Functional Analysis

Prof. P. D. Srivastava

Department of Mathematics

Indian Institute of Technology, Kharagpur

Module No. # 01

Lecture No. # 29

Total Orthonormal Sets and Sequences

We have discussed this concept of the orthonormal sets, orthogonal sets. So, let us revise what is the orthonormal and orthogonal sets?

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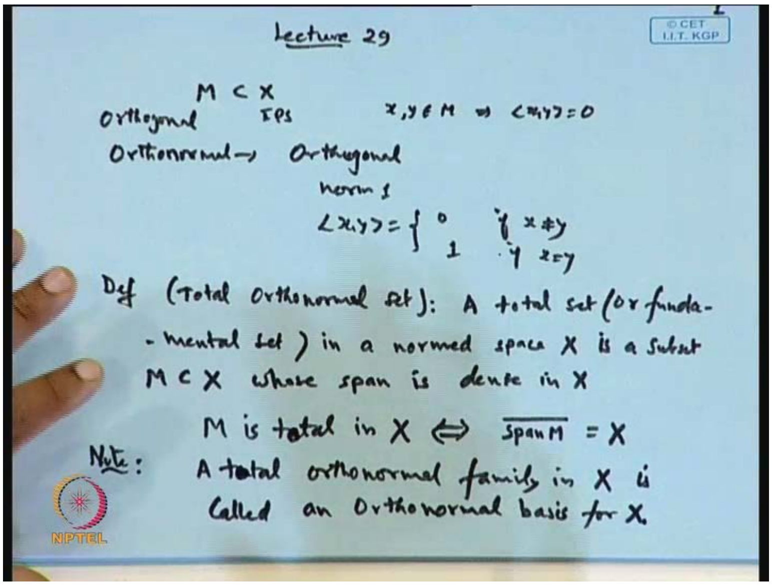

An orthogonal set M in an inner product space X, X is an inner product space and M is an orthogonal set. If an orthogonal set M in an inner product space X is a subset of X, in which the elements are pair wise orthogonal, that is, if we take x and y belongs to M, then the inner product of x y will be 0. And a set is said to be orthogonal, if it is orthonormal, if it is orthogonal as well as the each element has a norm 1. So, orthogonal means the pair, each pair of the elements satisfy this condition, inner product x y is 0. And orthonormal, we mean that it is orthogonal orthogonal as well as the elements, each element has a norm 1.

So, we say the inner product of x y is 0 if x is not equal to y, and 1 if x is equal to y, so such a set is said to be orthonormal sets. And if an orthogonal or orthonormal set is countable, then we can arrange in the form of the sequence, and that we call it as an orthonormal or orthogonal sequence respectively. Now, today we will discuss the concept of total orthonormal sets. The total orthonormal sets is defined as a total set, total orthonormal set, a total set or fundamental set, or fundamental set, fundamental set in an normed space X normed space X is a subset is a subset M of X, whose span is dense in X is dense in X.

So, a set is said to be a total set in a normed space X, if if its span is dense, and it is a subset of X and its span is dense in X. So, we say M is total total in X, if and only if span of M closure of the span of M is X, so that is what it defined. So, M is totally different and this is obvious from the definition, both if and only if part holds good. A total orthonormal sets, a total orthonormal family in an inner product space X is called is called an orthonormal basis for X is called an orthonormal basis for X. Now, this orthonormal basis is not exactly same as the basis in the sense of the algebra unless X is finite dimension.

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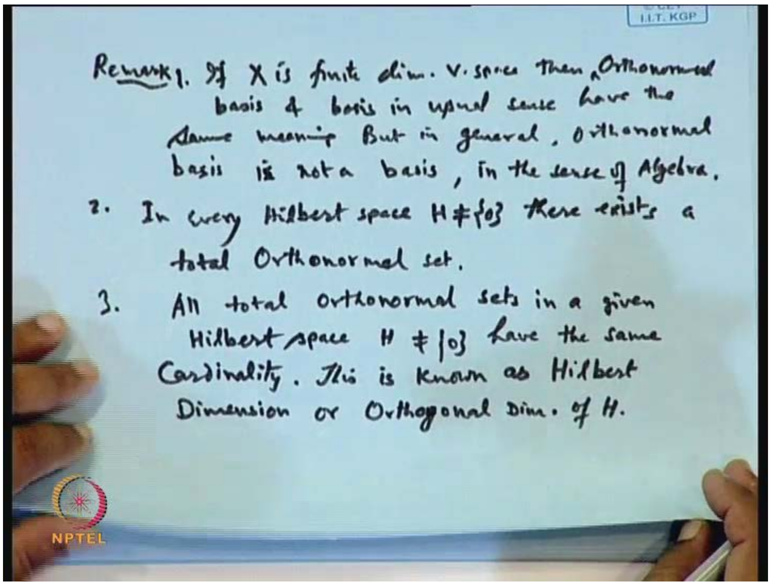

In case of the, if X finite dimension, both orthonormal basis and basis give the same concept. But if X is not finite dimensional, then this will be different concept than the basis of the vector space.

So, as a remark you can say that if X is if X is finite dimensional vector space, then the orthonormal basis orthonormal basis and the basis in usual sense have the same meaning same meaning have the same meaning, same sense. But in case of the infinite, this will be different. But in general, it is not it is not a basis in the sense of algebra, but in general orthonormal basis orthonormal basis is not a basis is not a basis in the sense of algebra, algebra this is remark.

Now, another important point, remark 1 and remark 2, just like every vector space which is different from 0, single transient 0 has a basis. Similarly here, every Hilbert space H in every Hilbert space H, which is different from the single transient 0, there exist there exist a total orthonormal set, there exist a total orthonormal there exist a total orthonormal basis. And this is then total orthonormal basis, orthonormal basis or total orthonormal basis, see total orthonormal normal set, total orthonormal set.

The proof is simple in case of the finite dimensional, it is clear because the finite dimensional, the orthonormal sequence, orthonormal set, we can have it a correspond linearly independent set and its span is also, we can get it as the entire X. But for the other Hilbert space which is separable, the proof follows from this our Gram Schmidt process, it can be Schmidt process, but can convert it to a linearly independent set and get this thing. And for the general case, the proof is using the John's Lemma for non separable case. So, we are not going in detail for that.

Then another remark is, all total orthonormal sets all total orthonormal sets in a given Hilbert space H, in a given Hilbert space H in a given Hilbert space H which is different from single transient 0 have the same cardinality cardinality the. This cardinality is called the Hilbert dimension or the orthogonal dimension of H, the later it is called the same cardinality and this is known as known as the Hilbert dimension Hilbert dimension, this cardinality is known as Hilbert dimension or orthogonal dimension of H. orthogonal dimension of h So, this is these are few things which we will keep.

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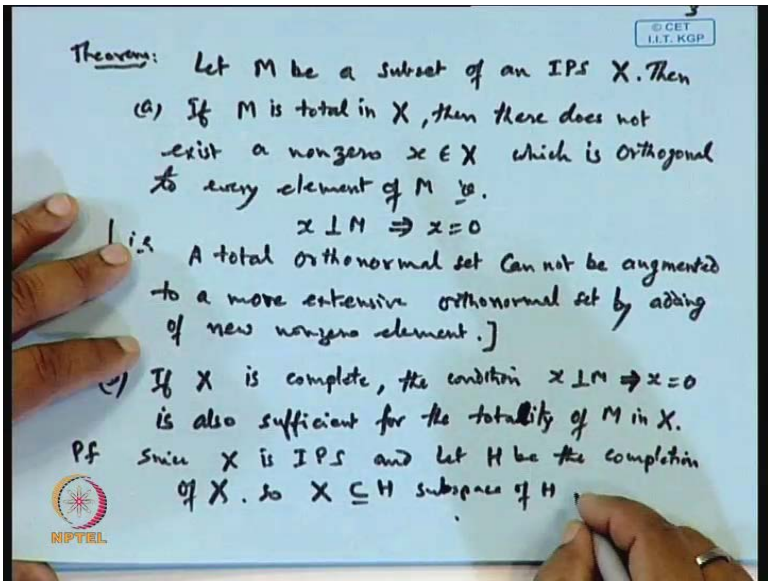

Now, we have, we wanted to extend our orthonormal sequence. So, let us take one results which is which says that the orthonormal set cannot be augmented to a more extensive orthonormal set by a simply adding one element. Suppose, a set is given which is a orthogonal sets or orthonormal sets, and we wanted to enhance this dimension by adding one more elements, then it will not be possible to add a non zero element in it. So, that the entire set with a larger dimension will be orthonormal.

So, the theorem exactly says is, let M be a subset of an inner product space X, I P S inner product space X, then if M is total in X, total in X then then there exist there does not exist does not exist a non zero x belonging to capital X, X which is which is orthogonal orthogonal to every element to every element of M, that is x is orthogonal to M will imply x is 0.

So, the meaning is here, that is hence, we can say that a total Roth a total orthonormal set cannot be augmented augmented to a more extensive to a more extensive orthonormal set by the adjunction, by adding new element by adding by adding of adding of new non zero elements. That if M is a total orthonormal set, then we cannot enhance it to a total orthonormal set, we are having the more element just by introducing one more non zero element because as soon as we introduce one non zero element, then that non zero element cannot be a orthogonal to any, to every element of M, that is it.

The second part of this theorem says, if X is complete, if the inner product if I P S is complete, that is if X has a Hilbert space is complete that that condition is also then the X perpendicular to X implies x equal to 0, then this condition then the condition x orthogonal to M implies X equal to 0, this condition is also sufficient, condition X is perpendicular is also sufficient for totality for the totality totality l i t y double l for a totality of M in X.

Let us see the proof of it, so first we want to show that M is total in X, then there does not exist a non 0 X which is orthogonal to every element of this. So, suppose M is total and x is belonging to M perpendicular, and then we will say x will come out to be 0. So, let us see, now X is an inner product space, since X is an inner product space and every inner product space has its completion, we can convert it or we can have a completion of X.

So, let H be the, and let H be its completion be, the completion of X. I think I have given the concept of completion. If X be a inner product space, then an H be a Hilbert space, then there exist a w, which has a one to one correspondence of n closure of w is h, then we say H is the completion of X. So, that is the meaning of this, and then X regarded as a subspace of H. So, since H is a completion of X, so X X is a subspace of H is a subspace of H and and it is dense in H, and is dense in H by definition dense in H. Because X is a has an isomorphic with the set w, which is subset of H and w (()) X. So, we can say assume that H is also dense in h, X is also dense in X, now so the span of M is dense in M normed.

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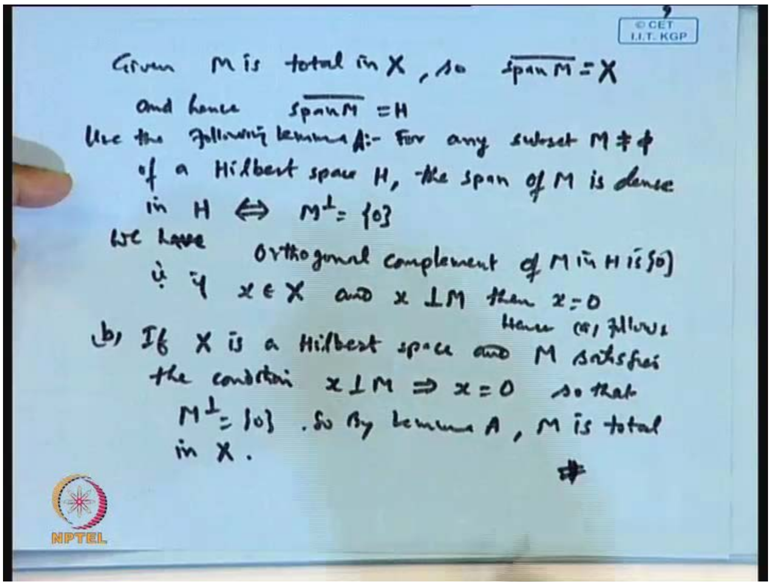

Now, what is given is M is total in X, this is given M is total in X. So, by definition of the total are set, a set is said to be a total or fundamental in a normed space, if the closure of the span is m. So, that is definition shows that since M is total in this.

Therefore, span of M is dense. So, span of M closure is h is X. Hence, in h hence and hence span of M closure is H, because H is the completion of X. Now, there is one lemma which was proved earlier, if not then we can prove right now. The lemma is says, the lemma says now using this lemma, use the following lemma. What the lemma says, for any for any subset M, which is not empty set of a Hilbert space Hilbert space H, for any subset M of H Hilbert space H, the span the span of M is dense in H dense in H if and only if orthogonal complement of M is a single transient 0.

So, this lemma let it be the lemma say A, the proof we will see later on lemma A. We can make use of this lemma and because of this lemma, we say what is lemma says, a subset M of a Hilbert space H is given, then span of M it dense in H, if and only if the orthogonal complement of this is 0. Here M is giving total, by span of this H and H is a Hilbert space H is a Hilbert space. So, this shows that orthogonal complement of m perpendicular is in H is 0.

So, using this lemma we have this lemma, we have have the orthogonal complement complement of m in H is single transient 0 and that is that is if x is an element of capital

X and x is orthogonal perpendicular, x is orthogonal to M, then x must be 0. Because of this here, then the orthogonal complement of H is a singleton 0, it means M perpendicular is 0. So, if x belongs to X and x is perpendicular to M, so according to this x must be 0.

So, what we proved, we stated it, we have stated this M, that M is total in this, then there exist a non zero x which is orthogonal to M, M imply which is orthogonal to every element of M, that is x is a there, do not there, does not exist, it means if there exist an x which is orthogonal to M, then x must be 0. So, we have started with an x belonging to here which is orthogonal to M. Now, according to this, we have shown that total of this M is total in X. So, span of this H, therefore by this lemma, the perpendicular orthogonal complement of M must be single transient and therefore, we get x to be 0. So, this proves the one side.

Sufficient part, let us see the sufficient part. For the sufficient part, that is part b, hence a follows for b, the proof of b if x is a Hilbert space Hilbert space and M satisfy condition and M, x is a Hilbert is a, if X is complete means, X is Hilbert space. And M satisfy the condition, condition that is x is orthogonal to M implies x is 0, x satisfy this condition so that according to this result, according to the lemma, that X, M is total in this so that M perpendicular to singleton 0, so by lemma A, why lemma A? That is, M is total in X and that is proves the results. So, only thing proved in up there lemma. So, let us see the proof of the lemma clear, so this completes the proof.

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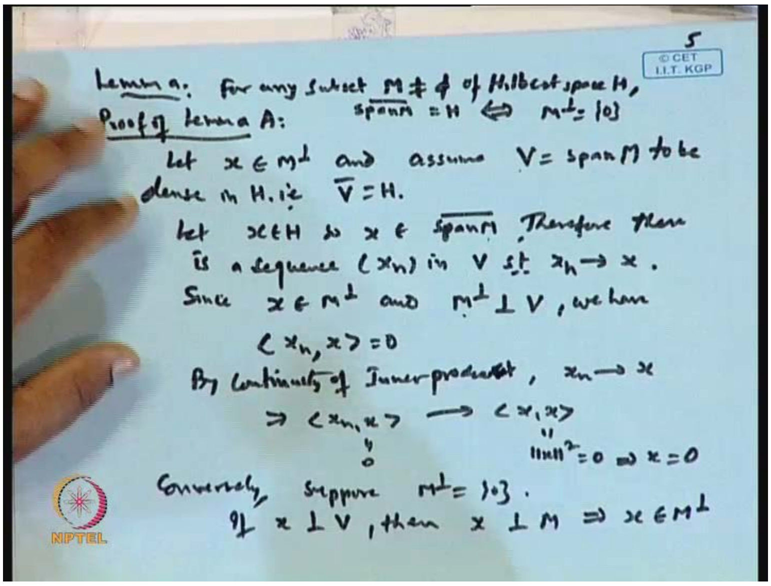

Now, proof of the lemma; now, what was the lemma is, the the lemma was for any subset M of a Hilbert space H, the span of M is dense in H, if and only if a perpendicular to this. So, the statement of the lemma was this.

So, let this was the, should I write lemma, the lemma was for any subset for any subset for any subset M which is non zero, non empty of a Hilbert space H the span of M dense in the span of this closure is H, if and only if M orthogonal is single transient 0. So, proof of this lemma, first is let us see, let x is perpendicular to this, let x belongs to m perpendicular and then we want to show x to be 0, and we assume V is the span of M to be dense in H dense in H that is the closure of V bar is H.

Now, let x belongs to H, so x belongs to the closure of this span M. Therefore, therefore there exist there is a sequence x n in V V such that x n will go to x because x belongs to the span of V M which is V, span of M which is V bar. So, we can take a sequence x n in V which converges to X. Now, since x belongs since x belongs to M perpendicular, therefore an M perpendicular is orthogonal to V because if closure of a span of a V, span of M is V and M perpendicular this will be orthogonal to V.

Therefore, we have x n inner product x n and x must be 0, because x n is an element of V, x is an element in the, x is element of the M perpendicular x is an element of the M perpendicular x belongs to, what is this, that x belongs to H no no this is something x belongs to....

So, x product of this will be inner product will be 0, but by continuity of this inner product, continuity of the inner product product this will give product we say x n converges to x implies, inner product x n comma x will converge to x x. But this is 0, so this part which is the norm of x square should be 0 and this implies x is 0. So, one way it is clear that if x is an element of M perpendicular and span of M is H then x must be 0.

So, this complete one side, the converse of this conversely, conversely supposes M perpendicular, orthogonal complement of M is 0. Now, if x is if x is orthogonal to V, V is the span of M, then x must be orthogonal to M, because V is the span of this so that, so this implies that x will be the element of m perpendicular orthogonal, but M perpendicular is 0. So, this shows that V perpendicular orthogonal complement of V is a single transient 0.

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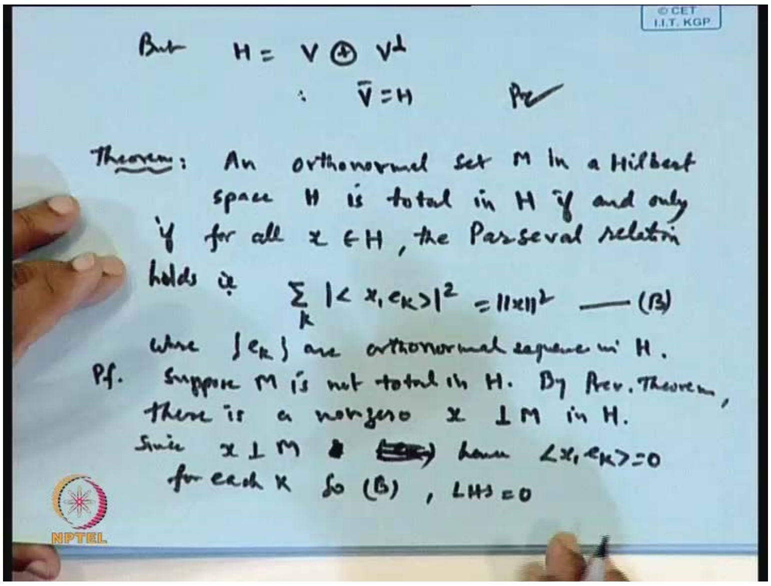

But any element of H, but H can be expressed as, but H can be expressed as V and V perpendicular by projection theorem, therefore, the closure of V is H, therefore this is 0. Therefore, closure of V is H, V is a subspace V closure of H is H and that is proves the results, so this proves the result clear. So, lemma is now, we shall go for this sorry now. So, this part is completion and is also sufficient.

Now, let us come to another result which is given a totality, another way also one can define the totality. So, that theorem an orthonormal set an orthonormal an orthonormal set M in a Hilbert space H H is total in H is total in h if and only if if and only if for all x belongs to H the Parseval Parseval relation Parseval relation holds Parseval relation, holds that is what is the Parseval relation is that is sigma of modulus inner product of e k whole square over k is norm x square, where e k are the sequence of where e k are the sequence of the orthonormal sequences, is it not? e k's are orthonormal sequence in that where e k's are orthonormal sequences in an in a in a Hilbert space H in H clear that is one.

So, now, this is a another way of defining the total set, one way we have introduced the total set as this is a M is total in X, if the closure of this one is H, and this is, this one fourth, this is third, the orthonormal set, this was the definition of the total set, a M is total if the closure of this one is M. And using this thing, we have it is proved one result and the result was that if M be a total in X, then there does not exist a non zero x such that which is orthogonal to every element of M. And if X is complete, then this is also a sufficient condition, now this is another way of defining the total. What is that, another set, an orthonormal set M is Hilbert space is total in H if and only if for all X Parseval relation holds.

Now, this is easy to verify rather than to go for that thing. Now, let us see the proof of this part, proof suppose M is not total in H suppose M is not total in total in H that is what we wanted, is it not? The orthonormal set M is total in H, suppose M is not total in H, then we will prove by a contradiction. Now, if M is not total in H, then by the previous result, there will exist a non zero element X which will be orthogonal to every element of this. So, by the previous theorem by the previous theorem there is a non zero there is a non zero element, non zero X which is orthogonal to M in H, which is orthogonal to every element of M, because it is not a problem.

Now, now now since X is orthogonal in this, therefore the inner product of this will be 0 for all k, since x is orthogonal to M and e k's these are the orthonormal sequences, is it not? Therefore, e k's we are taking to be what orthonormal sequence in M, is it not? And e k's are orthonormal.

So, we at in three, e k's are orthonormal sequence. So, in this result, let it be this will be B; this M is an e k as given is orthonormal. So, we get in this is a ok.So, x is orthonormal set, hence inner product of x e k will be 0, for each k for each k. So, from B, so in B, the left hand side is 0, each term will be 0. But the right hand side is non 0, but what is the right hand side? Right hand side is norm x square different from 0, because x is non zero vector, so this norm cannot be 0.

So, this is 0, this is non zero, a contradiction is this. Therefore, a contradiction is because our own assumption then the M is not total in H. So, a contradiction will help you.

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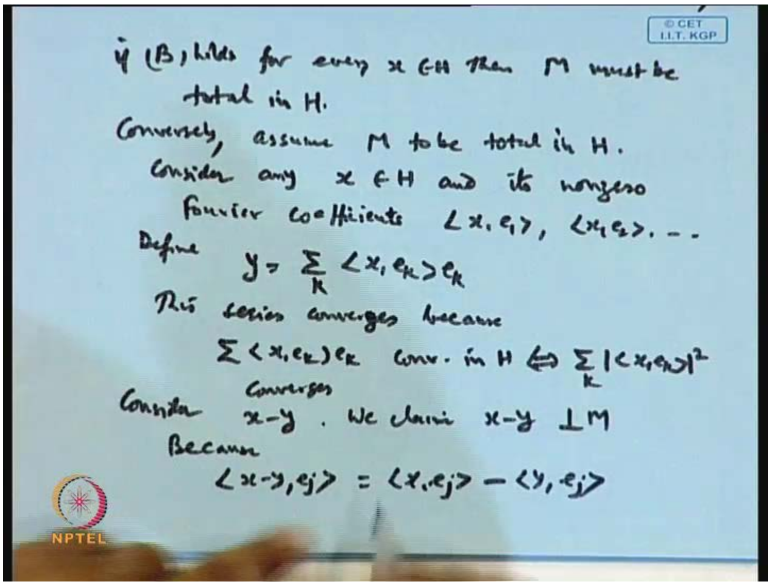

So, a contradiction proofs the proofs that proofs V e proves that Parseval relation holds, that B holds, is it clear or not? For all x, then M must be total that for short it if B holds for every x if b holds for a every x in H, then M must be total in H total in H, so this is simple. Now, conversely what we are assuming is suppose M is total in H then Parseval relation must hold good. So, conversely assume M to be total in H total in H.

Now, again there is a one result which we have not done it, but we will do it here. So, let us we will first write, suppose consider any x consider any x belonging to H and it is non zero Fourier coefficient and its non zero Fourier coefficient coefficients x e 1 x e 2 and so on. Now, Fourier coefficient arrange in an order, now define y as the sum of this inner product x e k e k over k, now convergence of this whole theorem, now this series converges. converges why ah.

Why there is converges, because our this result, because our result is a series of this form will converge if and only if this series converges sigma alpha k square, because because the series sigma x e k e k, this series converges in the norm of H if and only if the series of this term this term converges converges this term converges clear.

Now, it is already given that this series Parseval, this is convergent, is it not? Norm x square we wanted to, by Bessel's inequality sigma of this less than equal to norm x square. So, this series is convergent, therefore this will converge. So, this will given convergence is now, if I consider x minus y. Now, consider x minus y x is an H, why I am choosing this, and we claimed consider x y, now we claim that this vector x minus y is orthogonal to M, why because m e 1 e 2 n all an elements of this.

So, because the inner product of x minus y e j any arbitrary e j, this is equal to what x e j minus y e j, what is y is this series, e 1e 2 n they are the pair wise, they are orthogonal set orthonormal set basically. So, inner product e k e j will be 0 when j is different from k otherwise 1. So, basically you get only one term that is x e j e j and this will come out to be 0, is it or not? x e j minus x e j and that, so x minus y is orthogonal to j for every j. Therefore, x minus y will be perpendicular to M, so this is what M.

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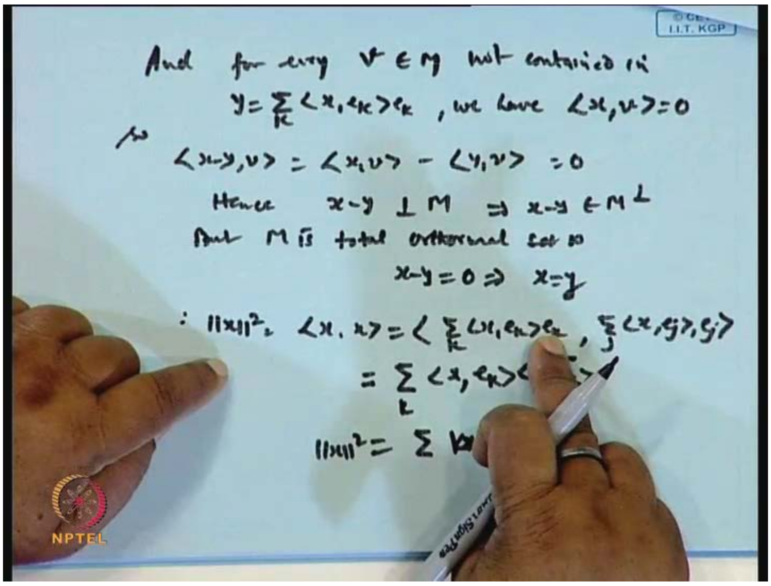

Now, for contain in (()) therefore, let V and for and for every v which is in M, not contained in form not contained in in this in not contained in this form not contained in y which is equal to k means not contained in this elements, we can say not contained, we have inner product of x comma v is 0, why?

Sir, as they are the perpendicular proof that in just last statement.

O i minus.

This is complete.

This x minus y is 0 and everywhere contained in it this is 0 x V will be 0.

That x minus y is V j and they are perpendicular.

Then.

So, from that.

What is our x is in H total arrange; now M is a sequence of orthonormal sets. So, any element we can put it in the form of e 1 e 2 n. Therefore, we can give it the coefficient may be different, why it is if the coefficients is not x e k, if the coefficient is something different say alpha k e k is still that inner product x be will be 0, is it not? Because that can be expressed in the alpha k e j and then particular with x we get 0. So, this will be 0 for this.

Therefore x minus y, so in x minus y comma v, this is equal to x v minus y comma v and this will be 0 again, is it or not? This is also 0, this is also 0. So, X minus y is orthogonal to M, therefore x minus y is an element of the orthogonal complement and there is a one proof which I have given if M is total orthonormal set, then x belongs to M perpendicular implies x is equal to 0.

But M is total total orthonormal set, is it not? So, x minus y will be a 0, therefore we get x equal to y. Now, once you take x equal to y, then the normality are using for and there, x equal to y, here then what will be the norm of this? So, norm of x square will be inner product of x x and inner product of x means sigma k x e k e k comma sigma j x e j e j, is it not? And that will be equal to y completion, this is equal to sigma x e k into x e k conjugate rest will be 0 and that is equal to sigma mode x e k, mod x e k whole square and that is equal to norm x square. So, possible relation holds, is it clear or not?

Now, x equal to y, so we can write the y is this one only, means x we can write in this form of this. So, no basically this starts to write x in terms of this, we have proved this result and this proves the result. So, this is now come to, now if H be a Hilbert space, then such space contradicts dense in H, then we have some results for the separable Hilbert spaces this.

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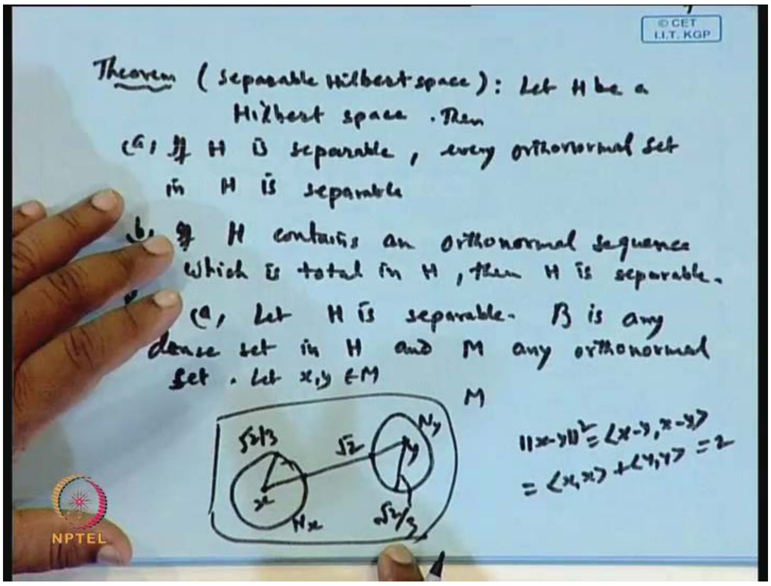

So, what is this result is theorem separable Hilbert spaces. Let H be a Hilbert space Hilbert space, then if H H is separable H is separable, then every orthonormal every orthonormal set every orthonormal set in H is separable and b part is, if H contains an orthonormal sequence orthonormal sequence which is total in H, then H is separable.

Proof: So, part a first, H if H is separable then every orthonormal set in H H is separable. So, let us assume H is separable, so it must contain a countable subset which is dense in its, so y and suppose B is B is any dense sets in H in H, and M any orthonormal set any orthonormal set, this is our M. Now, M is an orthonormal set, so suppose x is an element of M, y is an element of this, so let x and y be the two elements of M.

Now, if we take a distance of this, the norm of x minus y whole square, if we find then this is equal to inner product x minus y x minus y and if we calculate this, then it comes out to the inner product of x x plus inner product y y because x y will be 0, inner product will be 0. Now, x x distance plus y y. This two distinct element of this, a distance under root 2, now this will be 1 norm of x is y, because x is an orthonormal sequence. So, norm of x square is 1 norm of y. So, this is equal to 2.

It means the distance between these two is under root 2 distance between this is under root 2. Now, let us draw the neighbourhood around the point x, around the point y with a radius root 2 by root 2 by 3 and here also this is root 2 by 3, it means this distance we are dividing into three parts.

So, this is one part, this is two part, this is another part. So, they are disjoints; this is N x and this is N y. So, now you have got the two elements of x and a neighbourhood which are disjoint, now B is a dense in H. So, each neighbourhood will have an element of B, will it clear.

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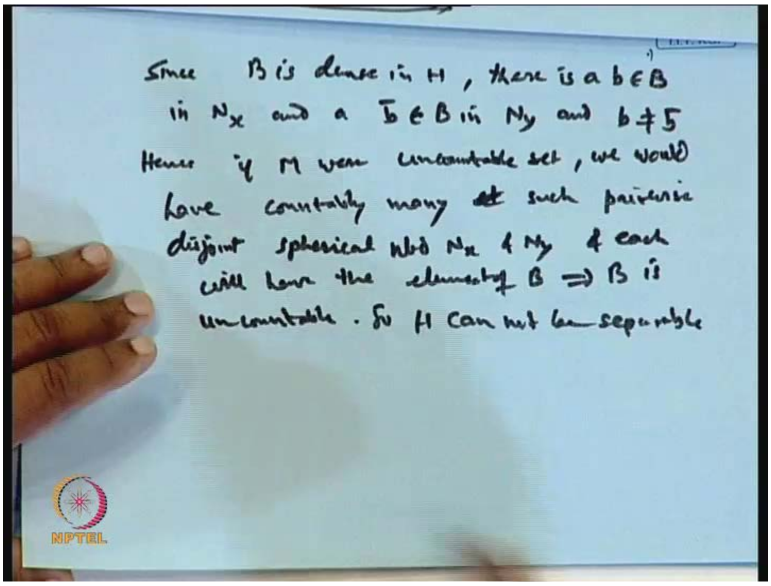

Now, N x N y it is since B is dense in H since b is dense in H. So, there is a b belonging to capital B in N x and c and a b bar belonging to B in N y N y such that b is different from b bar, one element b is here, another element b bar will be here of b clear.

Now, this will be, now since this intersection is empty this intersection is empty since and now what we want to show is that H is separable, we wanted H to be separable. So, it must have a countable subset which is dense in this, so we are assuming by contradiction. Suppose, it is not separable; it means, if set M orthonormal set M is a uncountable set. So, if M is our uncountable then we can find out the countable points where in the neighborhood of this as well as in the neighborhood of this.

So, since hence if M were hence if m were uncountable set, if m were uncountable set then we would have countable many we would have countable many such pair wise such pair wise disjoint disjoint spherical neighborhood N x and N y type respectively, is it not? Fully change and and each will have the elements of B, hence B is uncountable, is it not? Uncountable, hence B will be uncountable, but because B is dense, so that means, H cannot be a h can, so H cannot be separable. So, what we conclude is therefore, contradiction is that therefore, contradiction implies M is countable, that is what we wanted to show.

Let H be a Hilbert space then H is separable, then every orthonormal set is separable, orthonormal M is a orthonormal set, so it must have a countable and dense clear. So, ortho total orthonormal set clear. So, if we have assumed that is a uncountable, so it gives the contradiction H is becomes not separable. So, it means M must be a countable set that is good; second proof we will see next hour. Thank you. I think let us complete it, only one part is left that is all.