Functional Analysis

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Lecture No. # 03

Various Concepts in a Metric Space

There is considered the way will number of (( )) concepts, which play a role in connection with the metric space. So, we will discuss various type of concepts, particularly the open ball, close ball, open sets, close sets, convergence, Cauchy sequence, etcetera which will be used in the (( )) or in the deployment of these functional receipt.

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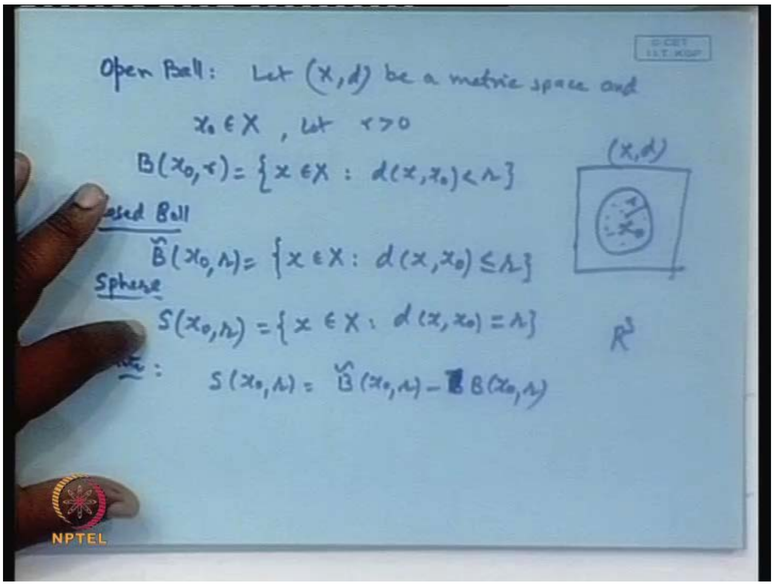

So, first see the what is the open ball. Open ball, let (X,d) be a metric space be a metric space and x 0 be a point in X. Let r be a positive real number, then open ball centered at x 0 and radius r is defined as the set of those points in x whose distance from x 0 remains is strictly less than r. Then this collection will be term as an open ball centered at x 0 and radius r. The closed ball in a similar way we define and denote it by B delta (x 0,r) as the set of those points in capital X where (X,d) is a metric space, such that the distance from x 0 is less than or equal to r. The sphere we denote this by S(x 0,r) that is the sphere centered at x 0 and radius r is the set of those points of a metric space X whose distance from x 0 is exactly r. So, here we have metric space say (x,d) and x 0 is an point in x, then if we draw a ball around the point x 0 with a radius r, then all such point which lie within this, is the open ball which includes the point of the boundary also closed ball and the points lying on the boundary only we call it as a sphere. Now, obviously, this is clear from this definition that - the sphere is nothing but the difference between closed ball minus open ball minus B(x 0,r). This is what we get this.

Now, we already have a concept in R 3, the concept of the open ball, concept of the closed ball and concept of the sphere. The open ball is the centered x 0 and radius r is as usual in the sphere. And we know that if r is a positive number or radius is non-negativity is positive, then we centered x 0 in a positive radius, the sphere cannot be a negative, ball cannot be a cannot be empty. So, this concept is available in R 3, but since we are dealing with the x d. So, the similar concept for the open sphere, closed sphere and the ball is not available in a general metric space. That is we cannot claim that a sphere is centered x 0 and positive radius, will always be nonempty. It may be empty also depending on the metric which we are using.

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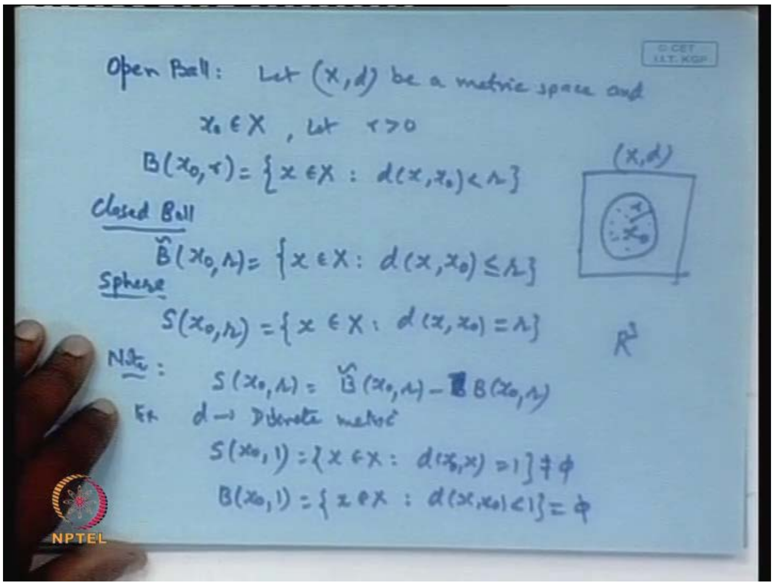

If we take the metric d as a discrete metric, then the sphere centered at x 0 and say radius 1 radius 1, this will be the set of those point x belonging to capital X whose distance from x 0 to x is 1. That is the set of those point whose distance under the discrete metric is 1 is nothing but the all the point which is different from x 0 that is the entire space. But if we take is not empty, but if we take a ball centered at x 0 and radius 1 in a discrete metric d, then we get that (x,x 0) which is strictly less than 1 comes out to be empty set, because there are no other point except the x 0 itself available in this class. So, that this becomes empty. So, this type of difference is available in a general metric. So, while we enlarge the concept, we should be beware about this thing that whatever the properties are enjoyed by the ball and the sphere in R 3 may not continue to hold good in a general metric space.

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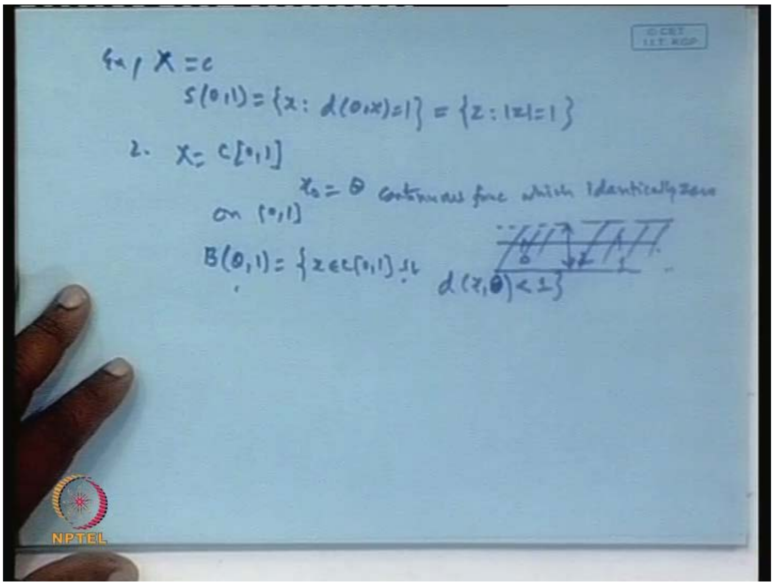

We can take few examples. We are these… Say suppose I take the c, X is equal to set of complex number. And the sphere centered at 0 and 1, the set of those points whose distance from 0 is 1, then this is nothing but the set of all those complex number Z where mod Z equal to 1. And similarly, if we take the open ball, it is mod Z less than 1 and like this. If we take another example, say X is equal to c(0,1) set of all continuous functions define over the closed interval (0,1). And suppose I take the point x 0 as theta, the continuous function which is identically 0 along the x is of x, that is the set of all continuous functions, say theta denotes the continuous function which is identically 0 which is identically 0 0 on this interval (0,1).

So, here this is the interval (0,1) and theta is lying here. Then that distance the ball centered as theta and with a radius 1 with the set of those points x belonging to capital X c a v c(0,1), such that the distance from theta is less than 1, this is the open ball; so basically, because this is a function. So, if we draw a strip around the point x, a strip which is of it is 2, length is 2 surrounding the axis of x, then all these points which falls within the all the functions which fall within this range will come in this class and open ball centered at theta n radius 1. So, these are the examples where you can having this.

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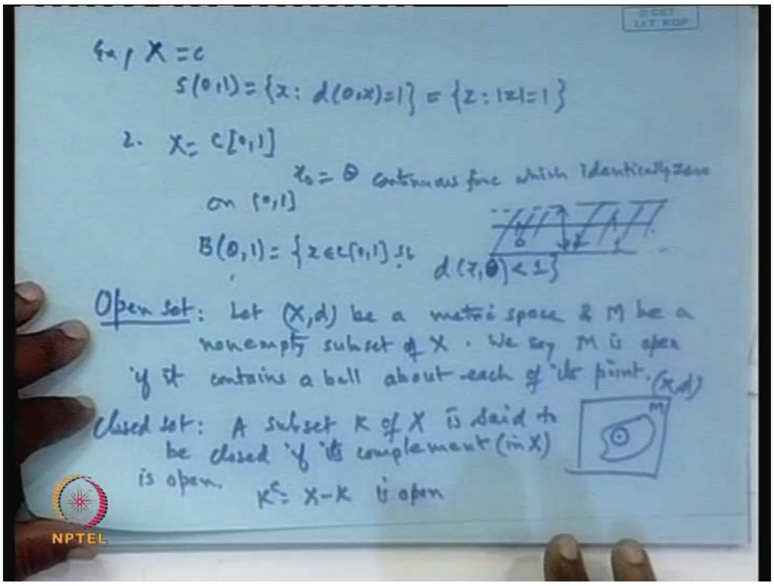

Now, if we go for a generalized form then we say an open set, which use the concept of the open ball. Let (X,d) be a metric space be a metric space and M be a and M be a non-empty subset of X subset subset of X. We say X c m is we say M is open open in x open, if it contains if it contains a ball about each of its point of its point. So, a subset M of a metric space x is said to be an open open, if it contains a ball about each of its point. That is - if we have this metric space and here is our M, then this set is said to be an open. If we pickup any arbitrary point inside it, then one can draw the ball around this point with a suitable radius which is totally contained inside M. This M is said to be an open set.

The compliment of these open sets is called closed. So, we define that closed set as a subset k of X - X d is a metric space is said to be is said to be closed. If its compliments it if its compliment with respect to X of course in X is open, if its compliment in X is open that is the k compliment which is X minus k is open. So, we define the this closed set as a compliment of an open set.

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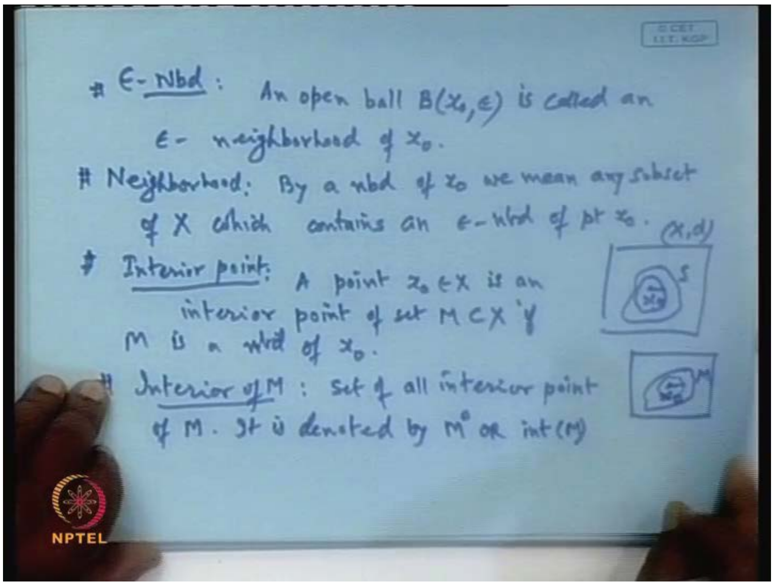

The epsilon neighborhood is defined as epsilon neighborhood an open ball an open ball centered at x 0 with a radius epsilon is called an epsilon neighborhood an epsilon neighborhood neighborhood of x 0. So, a set of all the points whose distance from x 0 remains less than epsilon, then those collection we call it as a epsilon neighborhood of x 0. And what do you mean by the neighborhood? Neighborhood we mean by a neighborhood of x 0, nbd I am using, by a neighborhood of x 0 we mean we mean any subsets of X any subset of X any subset of X which contains contains an epsilon neighborhood an epsilon neighborhood of the point x 0. So, if we have, say x d metric space, x 0 is this, then this set say s will be a neighborhood of the point x 0, if it contains a neighborhood around epsilon neighborhood of x 0 inside it, then we say this is a neighborhood of the point x 0.

The interior point interior point we define as a point x 0 belonging to capital X is an interior point is an interior point point x 0 in x is an interior point of the set M which is a subset of X; if M is a neighborhood M is a neighborhood of x 0; what is the meaning of this is that let M be a set and x 0 is a point, we say this is an interior point of the set M; if there are basically exist an open ball around the point x 0 with a radius epsilon which is totally contain inside in. Then only M will be the neighborhood of this. So, if such a point is available, then that point is called the interior of M. The collection of all such point x 0 forms a set which we call it an interior set. So, interior of M interior of M is the set of all interior points set of all interior point of M set of all interior points of M and it is denoted by it is denoted by either M 0 or may be some author used to write int M - interior of M. So, this is the concept of the interior points. Then we we have seen the open ball - open set, close ball - close set.

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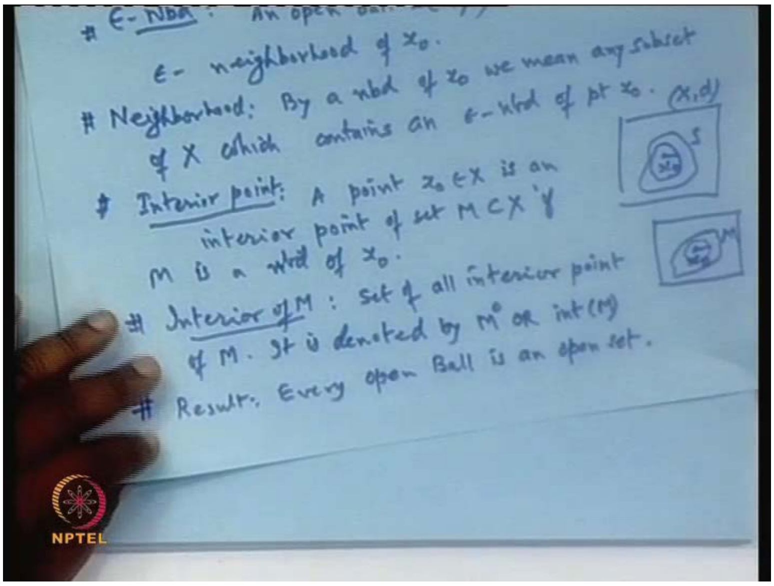

The every open ball basically is an open set, we have defined the open ball is the set of those points whose distance from x 0 is less than r. But an open set is the collection of those point, where every point is an around every point we can draw the open ball which is totally contained inside it. So, basically the open ball is also an open set. So, we can prove this result; every open ball is a open is an open set.

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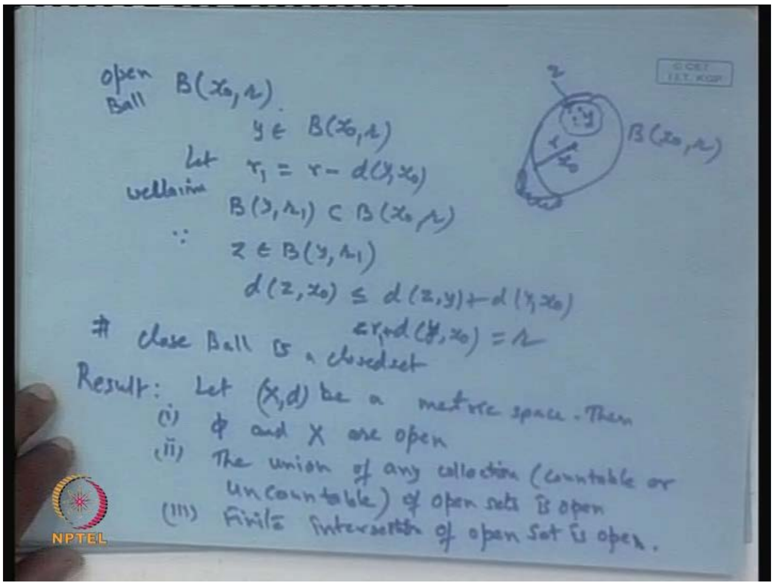

Let us see how; suppose we have an open ball centered at x 0 and radius r, this is our open ball. So, B(x 0,r), here is the point x 0 and this is our ball, of course this radius is r. So, I just take it like this. So, that this becomes r. We wanted this to be an open set. So, it means if we take any arbitrary point by here, then around the wide if we are able to draw an open ball which is totally contained inside it, then we say if ball is a open set. So, let y belongs to an arbitrary point belongs to (x 0,r). Let r 1 denotes the distance r minus d(y,x 0). This distance is complete a an r and here this distance is (y,x 0). So, I am taking r minus d(y,x 0) as r 1.

Now, if I draw the ball centered at y and radius r 1, we claim we claim that this ball is totally contained inside ball centered at x 0 and radius; means all the points of this ball will be available inside it. Why, because if we take a point say z, because if we take any arbitrary point z belongs to B(y,r 1); then what is its distance from x 0? Let this be a z point. So, this distance will be less than equal to d(x z,y) plus d(y,x 0). This is equal to d(z,y); what is the (z,y)? This is less than equal to r 1, because the distance from any point y belongs to (r 1,z) is the point inside you are taken. So, this is less than r 1, then plus this distance is (y,x 0). But r 1 plus this is nothing but r. So, if we pickup any z point here in this ball, then its distance from x 0 cannot exceed by r. It means the all the points of this ball lies basically within this d(x 0,r). Therefore, this becomes a open set. So, every open ball is an open set. Similarly, the close ball is a closed set is a closed set. So, we will not go we will prove like this.

Now, we can go for this some results again… Suppose (X,d) is a metric space; let (X,d) be a metric space, then the phonic results hold empty set phi and the entire space X are open. Second, the union of the union of any collection any collection, whether it is countable or uncountable of open sets of open sets is open. So, any arbitrary union of the open set is open. And third is the finite intersection of the open set is open of open set is open. So, we can prove this thing; first is empty set and x are open; this obviously true, because empty set you can say there no point is available. So, we can assume all any point around it, one can draw the open ball is totally contained inside it. And since X is a entire space is a universal space at that we are dealing with that. So, any arbitrary if we take any open ball inside it at any point, it is definitely a point cover in its. So, these two are the trivial things and we can solve.

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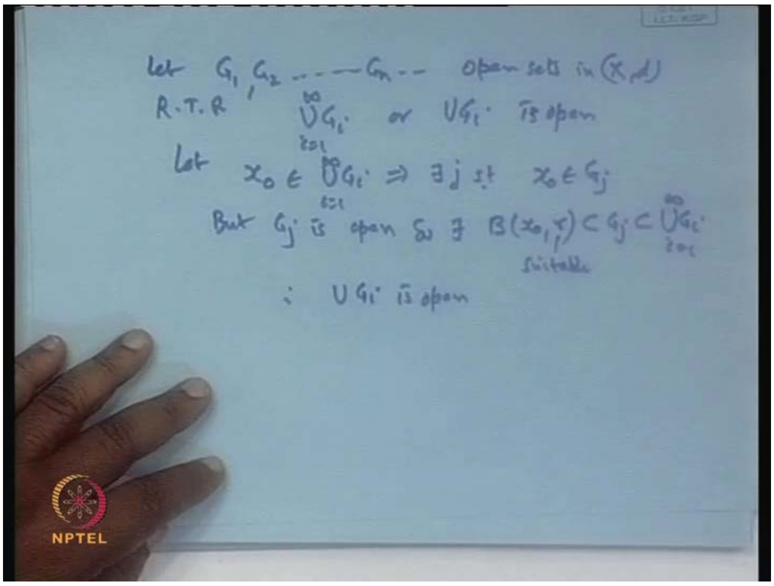

The second part, if suppose G 1, G 2, G n are the are open sets are open sets in a metric space (X,d), then we want it to show that union of this G i - countable union or may be an arbitrary union is open; this we want it to show. So, if I prove that around any point of this set, there is a ball which is totally lies within this set, then we say this collection is an open set. So, let x 0 be an arbitrary point in this collection. So obviously, there will exists a some j such that the point will belongs to this open ball G j, but G j is open. So, by the property of the open sets, there must be a open ball around the point x 0 with a suitable radius which is totally contained in it. So, there exists an open ball centered at x 0 with a suitable radius say r suitable radius such that which is contained in G j. Therefore, it will contain in the union of G i, i is 1 to infinity. So, this shows this is an open set - union of G i is open.

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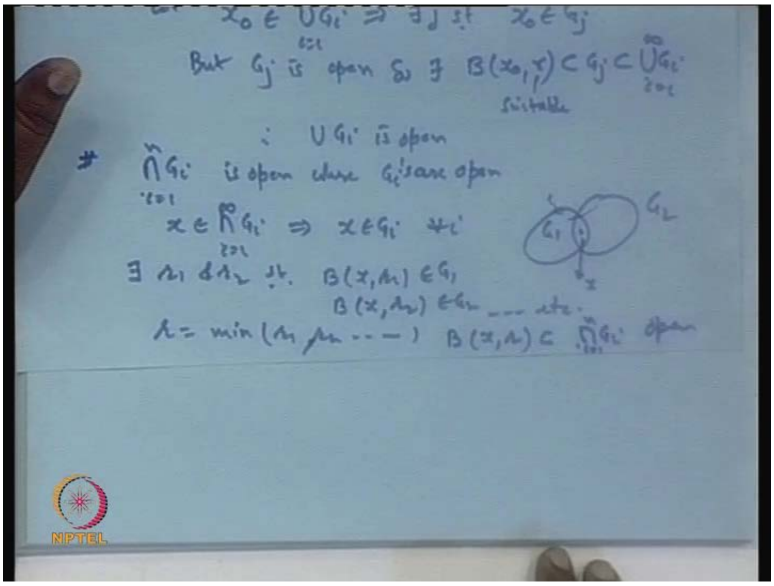

Thus proof of the second, the finite intersection of G i is open, we have G i’s are open. So, again the same treat; let us take a point x belonging to the finite intersection of this, then x must be the point in G i for every i. So, this is say G 1, this is say G 2, like this. So, if we take a point x 0 which x which is available in the intersection part. So, x must be somewhere here, which is a common point of both G 1, G 2 etcetera. So, many things are there so. Now, x belongs to e G i and G i is open. So, there must be a ball around the point each x which is available in G 1 then may be there another ball available in G 2. So, there exist a r 1 and r 2 such that so open ball centered at x and radius r 1 belongs to G 1, open ball centered at x and r 2 belongs to G 2, etcetera. Is it ok?

Now, if I pick up the r as a minimum of r 1, r 2 and so on. And draw the open ball draw the open ball centered at x and radius r, will it not include it inside the countable union of G i sorry finite union of G I, because it will be the smallest ball available in all the G i’s. So, it will. So, G i will be open. Clear? Here we have change the only the finite intersection of the open set is open. I am not taking the countable or any arbitrary intersection, because the countable intersection of an open set need not be open always.

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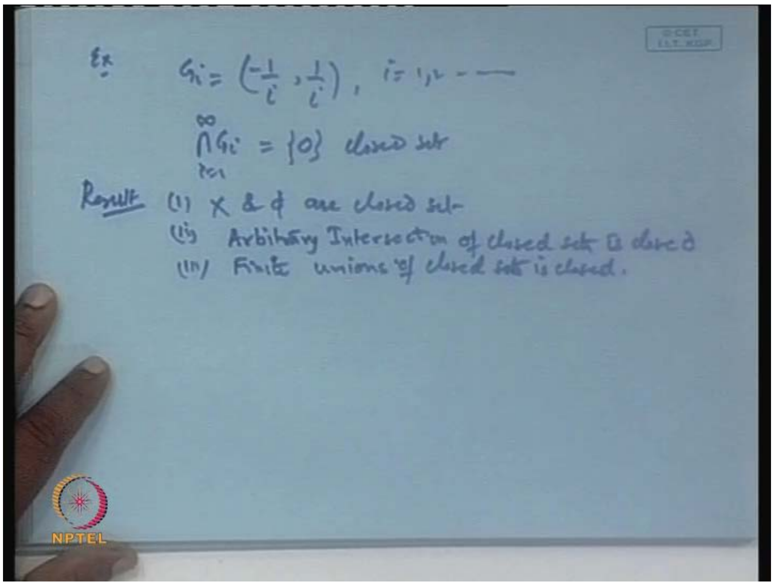

Example is, if we take G i say equal to 1 by i minus sign comma 1 by i where i is 1, 2, 3 and so on. So, this is the an open interval. So, it is an open sets - open ball. Now, if I find the arbitrary intersection of this, then what happens? This is nothing but the set containing only single term point; that is 0; intersection of this. So, a single term set cannot be a open set. We it cannot include any ball around the point it. Therefore, it is a closed set. So, what we conclude is that arbitrary intersection of an open set need not be always open.

The same type of results also available in case of the closed; x and phi are closed; a metric space (X,d); X and phi are closed set. Second intersection of the closed set is closed - any arbitrary intersection of the closed set arbitrarily intersection of closed set sets is closed; while the finite union unions of the closed set of closed sets is closed. So, this we can prove in a similar way, because the compliment of this must be open. Therefore, we can say that these results are basically the compliment of the previous ones. So, I think it is clear or any doubt.

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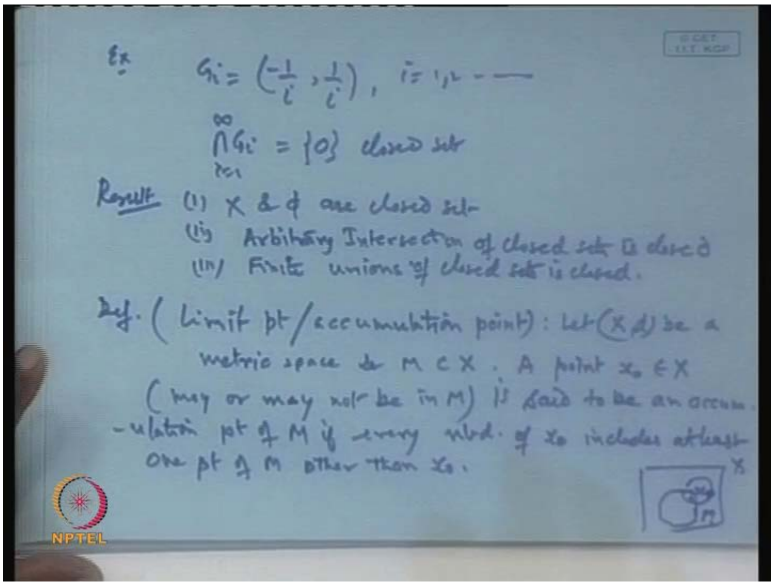

Now, we have another important concept; the concept of the limit point or accumulation point. We define the limit point or accumulation point as limit point or we say accumulation point accumulation point. Let suppose (X,d) be a metric space metric space and M be a subset of X. A point x 0 belonging to X, it may or may not be may or may not be in M is said to be an accumulation point of M; if it said to be an accumulation point of M or the limit point of M, if every neighborhood every neighborhood of x 0 includes at least one point of M other than x 0. Because if suppose x 0 is a point of M and then it includes except x 0, then it will not be considered as a limit point. So, limit point of a set M means if we draw a ball around the point x 0 and M is as such, this is our M. So, whatever the neighborhood you draw around the point x 0, it must include at least one point of M are distinct from x 0, then we say x 0 is a limit point of this. Otherwise, we would not say limit.

So, sometimes even that point if it is not available, then we can it is a boundary point. In fact, the boundary point is the point which has a property that if I draw the ball around the point x 0, then it will include both the points of M as well as the point outside of the M. Then that point is called the boundary point of this. So, this is point. Now, another important concept in this is our dense set dense set.

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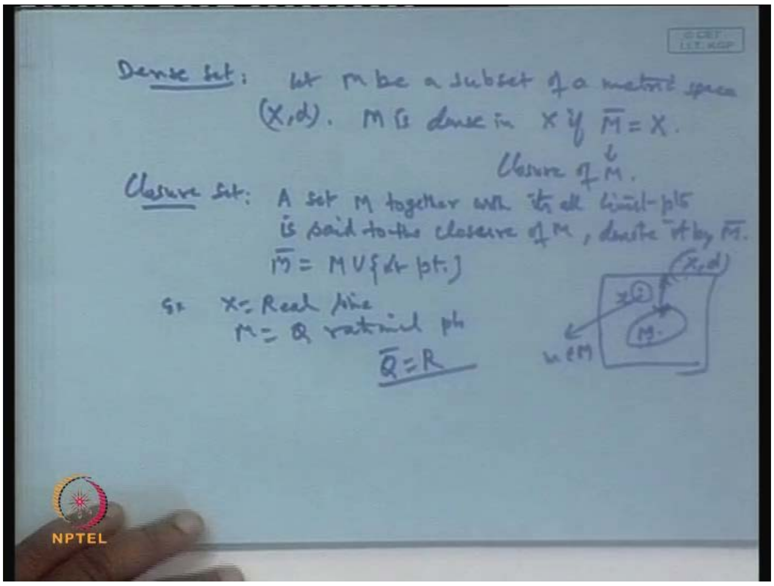

Let us suppose (X,d) be a metric space and M be a subset of M. Let M be a subset of a metric space (X,d); we say M is dense in X, if the closure of M is X. What is the meaning of closure? M bar denotes the closure of M. What is the meaning of closure? Closure set - a set M together with together with it is all limit points limit points. A set M together with this all limit point is said to be the closure of M is said to be the closure of M and denote by we denote it by M bar. So, basically M bar is the M union of all its limit point M union of all its limit point. So, this is the closure.

So, what it says is, a set M is dense in x means closure of this is x. So, if the meaning of this is that if this is our (X,d) space and here is M, clear. If we say closure of M is x, it means if we pickup any arbitrary point x belongs to capital X and draw the ball how is ever a small may be, then it must include the some point of M which is belonging to M. So, it means M and this x they are very close to each other; that is we cannot separate out these two points; whatever the point x you choose and draw a neighborhood, it will definitely include some points of x, then we say closure of this set M is x.

For example, if we take x to be the real line or real number, and say M is the set of all rational numbers - rational points. Then any real number if you pickup and draw the ball around the neighborhood around the point that real number, that is in open interval, it will definitely include the rational points. So, Q bar is R; that is one. Now, we have seen in case of calculus, the function continuous function plays the role. So, here also the concept of the continuity is defined in terms of the a over a metric space as follows.

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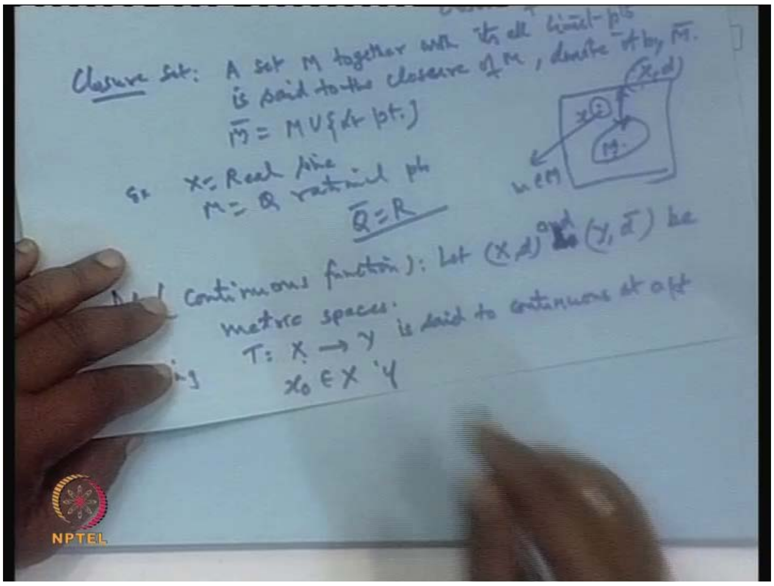

We take the continuity continuous function, let (X,d) and (Y,d) be two metric space. Let (X,d) and Y; let us assume another d bar a metric. Let (X,d) and (Y,d bar) and this is an interval and and Y, d bar be metric spaces, and a T is a mapping T is a mapping from X to Y. We say T is continuous then a mapping T from one metric space to another metric space is said to be continuous at a point at a point x 0 belonging to X if for every epsilon greater than 0, there exist a delta depending on epsilon and the point x 0, such that such that that when d bar (T x,T x 0) is less than epsilon for all x satisfying satisfying the condition that d(x,x 0) less than delta. What is the meaning of this here?

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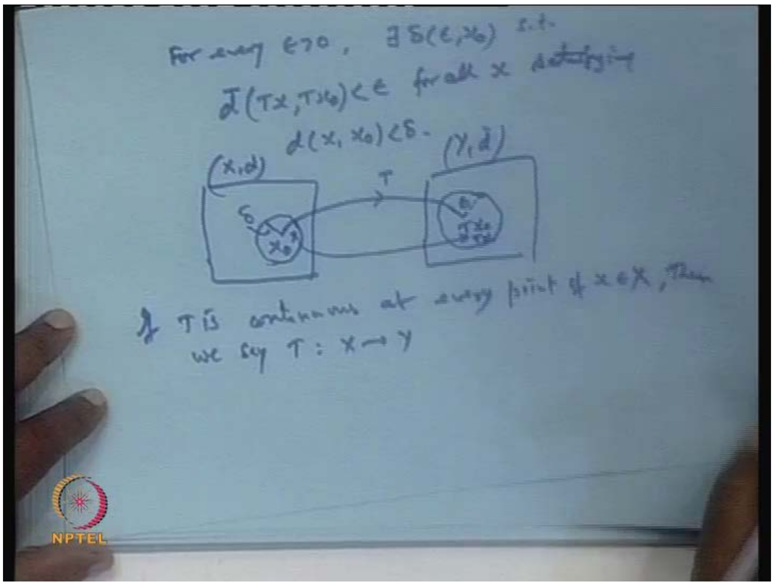

Let us suppose this is our (X,d) metric space, this one is say (Y,d) bar metric space and T is a mapping from X to Y. Then we say this is a continuous at a point x 0, if the image of this is T x 0. So, if we draw a epsilon neighborhood around the point x 0, then corresponding to this neighborhood, one can find a neighborhood around the point x 0 with a radius say delta, such that the image of any arbitrary point inside this neighborhood will fall within these stage. That is the T of x comma T x 0, the distance under d bar will be less than epsilon provided the point x and x 0 satisfy this condition or epsilon chosen then correspondingly you can choose the delta. So that, image of every point inside this neighborhood of the delta neighborhood of x 0 will fall within the epsilon neighborhood of T x 0. Then such a of data, such a function we call it as a continuous function. And if this result is true, if T is continuous at every point of x every point of x belongs to capital X, then we say T is a continuous function from X to Y. T is continuous over the entire range X to Y.

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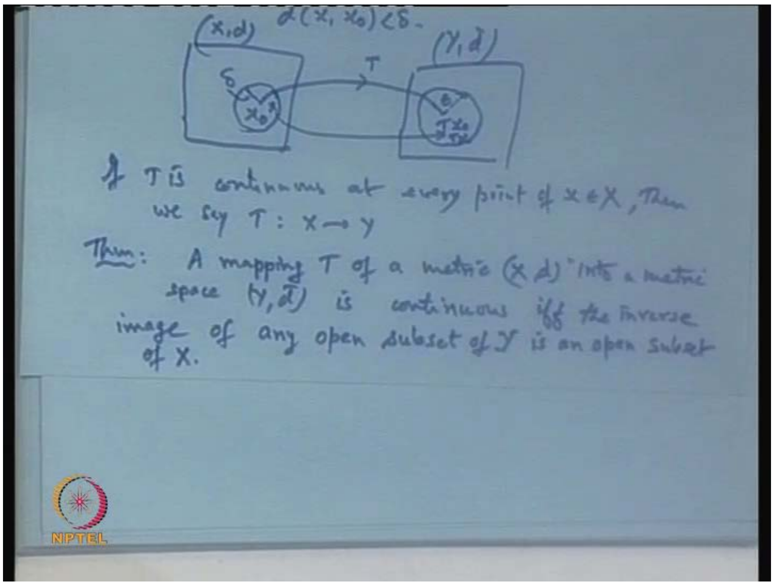

Now, there is a another definition of the continuous function in terms of the open sets and that definition is given in the form of theorem; what this theorem says is a mapping T a mapping T of a metric space x of a metric space (X,d) into a metric space into a metric space say (Y,d bar) is continuous is continuous if and only if; the inverse image the inverse image the inverse image of any open of any open subsets of Y subset of Y is an open subset is an open subset of X.

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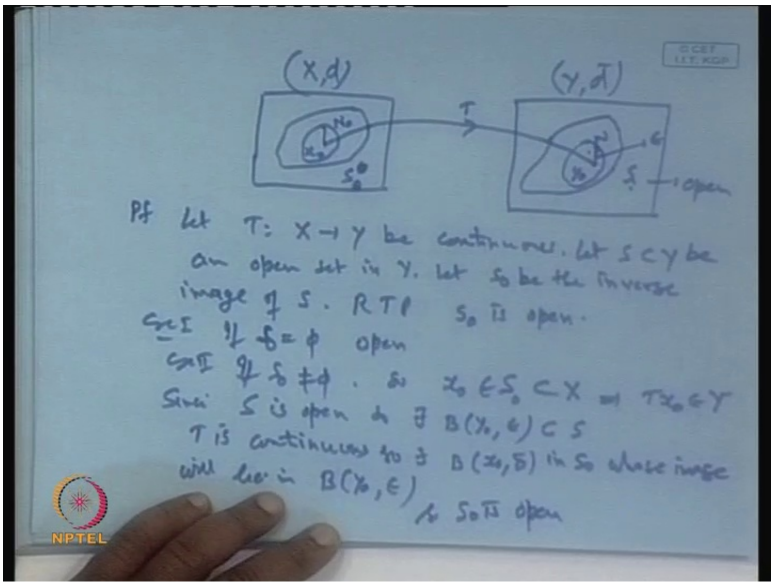

Let us see, what he says is that if T is a mapping from X to Y, X to Y d bar, T is a mapping from this to this, and then this mapping is continuous if the inverse image of any open subset of Y is an open subset of X. So, suppose if I take this S which is an open set and corresponding inverse image, we say S 0, this is. So, what is this says is if the inverse image of this open set is open, then T will be a continuous function, and vice versa, if T is continuous then inverse image of open set will be open set. The proof is like this.

So, let us prove with the one assuming first T is continuous and then we will show the inverse image of an open set is open. So, let T mapping from X to Y be a continuous function be continuous. Now, we want the inverse image of open set is open. So, let us take S which is a subset of Y be a open set in Y open set of Y. Let S 0 be the inverse image of this. Let S 0 be the inverse, here S 0 like this. Let S 0 be the inverse image of S under T, we want it to show S 0 is open - is a open set in this. So, there may be two cases; if S 0 is empty is an empty set, then obviously it is open, because every empty set is open. Second, if S 0 is not empty, then there must be some point. So, x 0 is a point available in S 0. Now, we want this S 0 to be open. So, if I prove that there is a neighborhood around the point x 0, which is totally contain in S 0, then it is open, because x 0 is an arbitrary point I am choosing.

Now, x 0 is a point in S 0 which is a subset of X. T is a mapping from X to Y. So, this implies the T x 0 must be a point of Y. Clear? T x 0 will be point. So, if this is the point say x 0, here is this is say y 0 which is T x 0 is a point in that. Now, S is open S is open, so, around the point y 0 we can get the epsilon neighborhood of y 0 which is totally contain inside it. So, since S is open. So, there exists a neighborhood around the point y 0 with a radius epsilon which is totally contained in S, clear. Now, T is continuous T is continuous, so by definition, if we take any epsilon neighborhood of y 0, then there will exist a delta neighborhood of x 0, say this is our N 0 whose image lies here, but so there exist a delta neighborhood, say B(x 0,delta) in S 0 which is whose image image will be will lie in the neighborhood of B(y 0,epsilon), clear B(y 0,epsilon).

Now, this is our clear. Now, S is contain this neighborhood N is contain in S; N 0 is image of S 0 is S inverse image and there is a neighborhood N 0 which contain in S 0. So, we can say that this x N 0 is contain a S, because S 0 is an open; is a open set, is it not? So, it will come to the open set is this, because the image of this when you take this point T, the image is already we are taking S 0 and there is a neighborhood N 0 which totally contain inside it. So, this inverse image S 0 will be an open set, lies in this. So, S 0 is open. Is it clear or not? Now, conversely in a similar way we can prove. Conversely it is given that inverse image of the open set is open.

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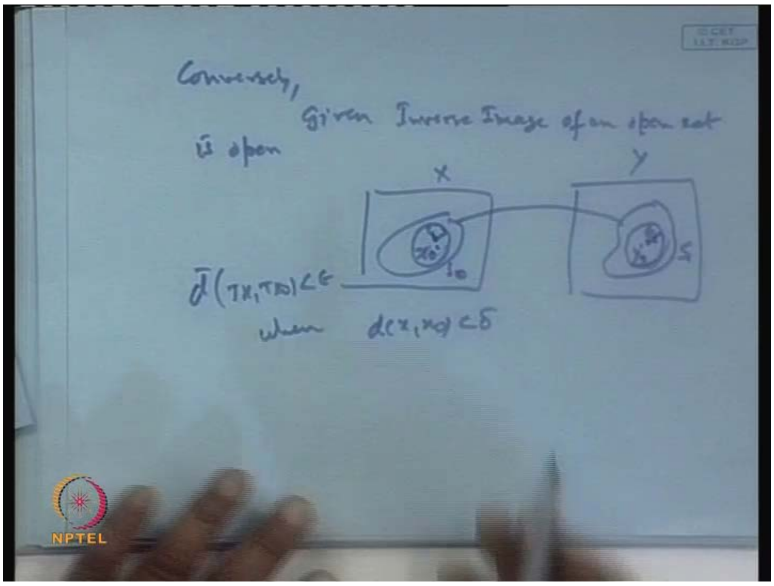

So, conversely given inverse image of an open set known; conversely given inverse image open set inverse image open set is open. This is giving. So, what is known is that if we take S, T is a mapping from X to Y and this is our open set, here is also open set, the T is a mapping, inverse image of the open set is giving to be open. We wanted T to be a continuous. So, what we do is that take any arbitrary epsilon here, x 0 belongs to S 0, the image will be say y 0. Now, if we take any delta neighborhood of this which is totally contained inside it. The image of this will be fall here, because T is giving to be continuous.

So, for a epsilon greater than 0 there exist a delta there exist a delta such that d bar (T x,T x 0) is less than epsilon whenever d(x), S 0 is less than delta. Is it not? That that we want it to prove, which obviously, it follows from here, because if this is open, take a ball which is a epsilon neighborhood of this, an open ball the image of this will be delta neighborhood here, clear. And since S 0 is open, it is already given. So, this is totally contained in it. It means basically we are taking an epsilon neighborhood is given then we can find a delta neighborhood whose image is lying here. So, T becomes continuous.

Is it not? So, that way we can find this here. So, this is very interesting concept of this continuous function and which will be used in a frequently many time.

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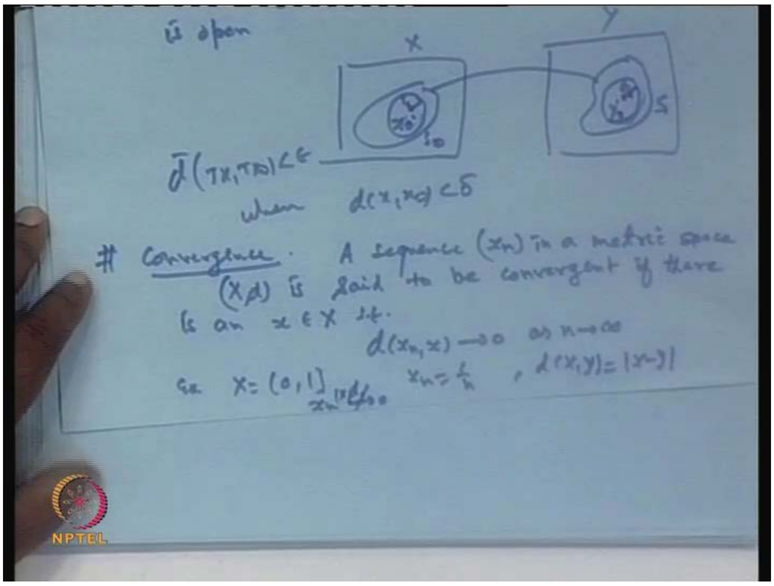

Then next concept which come from convergence part - convergence of a sequence in a metric space; what is the meaning of the convergence of sequence in metric space? Just like a sequence x n, we say in sequence of the real number or complex number, when it convergence. It means it there exists a some point x, converges to x, if for epsilon greater than 0 we can find N, so that, difference between x and minus x goes to 0 when any sufficient is large. So, there is a real line. So, one can identify the point near limiting point here. Same concept we generalize it for here, we say a sequence x n a sequence x n in a metric space in a metric space (X,d) is said to be convergence, if there is there is an x - the point x belonging to capital X such that the distance between x n and X tends to 0 as n tends to infinity as n tends to infinity. It means, there difference between x and a x, if it is a real sequence or in general if it is a arbitrary meet class, then the notion of the distance between x and a x, under the it must go to 0 as n tends to infinity clear. Then we say this sequence is a convergence.

So, existence of x is must. If x does not belong to the class then the sequence will not converge. And for example, if we take x to be 0, 1 and if we take a sequence x n to be 1 by n, and d(x,y) if we define as mod x minus y, then we say this sequence 1 by n though it goes to 0, but because 0 is not available in it, then we say this x n does not converge to zero in this metric space (X,d) in this (X,d). So, the point which is limiting point is must be available in (X,d). That is what we wanted for. So, I think this is ok, then we can go next time what is the concept of this bounded sequence and other. Thank you.