Functional Analysis

Prof. P. D. Srivastava

Department of Mathematics

Indian Institute of Technology, Kharagpur

Module No. # 01

Lecture No. # 26

Self Adjoint, Unitary and Normal Operators

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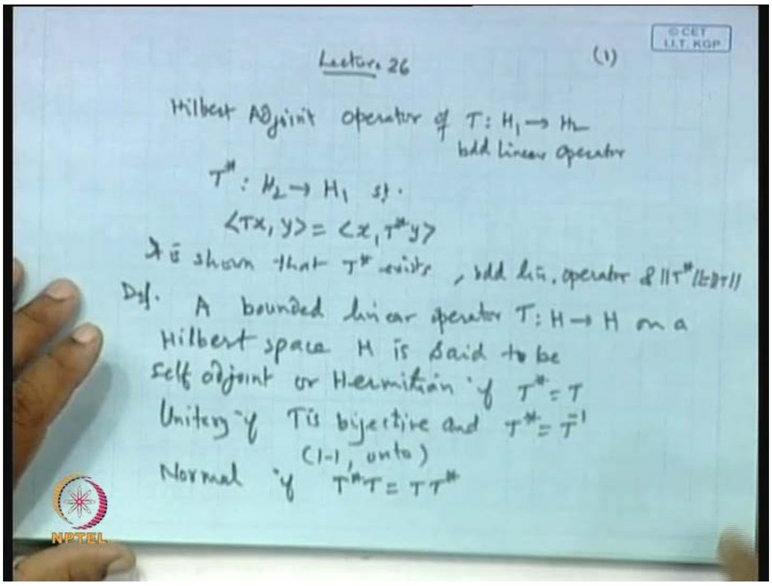

We have discussed the Hilbert Adjoint operator of a bounded linear operator T, of T, which is a bounded linear operator from Hilbert space H to another Hilbert space H 2; H 1 to H 2. And, we have introduced the Hilbert Adjoint operator T star is an operator from H 2 to H 1, such that, the inner product of T x y is the same as x T star y. And, this definition, if T is a bounded linear operator, then, it is shown that, T star will exist, is a bounded linear operator and both will have the same norm. This much we have discussed last time. Now, with the help of this Adjoint, Hilbert Adjoint operator we can classify the bounded linear operators and we can have it further, subclasses for the bounded linear operator, that we call it to be a self Adjoint operator, unitary operator, normal operator and so on so forth.

So, with the help of this Adjoint, Hilbert Adjoint operator, we can define the following for the bounded - the self Adjoint operator, a bounded linear operator T from the Hilbert space H to Hilbert space H, on a Hilbert space H, is said to be, is said to be self Adjoint or Hermitian, if T star is equal to T; that is, Hilbert Adjoint operator of T coincide with the original operator T, then, we say, such an operator is a Hermitian operator or self Adjoint operator. Then, the operator T is said to be unitary, if T is bijective; bijective means, it is 1, 1 and onto. So, injective and surjective both. So, bijective and T star should coincide with T inverse; then, it is said to be normal, if T star T coincide with T T star. And this, we have seen that, every self Adjoint operator is normal operator; unitary operator is normal. So, as a remark, we can, we have seen, in fact, that self Adjoint operators and unitary operators are normal operator, are normal, is it not. Just by definition, if we just apply the definition, you get the condition of the normal operator satisfy; but the converse is not true.

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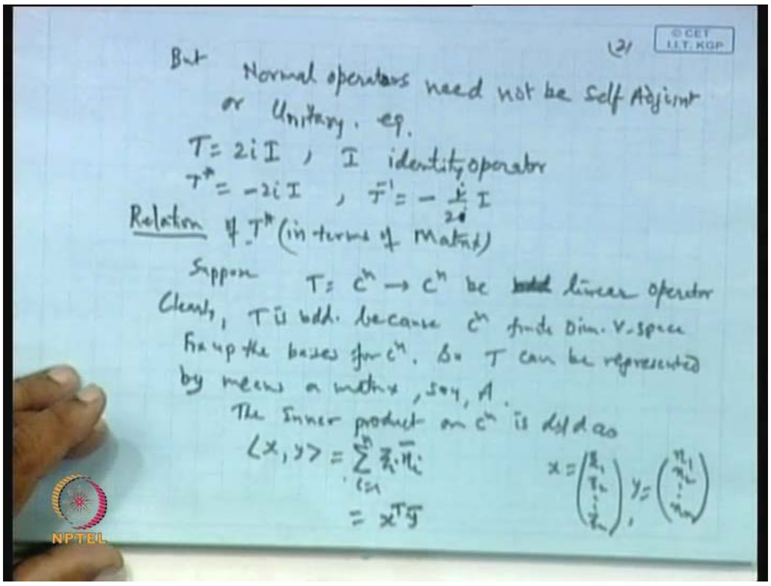

But, normal operators need not be self Adjoint or unitary. And this also, we have seen, for example, we have taken the operator T which is 2 i i, i is the identity operator; and this, we have seen that, this operator T, when we find the Hilbert Adjoint operator, this is, the star means 2 i i and the corresponding inverse will be minus 1 upon 2 i i minus i over 2 i, sorry, this inverse operator will be, product of this should be 1. So, this will be minus 1 upon 2 i up, is it not; then only, you are getting i square is minus 1 and half half gets cancelled and we get. So, T star and T inverse is this and it is easy to verify that, this operator is a normal operator, but not, neither a self Adjoint nor the unitary operator. So, this we… So, up to here we can see.

Now, as we know that, an operator, a linear operator defined on a finite dimensional space can always be represented by means of the matrix. And, if we fix up the basis, the order of the basis element, then, corresponding operator will be unique in nature. So, if suppose be an operator which is a bounded linear operator on a finite dimensional. In fact, every linear operator on a finite dimensional space will be a bounded linear operator, is it not? If T is a linear operator on a finite dimensional space X to Y, then, it must be a bounded linear, because continuity follow and continuity means, boundedness will be there.

So, if T is an operator from, say, one finite dimensional space to the same finite dimensional space, let us say C n to C n, then, it can be represented by means of the matrix. Hence, the question arise, what will be the corresponding form of the Hilbert Adjoint operator of such an operator? Can we express the Hilbert Adjoint operator of this operator T, in terms of the matrix, which may be the conjugate transpose of the previous matrix or something else, clear? So, we wanted to write the relation for a Hilbert Adjoint operator, when T is defined over a finite dimensional Hilbert space.

So, the relation, you can say, relation of T star, in terms of matrix, clear. Now, suppose, T is an operator from C n to C n, be a bounded linear operator. Suppose, we have this operator, which is a bounded linear operator. Since C n is a finite dimensional, n dimensional space, so, it is finite. So, clearly, even if I do not take the bounded, just if I take only linearity, then, clearly, T is bounded; because, because C n is a finite dimensional vector space. So, if an operator defined on finite dimension, it will be a bounded, linear operator will be bounded. So, it is a bounded linear operator. Let us fix the basis, fix up the basis for C n. So, it can be, so, T can be represented by means of a matrix, say, A, is it clear, by means of the matrix A. Now, C n is a n dimensional, where the inner product on C n, on C n, is defined as inner product of x y, equal to what?

(( ))

If x is, say, x i 1, x i 2, x i n; y is, say, eta 1, eta 2, eta n, then, the inner product, basically, is defined to be, sigma x i i eta i bar, i is 1 to n, and in fact, this is nothing, but what? This will be, in vector form if I write, it is the x transpose y bar, is it not? x transpose y bar. This is our way of defining the inner product. So, x and y are there. Therefore, if I take the T x, So, T is x.

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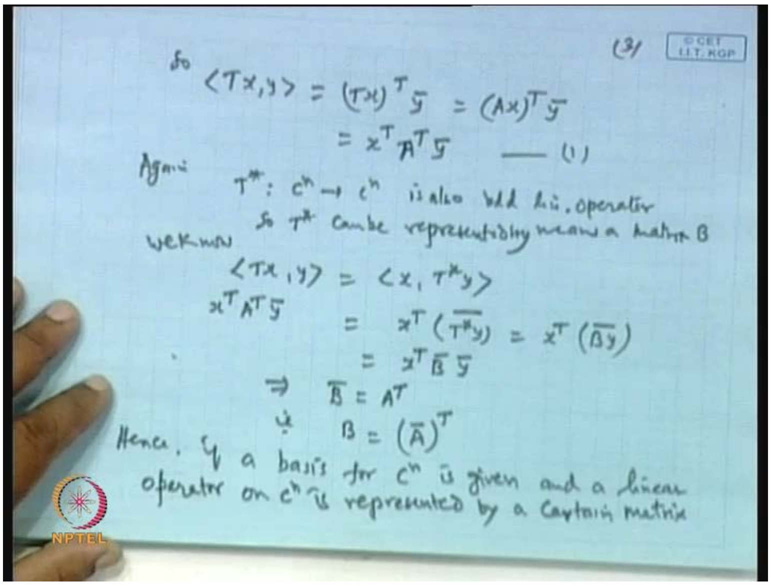

So, T x y, inner product of this, clear. T x y will be what? x is, x is replaced x transpose. So, T x will be T x transpose and then, y bar, is it not; but A B transpose is the B transpose A transpose. So, from here, we get x transpose T transpose, T transpose, or T, sorry, T is defined by A x, T is defined by A x. So, I will write, this is A x transpose y bar. T is, because T is a operator on a finite dimensional; it will be represented by the matrix A. So, it will be A transpose into y bar, is this clear or not. So, we are getting this one. Let it be 1. Now, again, T star is an operator from C n to C n, reverse page side. So, it is also, also, bounded linear operator, by definition; and since, it is finite dimensional, so, T star can be represented by means of a matrix, say B, by means of some matrix, say B.

So, it will be T star of x. So, what will be this, x T x y. We know the T x y is equal to x T star y. Now, T x y, we have already defined as x transpose A T y bar. And here, this will be what, transpose of this into T star y conjugate, by definition; but T x transpose T star y is what, is the B y conjugate; because T star is defined by a matrix B. So, it will be equal to B bar y bar. So, if we compare these two, this implies that, B bar comes out to be the A transpose. It means, that is, the matrix B is coming to be the conjugate of transpose of A, clear. So, what we get it, if T is represented by a matrix A, then, corresponding Hilbert Adjoint operator is represented by the conjugate transpose of this matrix. So, we conclude, hence, if a basis for C n is given and a linear operator and a linear operator on C n is represented by a certain matrix, then, its Hilbert Adjoint operator T star, is represented by, is represented by the complex conjugate, conjugate transpose of that matrix, is it ok.

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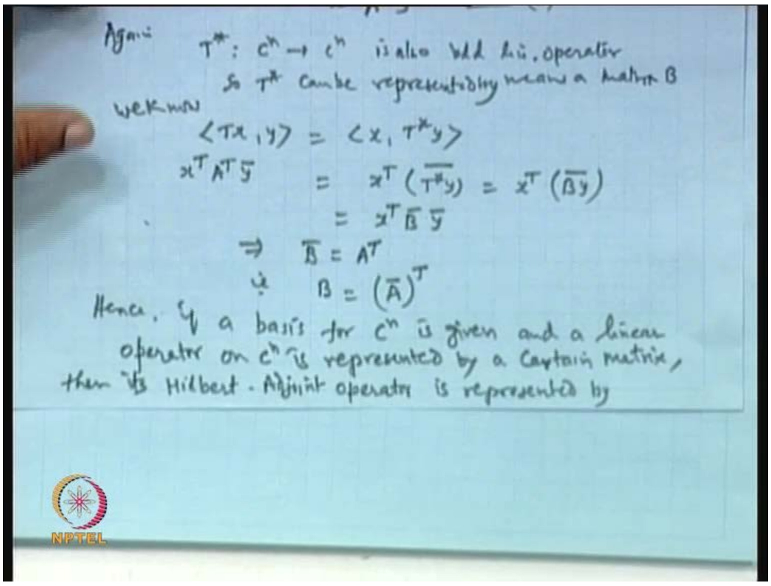

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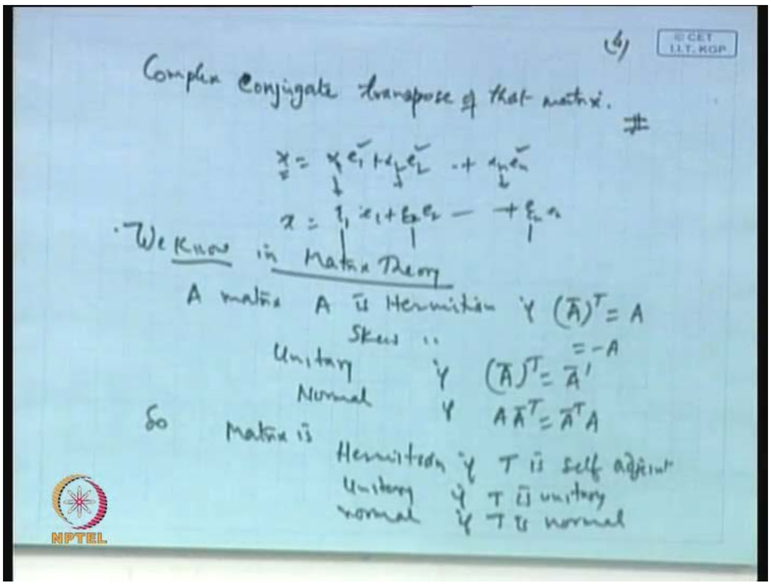

So, this is what we get it, representation of a Hilbert Adjoint operator, in terms of the matrix, clear. (( )) previously…

Sir, we know that, if we fix the basis, then, always we can represent some matrix.

Yes.

But, if you take the sender basis, then, the, any vector and the ((conjugate)) vector will be the same. So, in that case, directly any vector should be represented by (( )).

No, when we fix the basis, it means, it does not mean that, you are taking the particular basis, no. You are, basis, the elements of the basis, order is fixed. When you are taking the basis, it means, the order of that elements are fixed. So, throughout the calculation, you remain, keep that basis itself. So, every point x can be represented in terms of that basis and representation will be unique in that way.

(( ))

Slide is…

(( ))

What is your point? Achcha ok. So, what is the...

My point is, if you take the any basis, then the representation is something (( )) the coordinate of y, equal to that vector, matrix representation and the coordinate of x, coordinate vector. And here, we are taking x and y is not coordinate vector, just any vector.

No, no, it is not as, x itself is represented, when you say x equal to x i 1, x i 2, x i n, it means, x is represented in the form of… say, I…This is a… Suppose, I take any vector, 1 0 0, a, an element, it will be represent in terms of the basis element. What is this basis element? x is e 1, say alpha 1, alpha 2 e 2, alpha n e n. When these are fixed, it means, when x is given, then, this coordinates, alpha 1, alpha 2, alpha n cannot take any other value except only one. So, it means this x i 1 e 1, x i 2 e 2, these are the, x i n e n, this will be the representation of x. So, x i 1, x i 2, x i n are fixed and that is why we are taking in terms of x i.

Here x is also the coordinate, coordinate vector.

X is the vector, a coordinate vector and because, this coordinates are determined because of the basis is fixed.

Because order is fixed.

Order is fixed. So, that is why, clear. So, I think it is ok or...

Now, let us see the relation again, further. We know, in case of the general matrix or in matrix theory, if you remember, this is ok. Now, we know in the matrix theory, we say, a matrix A is Hermitian, if the conjugate transpose of this matrix coincide with A, is it not. And, skew Hermitian, with minus sign, if this is minus A. And, we say it is a unitary, if, if conjugate transpose of this equal to A inverse. And, we say it is normal, if A A bar transpose equal to A bar transpose A, is it not.

Now, what is our definition, if you remember? We have defined the self Adjoint operators, one more sheet, where is it? How did you define the self Adjoint? This is the way. Here, we have defined T star is equal to T. What is T star? T star was coming to be the conjugate, complex conjugate transpose of this matrix. So, is it not a T star and this is T. So, we can say a matrix is Hermitian, when T star is equal to T; and that is why, we have told self Adjoint or Hermitian, clear. Similarly, for the unitary we can say. So, we can say from here is that, a matrix… So, we can say, matrix are Hermitian, Hermitian, if the corresponding operator T, if corresponding operator T is self Adjoint, is it ok; and we say, it is unitary, if the corresponding operator T is unitary; and it is normal, if the corresponding operator T is normal, is it ok. So, we can relate the theory of matrix with the operators clear. Now, we have few properties of the self Adjoint operator in the form of result.

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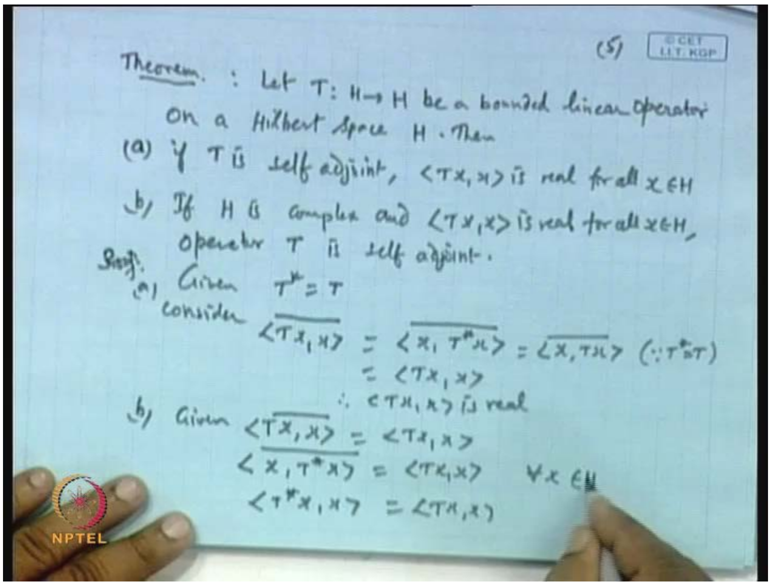

So, first theorem, that is on the self Adjointness. If let, T from H to H be a bounded linear operator, linear operator on a Hilbert space, on a Hilbert space H, then, if T is self Adjoint, self Adjoint, then, the inner product T x x is real, for all x belonging to H. And, second is, if H is complex, is complex and inner product T x x is real, for all x belonging to H, then, the operator T, operator T is self Adjoint; operator T self Adjoint. Solution or proof. What is given is, T is a bounded linear operator and T is self Adjoint. So, given T star is equal to T.

We want this to be real. So, consider inner product of T x x conjugate. If I prove, the conjugate of this is the inner product itself, then, it will be real. Now, this will be equal to inner product x T star x conjugate, is it not? And, this will be equal to, T is given to be self Adjoint; so, this is the same as x T x conjugate, because T star is equal to T, is it ok or not. Now, again, interchange the order. So, when you interchange the order, it is the same as T x x. So, this shows, the inner product T x comma x is real; nothing too much. For the part b, given H is complex. So, it means, the Hilbert space H should be a complex value; means, field of this scalar, should be a complex number. For real, it may not be, means, may not be true; we cannot get the conclusion. So, H has to be complex here. And, T x x real for all x, then, operator T is self Adjoint. So, this is given, T x x is real. What is given? Inner product T x x is real; it means, conjugate of this is the…But this, is the same as x T star x conjugate is the same as T x x; but this is the same as, T star x comma x is T x x; this is true for all x belonging to H.

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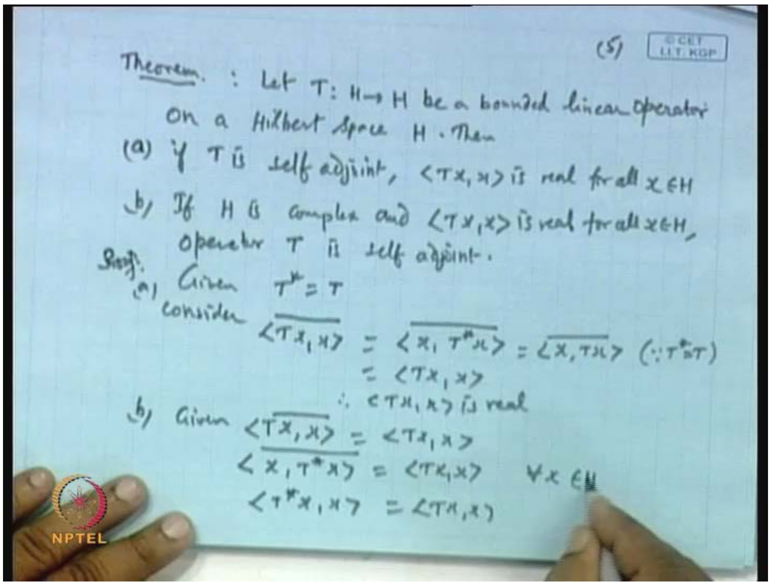

Now, from here, we can write it, inner product of T star minus T x comma x is 0 for all x belonging to H. Now, if q x x is 0, only when, implies q is 0, only when, H is a complex space; otherwise, in case of the real, you have seen that, q x x is 0, in spite of q will not be 0; because, if you take the rotation operator, then, it can rotate the x through the 90 degree and the inner product comes out to be 0, clear. But if H is a complex inner product space, then, in that case, this q x x is 0 for all x will imply, that q must be 0; that is, this is a 0 operator. So, this shows, T is self Adjoint. Is it ok? Otherwise, it cannot, clear. So, this two…

Now, another property, for this composite. The product of the two bounded, the product of two bounded self Adjoint, self Adjoint linear operators S and T, on a Hilbert space H, is self Adjoint, if and only if, the operators, the operators commute, commutes; that is, if S T equal to T S, then, the product S T will be self Adjoint and vice-versa, if product S T self Adjoint, then, S and T will come here. So, what is given is, given S star is equal to S, T star is equal to T, because of the self Adjoint, is it not. Both are given to be self Adjoints. Now, what we want is, product S T star should be self Adjoint. So, S T star, consider this.

Now, if you remember, yes, when you take the product UV star, it becomes V star U star. So, this will be equal to T star S star. So, start with this, then, this comes out to this, which is equal to T S. Now, if we assume the T S equal to S T, then, S T star becomes S

T. So, if we assume, this is equal to S T, this is given, T S equal to S T, then, it will imply that, S T star equal to S T, is it not; and, that will give you the solution that, S T is, is self Adjoint, is it ok. Now, let us see the converse space; suppose S T is self Adjoint. Then, S T must be equal to S T star. S T star means, we are getting from here, this side, S T star means T star S star and this is equal to S T. So, T S equal to S T and that will imply, S T. So, vice-versa, this two, clear. So, this proves the result, is it ok or not? So, we can say, S T star equal to S T, if and only if, S T equal to T S. So, this completes the proof.

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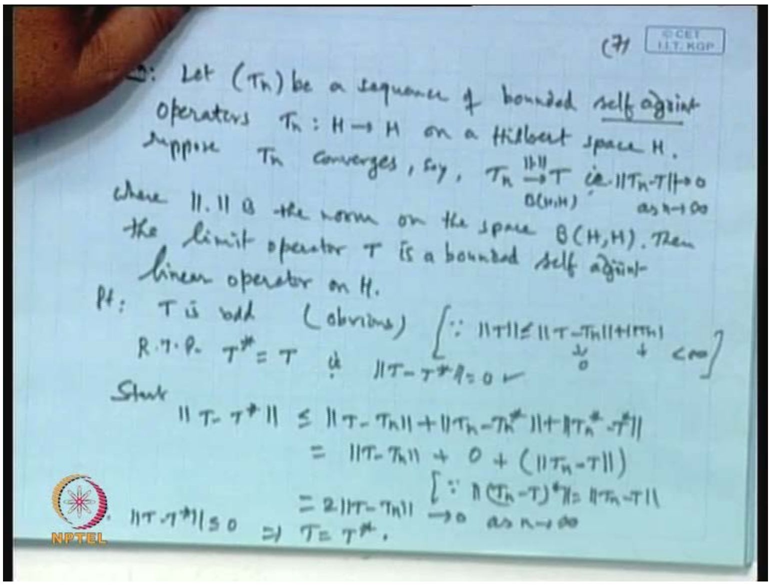

Now, next results shows, the sequence of the self Adjoint operator; means, if there are T 1, T 2, T n are the sequence of self Adjoint operator, then, the limiting point of this, limiting operator, whether will it be also a self Adjoint or not. Now, this is given by the next result that, if there are certain conditions on T n, then, it will, the limit operator will also be a self Adjoint operator. So, what is this is result. So, let T n be, be a sequence of bounded self Adjoint, bounded self Adjoint operators from H to H on a Hilbert space H, on a Hilbert space H. Suppose, that T n converges, converges; say, T n goes to T, under which, under the norm of B H H norm of B H H; this is the norm of B H H, bounded, set of all bounded linear operator defined from H to H and the norm is defined as the supremum norm of T x x over norm x, where x belongs to H.

So, T n converges to, that is, norm of T n minus T goes to 0 as n tends to infinity; that is true, where norm is the, is the norm on the space B H H, is it ok, B H H. now, what it says is, then, the, then, the limit operator T is a bounded self Adjoint, self Adjoint linear operator on H. So, what is given is that, T n is a sequence of bounded linear operator on H, where H is a Hilbert space and T n converges to T, under the norm of B H H; then, what it says is, T will be a bounded self Adjoint linear operator on H. Now, clearly, T is bounded. I think, it is obvious, or not. If sequence T n be a sequence of bounded linear operator, the limiting operator will also be bounded. Why, because, what is the norm of T; that can be written as, T minus T n plus norm of T n less than equal to…

Now, this goes to 0 and this is bounded, finite, is it not. So, basically, this is finite; or you can start with norm of T x, then, you can take T n x minus T x and both. So, that will be bounded, sequence will be bounded. So, this is nothing, is it ok. Now, T is bounded. So, what is left is, we have to show the T star is equal to T, T is a bounded self Adjoint operator, T star is equal to T, is it not. That is, instead of this, we can show, that is, if I prove that, norm of T minus T star is 0, then, we can say, because norm of x is 0 means, x must be 0. So, T is equal to T. So, instead of showing this, I will prove this, which is easy to prove that, because the sequences are converging in the norm. So, we will use the norm for this, is it ok, fine. So, you start with this, norm of T minus T star; add and subtract T n. So, this will be less then equal to T minus T n plus T n minus T n star plus T n star minus T star, is it ok or not, clear.

Now, this will be written as T minus T n plus, T n is a self Adjoint operators, is it not; given the T n be a operator, self Adjoint; this is given to be self Adjoint. So, T n star is equal to T n. So, this part will always be 0, and here, we, can we not write this thing as norm of T n minus T, that is all. Why, because, because of the reason, the T n minus T star, norm of this, is the same as T n minus T; self Adjoint operator, they have the same norm, is it not.

(( )) otherwise, T n minus T will also be an operator…

All be operator. So…So, you start… (( )). So, we are taking this thing, means, two times norm of T minus T n. Now, as n tends to, this goes to 0, as n tends to infinity. Therefore, norm of T minus T star is less than equal to 0; but norm cannot be negative. So, this implies that, T must be equal to… So, self Adjointness proved.

Sir, when you prove anything like this for the norm, (( )) we are taking T as a (( )) vector…

Here?

Yes, here. (( )). This is the norm of, (( )) we can see the triangular equality.

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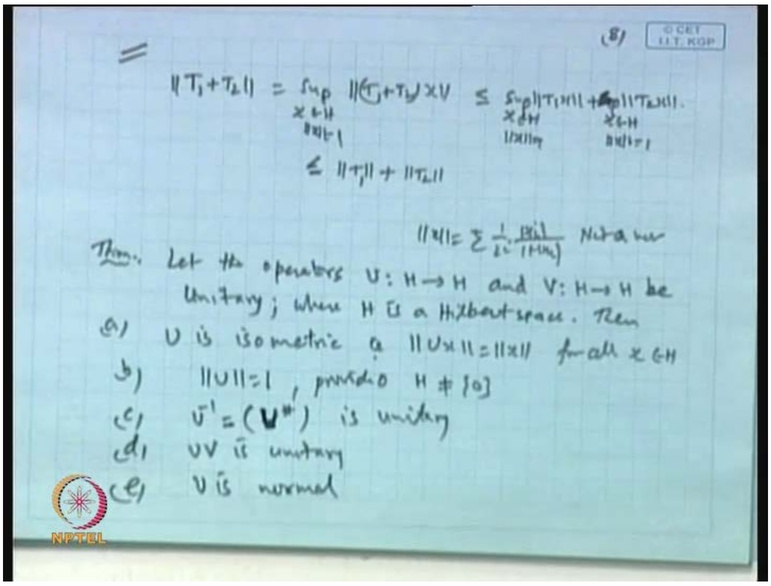

That is, no, it is not, because, we have proved, established this inequality, in case of the operator. How did I prove it, if you remember, T 1 plus T 2, norm of this. We have taken supremum, norm of T 1 plus T 2 x, where the supremum is taken over x and norm x equal to 1, is it correct or not. Now, this will be, because T 1, T 2 are bounded, so, this will be, less than equal to norm of T 1 x plus norm of T 2 x, is it not; and supremum of this and supremum of this, is it ok, of norm. Now, this, supremum of this, over x belongs to H, where the norm x is 1, where the supremum is taken as where the norm x equal to 1. So, this will be equal to norm T 1 and this will be equal to norm of T 2 less than equal. So, this shows less than equal to. So, basically, it behave as if it is a algebraic inequalities, but…

Sir, why cannot you say (( )).

Ok.

But otherwise, (( )) particular, this T 1 and T 2 is an element of B H H, and B H H is a vector space…

Yes.

So, T 1 and T 2 will be vector.

Yes, yes.

So, they can…

They can follow the triangular inequality.

But there every, but every vector space, that does not mean that, it will follow the triangular inequality. We have to establish, is it not. We have to establish, otherwise, the vector spaces are there; vector space itself require the triangular inequality, is it not; by the condition, if you go through the vector space definition, it requires the triangular inequality there.

(( ))

Only thing you can say, vector space, if T 1 T 2 are there, their linear combination all in there; but it does not mean that, an inequality will follow.

(( )) in a vector space, if we can define a norm, then, that norm will follow the triangular inequality; otherwise, (( ))…

No, norm, but we have to establish, because, it is not that every norm, is it not; sometimes, that, like our metric space, if I take, say X D is a metric space and if I define the norm in terms of the D metric, D x y over 1 plus D x y, it does not satisfy that one, that one, is it not, triangular, what is the norm. Norm, if I define in terms of sigma 1 upon 2 i and so on, it does not satisfy the triangular inequality, is it not. Every norm, that will be problems. You know, it is not necessary to be there, because norm of alpha x is not there. So, that itself is not a norm; if I define that way, triangular follow, but it will not define the norm. If I introduce the norm as sigma 1 upon 2 i mod x i, over say, 1 plus mod x i, then, this is not a norm.

Sir, this is not a norm.

So…

Sir, we cannot define (( )); this is only a metric.

Metric only.

Distance of metric space cannot be converge as a norm.

Norm. So, once it is not a normed space, that is why that; though it is a vector space. So, that is why, it is not necessary.

Sir, I was saying that, (( )) able to define a norm, then…

If you are able to define the norm, then, it will follow; you have to establish also. So, this is what… Now, another results, which is also interesting, unitary operator, for the unitary operator. Let the operator U, operator U from H to H, and V from H to H, be unitary operator; unitary means, they are 1-1 onto and U star is equal to U inverse; unitary. Here, H is a Hilbert space, Hilbert space. Then, the following property holds; then, U is isometric, that, that is, norm of U x equal to norm of x, for all x belongs to H; b, isometric means, they are having the same norms, and norm of U equal to 1, provided H is not equal to null vector.

c, U inverse equal to, equal to U star, U star is unitary, which is equal to U star is unitary and UV is unitary; e, U is normal and finally, we get, the last one, that is, a bounded linear operator T f 1, f, a bounded linear operator, operator T on a complex, on a complex Hilbert space H is unitary, if and only if, T is isometric and surjective; surjective means onto.

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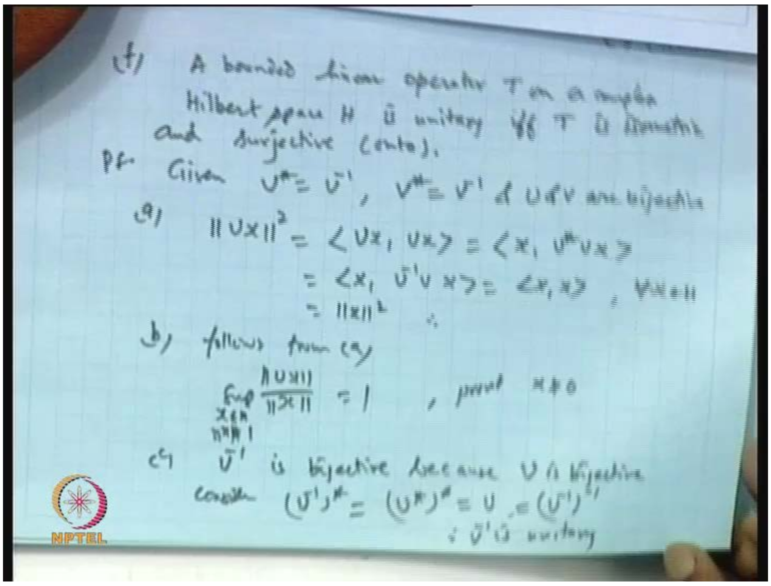

So, let us see the proof; proofs are simple. So, what the first, U and V both are given to be unitary. So, it is given, U star is equal to U inverse; V star is equal to V inverse; then, H is a Hilbert space, fine. Then, we have to show, U is isometric; isometric means, the norm of U x and, and norm x will be the same; the U does not transfer the distances; maintain the distances between x and y, the distance and the corresponding image will have the same distance. So, to prove the part a, consider U x norm square. This can be written as U x comma U x, in terms of inner product, but this will be equal to x comma U star U x; but U star is equal to U inverse. So, this will be equal to x U inverse U x, which is equal to x x; and this is true for all x. So, this shows, this is equal to norm x square; hence, for, follows.

b, U in, norm of U is 1; it follows from here. From, follows from a; why, if I divide by this, norm U x by norm x, take the supremum over all x belonging to H, x is, norm x is 1, then, it is 1; follows from here. Nothing to…provided x is non-zero; norm of… otherwise, it will not. That is why, H is different from 0. Then, U inverse is unitary. To show U inverse is unitary. So, what we have wanted to show, first is, clearly, U inverse is bijective; why, because U is bijective. Because this is given and U and V are bijective, is it not, by definition. So, if U is 1-1 onto, the inverse will exist and inverse will also be 1-1 onto. So, it is a bijective mapping.

Then, next is, if I prove that, U inverse star is equal to U inverse, then, our result is ok. So, consider U inverse star, but U inverse is equal to what, U star star; because U is given to be unitary. So, U inverse is equal to U star. So, U star star is U; that can be written as, U inverse inverse; therefore, U inverse star is equal to U inverse inverse. So, U inverse is unitary, is it ok. Clear, U inverse is bijective as well as U inverse star becomes the U inverse inverse; because this is a U star is equal U inverse, no. So, that is all.

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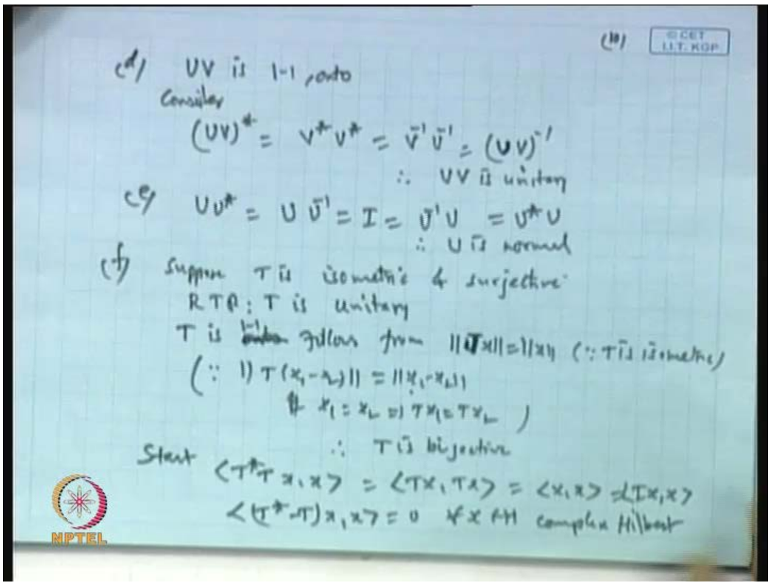

So, this follows. Then, next is, U V is unitary; U V is 1-1 onto follows because U is 1-1 onto; V is 1-1 onto; the product will be 1-1 onto and...

Now, consider UV star. This is by definition V star U star. V star is equal to V inverse; U star is equal to U inverse, but by definition, this is UV inverse. So, UV is unitary, clear. Then, e, e is U is normal. So, U is normal means, what is to be proved is, U is normal means, U U star must be U star U. So, U U star, U U star is equal U inverse, because U given. So, this is I, which can be written as U inverse U; but U inverse is U star U. So, U is normal. Now, last one, what is needed is, this, a bounded linear operator T on a complex Hilbert space is unitary if and only if, T is isometric and surjective, this. So, suppose T is isometric and surjective; isometric means, norm of U x equal to norm of x and surjective means, it is onto. Now, what is required to prove is, required to show that, T is unitary, T is… So, we wanted to show T is bijective and T star equal to T inverse, is it not. So, T is onto follows from, from this relation, sorry.

Let us take, U, instead of this, let us take T, because T is isometric, because T is isometric. So, it satisfy this, norm of T x equal to norm x. It means, each point will carry to the same, the length, the length of this vector of each point, under the image will be the same, clear, and this is true for every x. So, there will be a 1-1 mapping; corresponding to each x, you are getting T x. So, two different x 1 and x 2, their length will be different; therefore, T x 1 will be different from T x 2, is it not. So, the one ontoness will follows from this, is it clear. So, we get, onto follows from, because this is given, now 1-1, sorry, 1-1ness.

Sir, T is isometric…

T is 1-1; T is 1-1, follows from here, yes.

Sir, T is isometric; isometric means, that, (( )) space, if the operator is isometric means, there is a isomorphism between the two spaces…

Isometric means, that…

Isomorphism between the two spaces…

No, isomorphism, them towards isomorphism; isomorphic is, isomorphic is a 1-1onto mapping, which preserve the operations.

Which preserve the operations. (( ))

So, whenever we go for the normed space, then, it becomes the isometric.

(( ))

Isometric. So, length is the same.

(( )) bijective…

Ok.

(( )) bijective and there is a (( )) this exists, the dimension of the two spaces must be same, from the…

Otherwise 1-1ness; you wanted the 1-1ness, no. You can say from here, suppose, why, because, I am writing T of x 1 minus x 2, I take it; T is isometric. So, this will be equal to…Now, if I take this is equal to 0, x 1 equal to x 2; T x 1 equal to T x 2. So, if x 1 is equal to x 2, will implies T x 1 equal to… 1-1ness. So, basically T is 1-1; T is onto is already given in the problem; it is surjective. So, T is surjective; T is injective; therefore, T is bijective. 1-1 onto is a… Now, only thing we have to show, the T star T is the same as T T star. So, start with this, inner product T star T x comma x. Now, this can be written as T x comma T x, because of the property Adjoint, but T is isometric. So, this is the same as x x and this will be equal to i x comma x; but this implies, x comma x is 0, for every x belongs to H, where H is a complex Hilbert space; this is given; H is a complex Hilbert space. So, complex Hilbert space.

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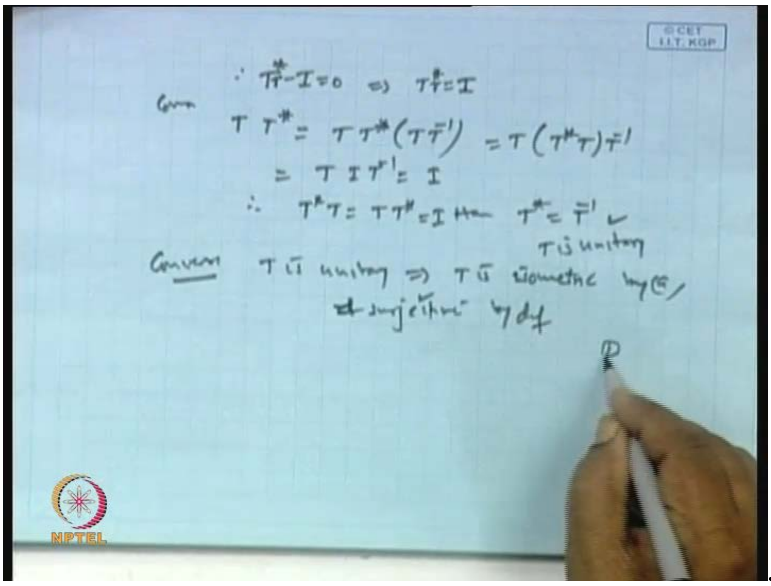

Therefore, T star must be equal to T. Therefore, T star minus T 0. So, T star will be equal to T, is it ok or not? Oh sorry, T T star, this is, I am sorry, this is T star T, T star T and this is I; you are transporting this I. So, sorry. So, this is T star T minus I. So, T star T must be I. Now, we consider T star, T T star; this is equal to T T star and then, I, we can write T T inverse; this will be equal to T T star T T inverse, but T star T is already proved to be I. So, this is equal to again, I. So, what we conclude is, T star T is equal to T

T star; hence, T star will be the inverse of T, because those product and which is coming to be I. So, one should be the inverse of the other, and this shows, T is our unitary, T is unitary, is it ok.

Now, let us see the converse part. Converse follows easily. If T is unitary, then, it is isometric, because of the first property; by the first result. And then, second one, what we said is, second, what you have, if isometric and surjective, and surjective follows and surjective by definition; because unitary means, it is bijective. So, definition, by definition, it follow surjectiveness and because T is unitary, in the first case, you have already proved, T is isometric. So, this. So, this completes the proof. Thank you. Clear.