Functional Analysis

Prof. P. D. Srivastava

Department of Mathematics

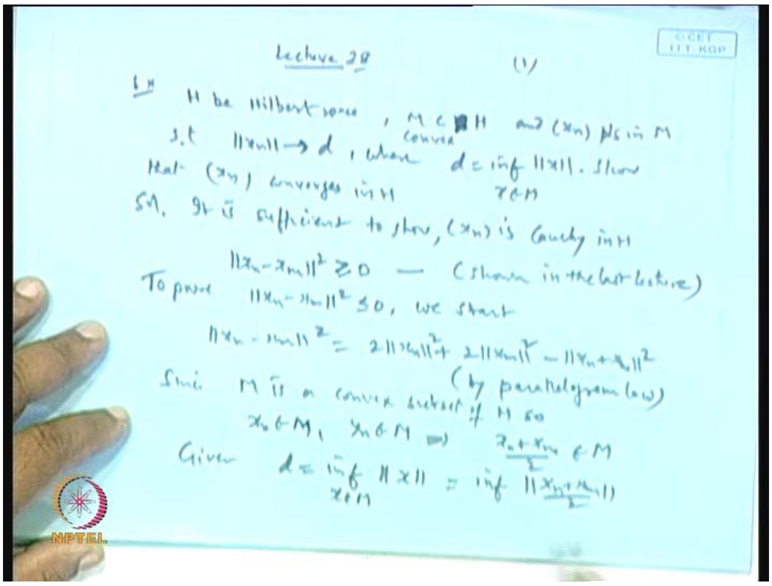
Indian Institute of Technology, Kharagpur

Module No. # 01

Lecture No. # 28

Annihilator in an IPS

(Refer Slide Time: 00:26)



Today class, we have discuss one problem then that problem we have slit the one side inequality, the problem was let H be a Hilbert space, and M be a convex subset of x, H M be a convex subset of H H and x n be a sequence of points in M and x n the sequence of points in M such that, norm of x n goes to d as n tends to infinity, where d is the infimum of norm x, when x belongs to M.

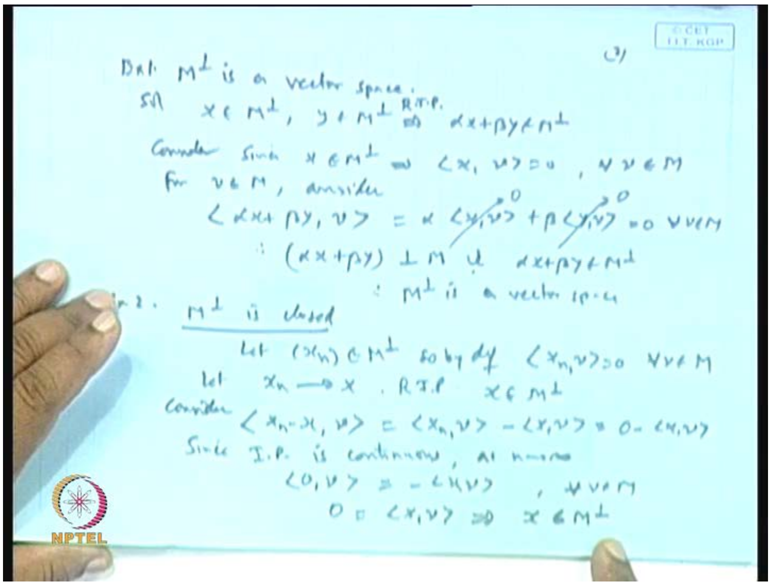
Then, it is required to show that x n converges M, show that x n converges x n converges in H that was the problem, this was the exercise last time. So, what we have discussed is, we have established one equation instead of showing this converges, we it is sufficient to show the sequence x n is Cauchy, is it not?

Once it is Cauchy in H and H is complete, so it must be convergent, because H is complete. So, in to establish this, we have started with x n minus x m whole square and using the, our parallelogram laws we have proved that, this is greater than or equal to 0. This was shown shown in the last lecture.

In fact, we require now the less than equal to sign also, so to prove norm of x n minus x m whole square is less than equal to 0, we start with this again norm of x n minus x m whole square that is equal to 2 times norm x n 2 times norm x m square, x m square minus norm of x n plus x m whole square, this is by parallelogram law, is it not? Law.

Now, in the earlier case, we have not made use of this condition that M is a convex subset of this, M this is a convex subset of H we have not made use of this, so since, M is a convex subset of H. So, by definition of the convex set, if we pick up the two points x n and y n or x and y, then any line segment joining x y must be available inside it, that is, so if x n belongs to M, y n belongs to M, then the point x n plus x m which is the middle point joining the line segments must also be the point of M. Now, this is given d is the infimum of norm of x when x belongs to M, so this is also a point of M. So, it is the infimum of norm of x n plus x m by 2 is it not? when this point belongs to m.

(Refer Slide Time: 04:27)



so if I remove the infimum sign then what we get from here is, so we get the x n plus x m by 2 norm of this is greater than equal to d, because of the infimum, because infimum this is the smallest number d, so this value will be greater than equal to d. Therefore, norm of x n plus x m whole square is greater than equal to 4 d square, so minus times of this is less than equal to minus 4 d square, and then now we substitute it here, substitute in A. So, if we put it in A then what we get here is, x n minus x m whole square this is equal to 2 times x n square 2 times x m square and minus of this means 4 d square and sign will come less, is it clear? So this was the correction, because m is a convex set that is why we are getting this, now as n tends to n m goes to infinity, this is given that norm of x n goes to d norm of x m will also goes to d.

So, when n m tends to infinity this x n minus x m norm of this will go to 0, in fact it is less than equal to 0, is it not?. In fact, this will be less than equal to 0, so this will be x n minus x m whole square is less than equal to 0 for n m greater than equal to capital n and there it was greater than equal to 0. So, basically it will tends to 0 as n m goes to infinity, but norm cannot tends to 0 means norm is 0, so basically x n is a Cauchy sequence in what, x n is a point in M, so it is also point in H. So, Cauchy sequence in H and H is complete, H is complete, so it must be convergent clear, so this was the argument which was left in this that lectures. In fact, one side only we have established other side we have to establish using the convexity of the M clear, it is clear.

Now, let us look some other problem before moving for this, we have also discussed some orthogonal compliment is it not? And this, so we have discussed the orthogonal compliment or a special orthogonal we call it as a annihilator, annihilator. Orthogonal especially is a special annihilator, where the annihilator M perpendicular, annihilator M perpendicular of a set M, which is not empty in an inner product space, in an inner product space capital X is the set x belongs to capital X such that, x is orthogonal to M.

So, orthogonal compliment is a particular case of the annihilator, orthogonal compliment when we go for the projection theorem then we have used the by perpendicular that is orthogonal compliment of y, so annihilator the orthogonal compliment special case of the annihilator. Now, this annihilator the problem is the M perpendicular is a some vector subspace, is a vector subspace is a vector space, why it it was not discussed vector space, first thing is we wanted show this is a vector space.

So, if a linear combination of x belongs to M, M perpendicular y belongs to M perpendicular, if it implies that alpha x plus beta y belongs to M perpendicular then, so required to prove this part M, to show it is a vector space clear. So, consider the inner product of this with the elements of M, since x belongs to M perpendicular, so it implies the inner product of x v is 0 for every v belongs to M similarly, y belongs to perpendicular. So, for v belongs to M consider inner product of alpha x plus beta y v, now this can be written alpha times inner product of this, beta times inner product of this, now x is an element of M perpendicular, so this product will this inner product will be 0, y is an element of the m perpendicular, so this inner product will be 0 totally 0 for every v belongs to M.

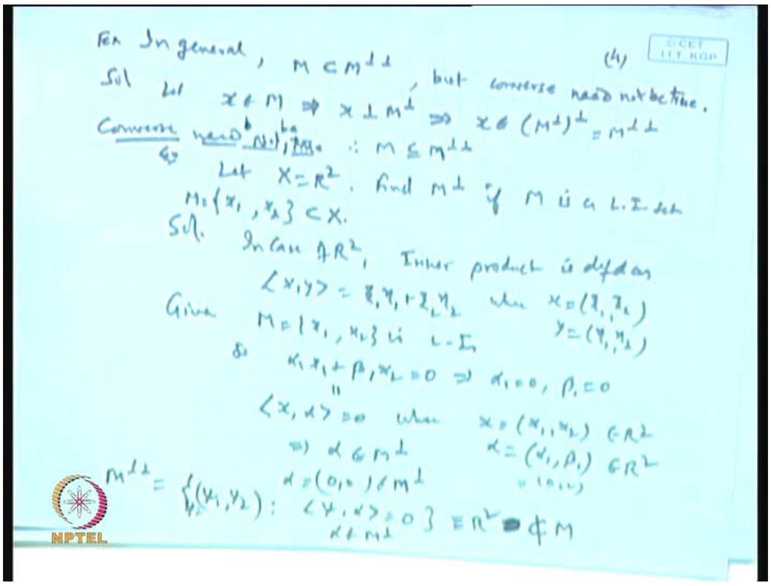
Therefore, alpha x plus beta y will also be perpendicular elements of M that is it belongs to M perpendicular. So, M perpendicular is a vector space clear, and not only this vector space M perpendicular is also closed closed it is a closed subspace of say H, whatever the element belongs to that, why it is closed, it means all the limit points of the sequence of this M will be a point of M perpendicular. So, if we take any sequence x n which is in M perpendicular, the limit point has to be in M perpendicular if it is closed.

So, let us start, let x n be a sequence of the points belongs to M perpendicular. So, by definition the inner product of x n v is 0 for every v belongs to M; let us suppose, let x n converges to x in the inner product defined on M, then it is required to prove that x must be the point of M perpendicular; that is inner product of x be must be 0. So, consider inner product of x n, x x n minus x this can be written as x n, v minus x, v.

Now, since the inner product is a continuous function is it not?, so as n tends to infinity x n converges to x, so this limit will go inside, so the left hand side will be 0 v clear, 0 v and the right hand side yes, before this let us go this can be written as 0 minus x v, why because x n is a point in m perpendicular, so x n v must be 0. So, now, you take the limit as n tends to infinity, so this side will be 0 will be is nothing, but the minus x v, but v is any elements belongs to M when one coordinate 0, so it will be 0, will be x v so this implies x belongs to M perpendicular.

So, if we take a sequence x n in M perpendicular the limit point has to be in M perpendicular therefore, M perpendicular is closed exist, because inner product is a continuous function, so limit can be given inside. When you take the limit of this as a limit will go here and it will be continuous clear; so that is what, I think it is ok. Now, this also suggest one another example.

(Refer Slide Time: 14:03)



In general M is a subset of M perpendicular perpendicular, but converse need not be true, converse need not be true in general M is contained in the annihilator of annihilators that is perpendicular perpendicular, but M perpendicular perpendicular may not be the subset of M always, the solution is this very simple suppose let x belongs to M; it means what do you mean x belongs to M, so x and M they are at right angle M perpendicular, at right angle to each other orthogonal compliments of M.

Once it is orthogonal compare, so it means x must be the point of M perpendicular perpendicular is it not?, again the orthogonal compliments, so that is equal to M perpendicular perpendicular. So, M is a subset of M orthogonal compliment orthogonal, converse need not be true; it means we have to give an example, counter example where converse need not be true. So, let us take an example here, suppose I take say let x is equal to R square, R square find M perpendicular, M perpendicular if if M is a linearly independent set x 1, x 2, x 1 x 2 which is subset of x.

Now, for converse converse need not true, need need not true need not be true need not be true. So, M is a linearly independent set x 1, x 2 this is our M, x 1, x 2 be subset of x. What is R 2, in case of R 2 the inner product is defined as inner product of x, y is is x i 1 eta 1 plus x i 2 eta 2 where x is x i 1 x i 2 y is eta 1 eta 2. So, x is this point x is this point x 1, x 2 this is the point M x is now, if this set is linearly independent given x 1 M which is x 1, x 2 is linearly independent. So, alpha 1 x 1 plus beta alpha 1 or alpha plus beta 1 x

2 is 0 implies alpha 1 is 0, beta 1 is 0. Set of vector is linearly independent, if the scalar alpha 1 x 1 plus alpha 2 x 2 is 0 implies all alphas are 0, so alpha 1 is 0, beta 1 is 0; it means this inner product, this is basically the inner product of what, if can we not write this thing H, the inner product of x, say alpha where is 0, where x is x 1, x 2, alpha is alpha 1, beta 1 this is also in R 2, this is also in R2 clear; it means if x is given we have got an element whose inner product is coming to be 0. So, this implies that alpha must be belongs to M perpendicular is it not?, clear, but what is alpha 1 beta 1 is coming to be 0 0.

So, this is coming to be 0 0, so it means 0 0 is a point in M perpendicular agreed, now if we want, the perpendicular perpendicular, what is the, this set perpendicular? M perpendicular is the set of those points, say I take psi 1, psi 2 such that, inner product of this that is psi inner product of psi alpha is 0 is it not?, because alpha belongs to

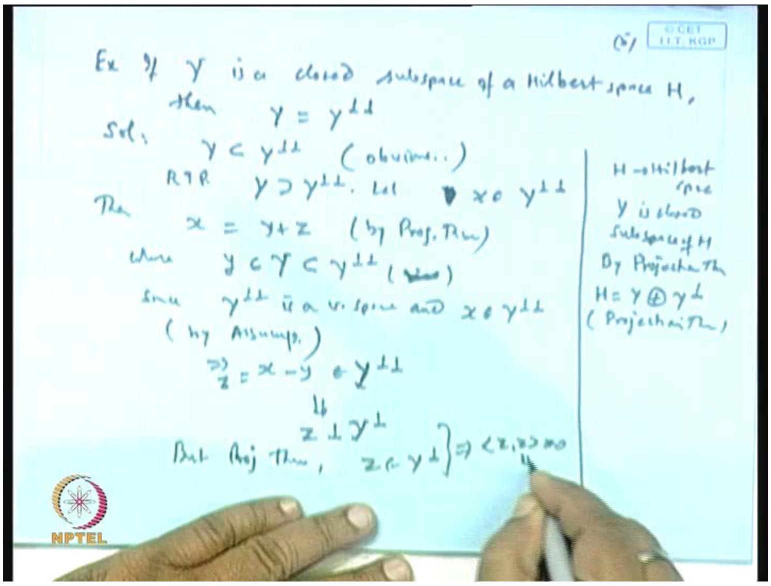
M perpendicular m perpendicular perpendicular, set of those point psi belongs to R 2 whose inner product with this is 0 now alpha is what 0 0, so psi what should be the value of psi 1, psi 2.

Alpha

Alpha alpha is already coming, this is alpha is 0 0, we want the inner product of psi alpha to be 0, so what type of the psi may be, whatever may be it means this is the entire R 2 is it not?, any x i 1 x i 2 will solve our purpose, but this does not contain, this does not contain M, this is not M perpendicular is not the subset of M, is it all correct or not? Because, M is our what, m is the set only linearly independent vectors while it does not contain it may contain any, may be linearly independent dependent, so that is why it is not subset. So, this in general if x is a, M is a set then its orthogonal compliment compliment will cover M, but converse need not be true always, then what should be the restriction on M, so that if the both are equal.

Because, normally we go for just like a dual, when we go set x is dual x prime then second dual x double prime, and when second dual coincide with x, we say it is a reflexive and all these property will comes out. So, here also if it second of the compliment or compliment coincide with M then you get a better structure instead of M, because otherwise every times you have to search for what is the second orthogonal compliment orthogonal then what will be the third and so on.

(Refer Slide Time: 21:44)



So, the restriction is the exercise is this, if y is a closed subspace, if y is a closed subspace of a Hilbert space H, of a Hilbert space H then y is equal to orthogonal compliment compliment, it means it should be the closed closeness is must. Solution, now one way it is obviously true, y is subset of y orthogonal or it is obvious, is it not? or it is true always, there nothing to prove for it, that now in order to prove the y contains y perpendicular perpendicular this is our require to prove.

So, let us take an element y belongs x, let us take an element x belongs to y perpendicular perpendicular, now y is a closed subspace of the Hilbert space H. So, once H is a Hilbert space y is a closed subspace of the Hilbert space H, so by projection theorem by projection theorem H can be expressed as a direct sum of y and y 1 is it not?, clear it means whenever you take any point x belongs to y perpendicular perpendicular it must be a point of H also. So from here, so we can write x as y plus z, by projection theorem, this is by projection theorem is it correct or not.

Yes by projection theorem by projection theorem where what is y, where y, y which is subset of y perpendicular it is given, in fact it is already there, so nothing is, now since the orthogonal compliment annihilators they are vector spaces is it not?, so since y perpendicular perpendicular is a vector space. So, we get from here and x belongs to and x belongs to y perpendicular perpendicular by assumption, this is by assumption is it not? Assumption.

So, x belongs to y perpendicular perpendicular, y is also a y perpendicular perpendicular therefore, it implies x minus y this will be the point in y perpendicular perpendicular, is it or not, but x minus y is nothing but z, is it ok z.

So, this shows this implies z must be orthogonal to y perpendicular is it, because it is belongs to y perpendicular perpendicular, so z should be the orthogonal to y perpendicular, but from this theorem if y belongs to this then z must be the point in y orthogonal, but from projection theorem, projection theorem z should be the point in y perpendicular, because this is belongs to y and this is y perpendicular; so z must be the point in y perpendicular here z is coming to the orthogonal to the y perpendicular. So, combine this thing we say the inner product of z, z is 0 this implies that z is equal to 0 agree.

Once z is 0 then x minus y equal to 0, so this implies x is equal to y, but what is our x, x is a point y perpendicular perpendicular; what is y, y is a point in capital y, so it means this shows that x is a point in y. So, y perpendicular perpendicular, therefore y perpendicular perpendicular is a subset of y, so this proves is it correct or not, so this proves this clear or not, where x is arbitrary, so we can say y is, so this is a interesting result and then another higher.

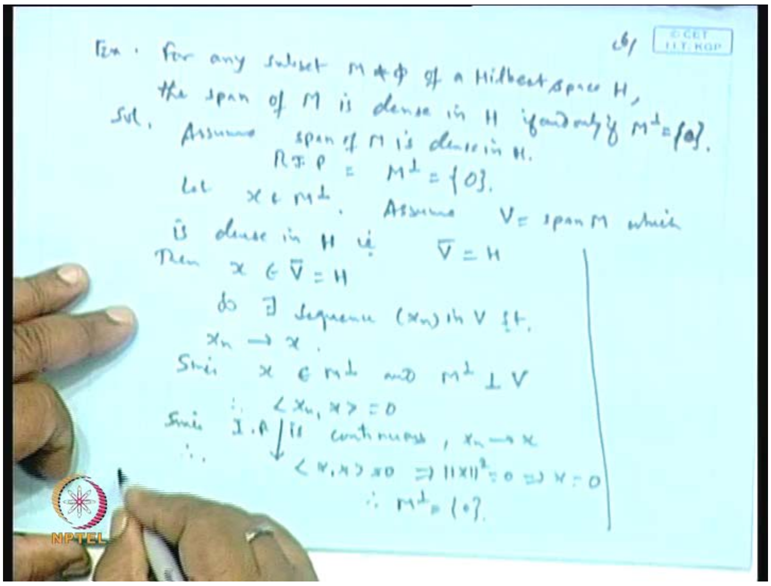
(())

Ah

(())

Annihilators may, because it transfer the point x to the perpendicular means 0, is it not? A null operator also the same, so that is why this word is taken from that otherwise it is a…

(Refer Slide Time: 27:44)



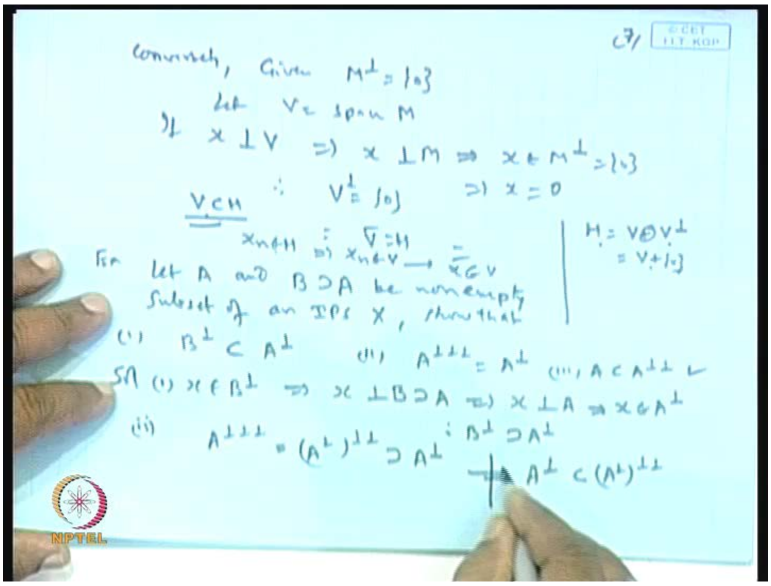
Then another example, that is also interesting for any subset M which is not empty of a Hilbert space, of a Hilbert space, Hilbert space H, the span of M is dense in H if and only if if and only if M perpendicular is the singleton set 0, solution so if an annihilator of a set M comes out to be 0, then the that set a will span the entire class H, that is closure of the span of that M will be H. So, second part of this becomes 0, because H can be expressed sum of the two, one is closed subspace, another one is z, so here, z behaves as a singleton set 0.

So, just I am identifying that annihilator of a set M is 0, it means it is very easy to find out H just find the span of the M and get it, that is what is resulting. So, let us see the solution for this, so first is assume assume that span of M is dense in H span of M is dense in H this is required to prove is the M orthogonal compliment is 0. So, let x be an element of M orthogonal compliment of x, now suppose assume span of this V is the span of M, which is dense in H which is dense in H, that is the closure of V bar is H, that is what is span means closure of V bar is H. Now, if x belongs to M perpendicular, M what is M? M is a subset of the Hilbert space H, M perpendicular is a closed always closed orthogonal compliment is always closed, so by projection theorem x can be expressed as M perpendicular and this something, so it must be the point as H also; so let x, x belongs to then x belongs to V bar which is H V bar belong, then V bar by theorem is it clear or not, this is H.

So, now if a point belongs to the closure of a set, it means there must be a sequence of the points in V which converges to the point x, so there exist, so there exist a sequence x n in a V such that, x n will go to x x n will go to x. Now, since x is a point belonging to M orthogonal compliment of M, and M perpendicular is perpendicular to V, why? X belongs to M perpendicular this is our assumption m, V is the span of this V is the span of M, so any element of V will be the elements linear combination elements of M. So, if any element belongs to the orthogonal compliment will be the orthogonal to M, so that is why the x in an orthogonal it will be orthogonal to the, so M perpendicular will be perpendicular to V, therefore inner product of x n v x will be 0 therefore, inner product x n is 0, but since the inner product is continuous continuous and x n goes to x this is our…

Therefore, it will imply, this will implies the inner product of x x is 0, inner product x x 0 means norm x square is 0 this implies x is 0, so we have assumed that x belongs to M perpendicular. So, this shows M perpendicular is 0. So, one way it is clear x is an arbitrary, therefore M perpendicular as x is arbitrary, so we get this clear.

(Refer Slide Time: 33:26)



Now conversely, suppose given M perpendicular is singleton set 0, and then we have to show that let this span of M is dense in H. So, what we say is, let V is the span of M clear that is what V… Now, x is perpendicular then, if x is perpendicular to V, if x is perpendicular to V then x must be perpendicular to M, because V is the span of M. So, that x belongs to M perpendicular, but M perpendicular is 0, but which is 0, so this implies x is 0, it means therefore, our V perpendicular is 0 V perpendicular is 0. So, V is in the subspace of this and V bar is. Now, let us take this is our H, H can be written as V plus V perpendicular, V perpendicular is coming to be 0, so this will basically V plus 0 is it not?, but the closure of V must must be H all the limits point must be, so this shows closure of V is H, so we get.

(())

Ah

(())

Means any element will come it will be 0 only that is all

And 0 is the common point for v n and v perpendicular both

So, if we take any arbitrary sequence in x n that limit point where it go it will go to the v only.

Yeah.

So, it is a closure will be H you follow me, suppose I take an arbitrary sequence x n in V then limit point of x n either it will be from V in V or may be outside of V, outside of V means it must be V perpendicular, but V perpendicular contains only the singleton set 0 which is also available in V; so only possibility is that it must be the point in V itself, therefore V bar will be equal to H, so this clear, is this clear now. So, this is now let us taking some more examples is it clear or not? Let A is, A and B which is a superset of a B non empty subset, because why is making, from here it will go V is a subset of H is it ok, V is a subset of H any H can be expressed direct sum of this two; now one element is 0, now we wanted any sequence x n belongs to H, then it must be the point in V or V perpendicular, if it is non 0 then it must be the point in V only.

The limiting value of this is x, since it is a closed, so it must be the point in V again x therefore, V closure will be H, what is V, is the span of this M, span of this, so closure of this span will be H, because if you do not take it this one then what happens the points which are limiting point will not belongs any where, if it is not H…

(())

V is subset of H Vis not equal to H V, because V plus 0 is H.

(())

So, V unless you take the, there also in limiting values there you have to justify the limiting value is also here, because V and V closure, these two are different thing here it is coming to be the same.

(())

Yes here it is coming to be the same, so this V subset non empty subset of an, non empty subset of an inner product space X, then show that B perpendicular is subset of A perpendicular. Second is A perpendicular this is obvious of course A perpendicular, and third one which is already prove, we have start A subset of A this is we have already discussed.

So, let us see this one, solution, let us take x belongs to B perpendicular first part, then x belongs to V perpendicular means the x is A perpendicular to B, but B is a subset B is a subset of A, so x is perpendicular to B orthogonal to B means, for every point of B the inner product x B will be 0, so that will include the point of A also, so this shows x is orthogonal to a also; it means x must be the point of, x perpendicular must be the point in a orthogonal compliment, is it not? So, this shows B perpendicular is covers A perpendicular.

Second part is also simpler, what is the A triple means, A perpendicular double, is it not?, now, this one we have seen according to the third a perpendicular perpendicular covers A, so this covers A perpendicular, this covers a perpendicular.

So, A triple perpendicular covers A therefore, what you can say about this A by B this will be B perpendicular subset of a if B is a subset of A, then B perpendicular is contain in, so if we take this this one A perpendicular to perpendicular, then this will contain, this part, so where we want you wanted a perpendicular yes it is clear. This A perpendicular perpendicular covers A and A is covers A perpendicular is it not? contain in this, no covers this is, so this implies this this will be this, this is one side next otherwise how do is, what you do is, you start with this third, what is the third; third one is A perpendicular will covers A perpendicular perpendicular perpendicular, so these two combined, these two will give you both are equal is it not?.

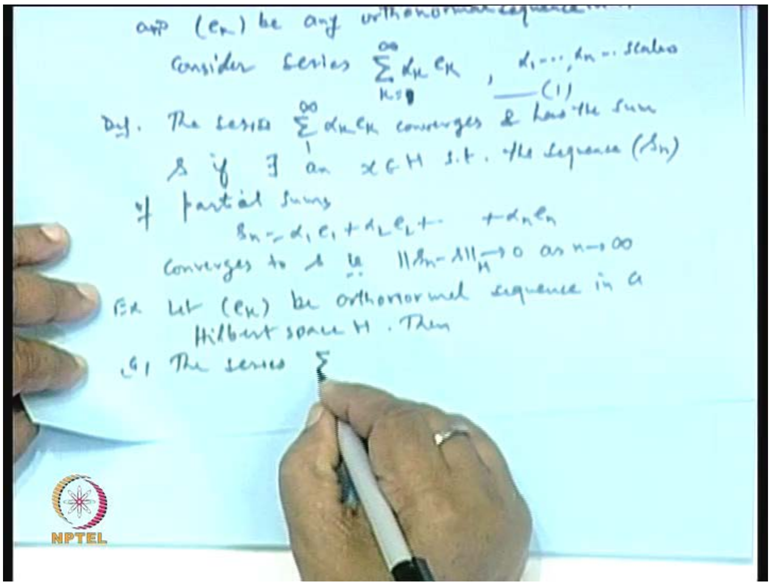
Sir

Um

Checking

No no again I am checking third a subset of this using this result B is superset or when you take the perpendicular the order reverses. So, here take the perpendicular order reverses, so here now these two are different opposite is it not?, so you will get this, so this proves clear then. Now, we get some more things, some more examples let us take from orthogonal series related to orthonormal sequence.

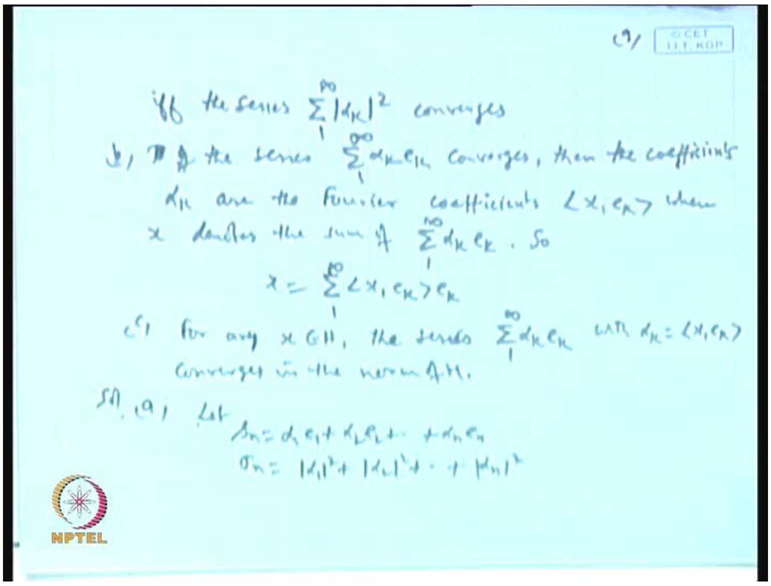
(Refer Slide Time: 42:57)



In fact, yeah this one, so we will discuss something here, before going for the series related to orthonormal sequence. Suppose we have a given orthonormal sequence let I will just get a series related to orthonormal sequence. Series related to orthonormal sequence, then some problem will come here, suppose H be a Hilbert space, let h be a Hilbert space and orthonorm and e k be an and e k be any orthonormal sequence in H, in h. Consider the series, consider series sigma alpha k e k, k is say 0 to infinity or 1 to infinity, 1 to infinity, where alpha 1 alpha 2 to alpha n these are scalars, any scalar real or complex unit.

Now, the series converge we define, we say the series sigma alpha k e k 1 to infinity, converges and has the sum s, if there exist an x belongs to H, such that such that the sequence s n such that the sequence sequence s n of partial sum partial sum, sums that is s n is alpha 1 e 1 plus alpha 2 e 2 plus alpha n e n converges to s, converges to s that is norm of s n minus s goes to 0 as n tends to infinity. Now, this is the norm of H, remember this is the norm of H, because e k are any orthonormal sequence in H, we want this infinite series is said to be convergent, when the sequence of their partial sum goes to a value s, will goes to value s under the norm. Then we say s is the sum of the series and the sequence converges, now the criteria is the exercise, let e k be a orthonormal sequence in Hilbert space H let this series be 1. Let e k be an orthonormal sequence in a Hilbert space H then number a, the series sigma of alpha k e k, k is 1 to infinity converges in the norm of H.

(Refer Slide Time: 47:10)



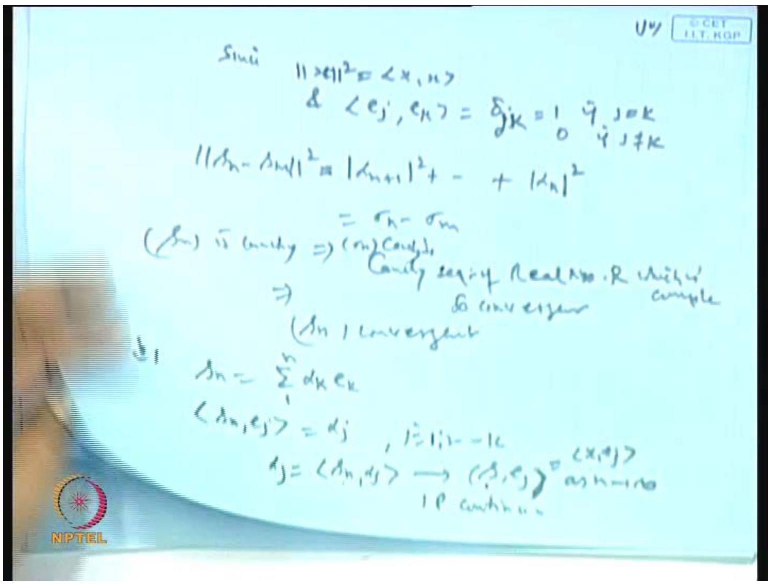
If and only if, if and only if the following series converges, if and only if the series of its scalar that is sigma alpha k square k is 1 to infinity converges, that is what. And second one is B the series converges, if the series sigma alpha k e k 1 to infinity converges, then the coefficients alpha k, coefficients alpha k are the flourier coefficients coefficients x e k where where x denote the sum of s, where x denotes the sum of the series sum of the series, hence in the case six can be written as this. So, x so x can written as, so this series can be written as sigma x e k e k k is 1 to infinity and third one is see for any x for any x belongs to H the series sigma alpha k e k k is 1 to infinity with alpha k equal to inner product x e k, converges in the norm of H. So, let us see the first proof then we will discuss, what is the flourier coefficient, so solution is first a what is given is, let e k b an orthonormal sequence in H and the series sigma alpha k e k converges, in the norm of H, if and only if the following series converges.

So, let us take s n alpha 1 e 1 plus alpha 2 e 2 plus alpha n e n and sigma n, let us correspond to the mod of alpha 1 square plus mod of alpha 2 square plus mod of alpha n square. Now, we want this result that the series this convergent converges in the norm of H, if and only this converges it means sequence of their partial sum, minus s goes to 0 when as n tends to is a convergent or you can say s n minus s m this goes to 0 when n m goes to infinity it becomes Cauchy, and the Cauchy will also Cauchy in alpha k then it is ok.

Because, if the Cauchy sequence of real and complex number it is always be convergent. So, we will find out the relation between s n and sigma n, if I show s n is Cauchy if and only sigma n Cauchy and sigma n is a sequence of this scalars, therefore every Cauchy sequence of a scalar is convergent, so s n will converge.

So, let us consider norm of s n minus s m you follow me, square of this what is norm of s n minus s m square means, norm of alpha m plus 1 e m plus 1 plus alpha n e n whole square, because s n first n terms of the series, s m is the starting with n and then up to m, so s n minus m means starting with m plus n 2 n this.

(Refer Slide Time: 51:53)



Now, norm of square is inner product of x n x n, so since since the norm of x square is the inner product of x, x and e j, e k inner product is delta i k j k that is 1, if j is equal to k and 0 otherwise. So, if I open this part, then what happen e m plus 1 with rest of the thing it will b 0 except 1, so it will give basically norm of s n minus s m whole square, this will come basically mod of alpha m plus 1 whole square up to alpha n square, and that is nothing but, sigma n minus sigma m.

Now, the right hand side series is a Cauchy Cauchy sequence of real or complex real numbers is it not? So it is convergent so convergent therefore, s n is Cauchy if and only if sigma n is Cauchy. Is it correct or not?.

Right hand side is Cauchy then left hand side.

Then right hand this also, now if this is convergent, this is convergent therefore s n will be convergent. So, this proves the result is it correct or not, because this is real number which is complete, so it is convergent therefore s n is convergent nothing to prove.

Now, for the part second, what you have to do is just orthonormality you have to use, what is our s n, part b s n is sigma alpha k e k k is 1 to n, so if I take the inner product of this s n e j what will be is it not? alpha j, j is 1 to k. So, basically you are getting s n just substitute here, so an alpha j this will go to, alpha j which is s n, e j this will go to s e j as n tends to infinity, because inner product is continuous. So, n s n goes to s, so this s j means nothing, but the inner product of x e j, so this again converges.

Second one if I take limit of this, what happened is s n as n tends to infinity is s e j, s is nothing but x, so as n tends to infinity this alpha j will go to this point. So, corresponding series second part that is x will be equal to sigma x k e k will be, and third one is, it converges in this form it follows from the first if the series converges sigma square is convergent, what is this sigma? Sigma of mod inner product of e k square, this is convergent by Cauchy Schwarz inequality no sorry by Bessel's, inequality by Bessel's is it not?, so this will give the thank you now I will stop, is it not? it is fl.