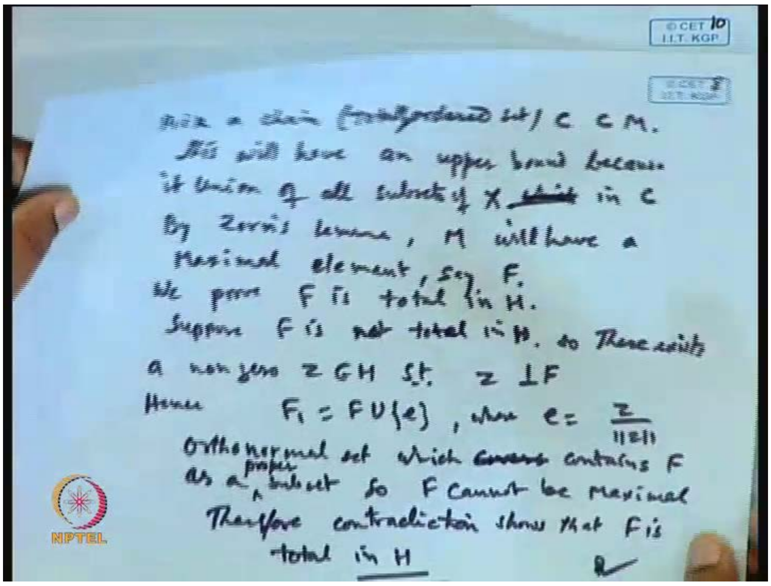
What is the total orthonormal set? We know the orthonormal set is the collection of points, where the inner product x y will be 0, if x is different from y. And 1, if x equal to y, so this is called the orthonormal sets and whenever the set this is orthonormal.

What is the total orthonormal set? A subset M when the closure of this span M is x then, we say it is a total orthonormal set is it not, then in fact this we we are define this thing as like this, that M is totally renewed the closure of the span of M is x. So, this we wanted in Hilbert space there exist a total orthonormal set, that closure of the span of M will be H itself.

So, let M be a proof again the proof follows on the same line, let M be the set of all set of all orthonormal subsets of H. set of all all orthonormal subset orthonormal subsets of H orthonormal subsets of H Now, since H is not equal to 0; so this shows, it has an element x which is different from 0, because x is x is not equal to 0. So, there will be an element x which is different from and the corresponding orthonormal subset of H and the orthonormal subset and an orthonormal subset of H in an orthonormal subset of H is this y, where y is equal to x over norm x, so that the norm of y becomes 1.

Now, this implies that M is not equal to phi, M is nonempty set, set of orthonormal subsets we have chosen and because of this condition M becomes nonempty. Now, let us consider the set inclusion defines a partially on M define set inclusion relation. So, M becomes a partially ordered set, so M is a partially ordered set, so this gives M a partially ordered set.

(Refer Slide Time: 45:00)




Now, let us pick up a chain, so let us consider a chain, then this chain will have a upper bound picked up the chain pick a chain that is totally totally ordered set ordered set pick a chain c, which is subset of M; now this c will have an upper bound, this c now this every chain now this will have an upper bound, because because why it will have a upper bound? Because, the union of all subsets of x, which are the elements of c. Because, union of all subsets of x, x are the elements of c, which are the which are elements of c this is upper bound, because union of all subsets of x is again in x. So, as an upper bound c because, it is the union of upper bound c or subsets of x this ok. So, an upper bound, because it is the union of all subsets of x in c, in c because it is means c is a set consider the union of all subsets of c then, that is will give the upper bound of it. So, this is an upper bound for this. Now, by by Zorn's Lemma this M will have an maximal elements.

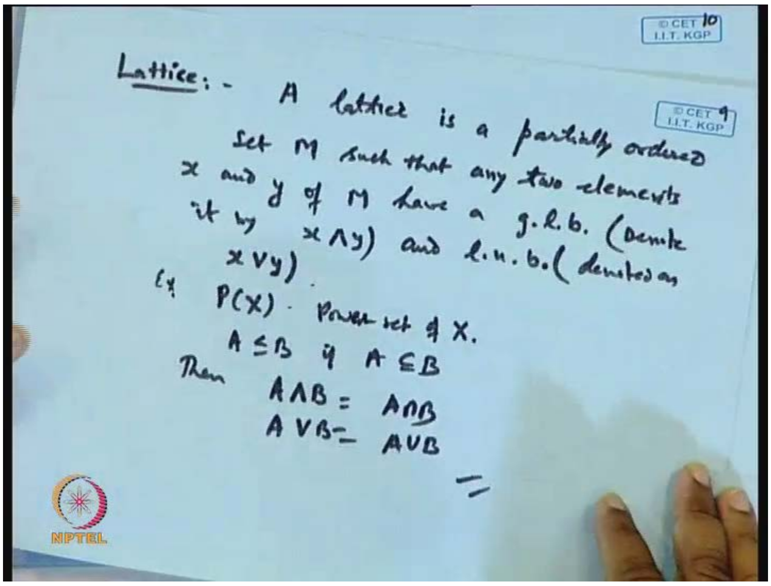
So, Zorn's Lemma M will have have a maximal element, say F. M will have a maximal element maximal element say F ok.

Now, we prove F is a total we prove now F is total in F is total in H then, our results complete. Suppose, it is not true suppose F is not total suppose F is not total in H. Now, there will be one result that, if a set F is total in H, then there will not be exist any nonzero x, which is orthogonal to every element of F.

So, when we say F is not total in H it means there must exist nonzero elements in F, which is H which is orthogonal to each element of this. So, we can enlarge the F, so what we get; so there exist So, there exist exist a nonzero z belonging to H such that, z is orthogonal to F this is by the result, which we have proved already that a set is total, if and only if, there exist no no nonzero element exist, which is orthogonal to this or x belongs to orthogonal implies x must be 0, that is what we have shown that result.

So according to there will be a nonzero z, which is orthogonal to F it means we can take F 1 as F union e, where the e is z over norm z and since, z is orthogonal to F. So, F 1 will be now a orthonormal set, this will be orthonormal F 1 will be orthonormal set which covers which covers or which contains F as a subset as a subset. So, this is as a proper subset rather than not exacting proper subset, it means F cannot be maximal, if it is so. So, M is not total in H, so this contradicts the maximality of F. So, F cannot be maximal element F cannot be maximal element therefore, a contradiction contradiction shows that, F is total that F is total in H that proves the result, so this is one.

(Refer Slide Time: 50:53)




Now, there is some one more concepts are which we know lattice. Now, lattice that may also be use something what is the lattice? A lattice is a partially ordered set, a lattice lattice is a partially ordered partially ordered set M, such that any two element any two element any two elements any two elements x and y x and y any two elements x and y of M have a greatest lower bound have a greatest lower bound we write it as, denote as, denote it by x meet by y and a least upper bound least upper bound denote as denoted as, x join y. So, a lattice is a partially ordered set in which, if we take any two element of this, then that meet and join, they must be the elements of M then such a partial ordered set is called the lattice.

For example, if we take x to be p x as a power set of x as a power set of x and let us define the first ordering as subset, A is less than equal to B, if A is a subset of B, then the meet and join both are available then; obviously, the meet is A A B is A intersection B, where the A join B is the A union B, so this meet and join are available. So, it will be a lattice for this, thank you that is all. So, we will discuss next time the remaining, thanks.