Functional Analysis

Prof. P. D. Srivastava

Department of Mathematics

Indian Institute of Technology, Kharagpur

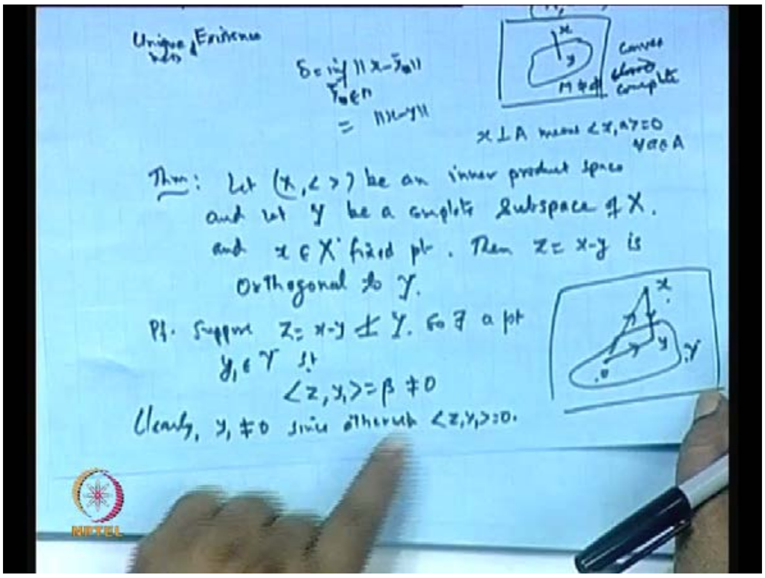
Module No. # 01

Lecture No. # 23

Projection Theorem, Orthonormal Sets and Sequences

We were discussing about the existing, existence and uniqueness problem.

(Refer Slide Time: 00:20)



Uniqueness and existence problem and in fact, what we have shown is that if X be a inner product space and M be a non empty convex subset which is closed, convex and closed subset then, for any point x belonging to capital X, there will exist a point y in capital M such that, the infimum of this x minus y n, where y n belongs to or y bar, y bar belongs to m will exist and (()) it will be x minus y.

So, this result we have seen, when m is considered to be a non empty convex close subset convex and complete, this is complete in the metric defined by the inner product. Now, this result in this result, even if we change m instead of checking the convex set, if we consider a subspace which is complete then also one can prove the result. So, we have the next result, where M is replaced by a (()) subspace of x which is complete or complete subspace of x.

So, we have this result let X be an inner product space be an inner product space and and let capital Y be a complete subspace of X. Let x is fixed belongs to capital X is a fixed point suppose then, what this result says is then the vector Z which is x minus y is orthogonal to y.

So, basically what is the meaning is that, if suppose we have this is our y here is x, this is our say x then, we are getting a point y here, such that this vector x minus y is perpendicular to y is orthogonal to y, it means if suppose this is origin and if I join this as vector, then this vector x minus y, this vector x minus y will be orthogonal to y.

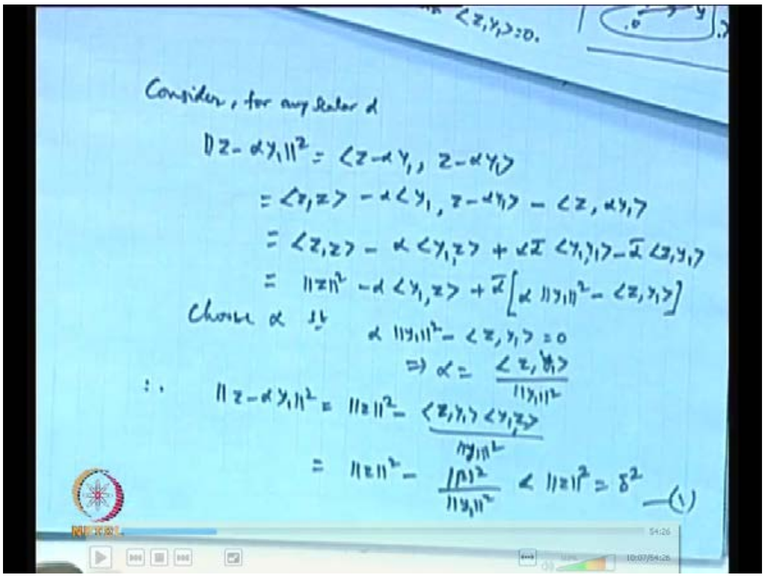
So, this (( )) or this (( )) and this confirms our previous hypothesis that, if we want to find a shortest distance from a point x to the set m, then it can be obtained by dropping a perpendicular from x to that point, is it or not? So, that gives the confirmation for that (( )) this is which we had in mind clear, so this is true.The proof is simple. What we are interested is the vector x minus y should be orthogonal to y.

So, suppose the z which is x minus y is not orthogonal to y, let us prove by contradiction, it means once it is not orthogonal to y, so by definition of orthogonality a point x is said to be orthogonal to a if the inner product of x is 0 for all a belongs to x, is it not? That is the orthogonality of x.We say x is orthogonal to a means the inner product of x a is 0 for every a belongs to a.

So, when we say x is not orthogonal to a it means there must be some point available in a whose inner product with x will not be 0. So, let let us suppose z is not orthogonal to y. So, there exists a point say y 1 belonging to capital Y such that, the inner product of z, y 1 which is say beta is not equal to 0, if it is 1.

Now, clearly this y 1 clearly this y 1 is v different from 0 will be different from 0 why? Because, if it is 0 then this inner product will be 0, so a contradiction, so it cannot be 0. Otherwise, the inner product of z y 1 will be 0 a contradiction will not cursive it. So, let us assume that the inner product z y 1 is beta which is not 0 where y 1 will not be equal to 0.

(Refer Slide Time: 06:28)



Now, consider for any scalar, alpha the norm of z minus alpha y 1 whole square clear. So, this can be written in terms of the inner product as alpha y 1, z minus alpha y 1 and this will be equal to inner product of z z, then minus alpha times y 1 inner product of z minus alpha y 1 for z.

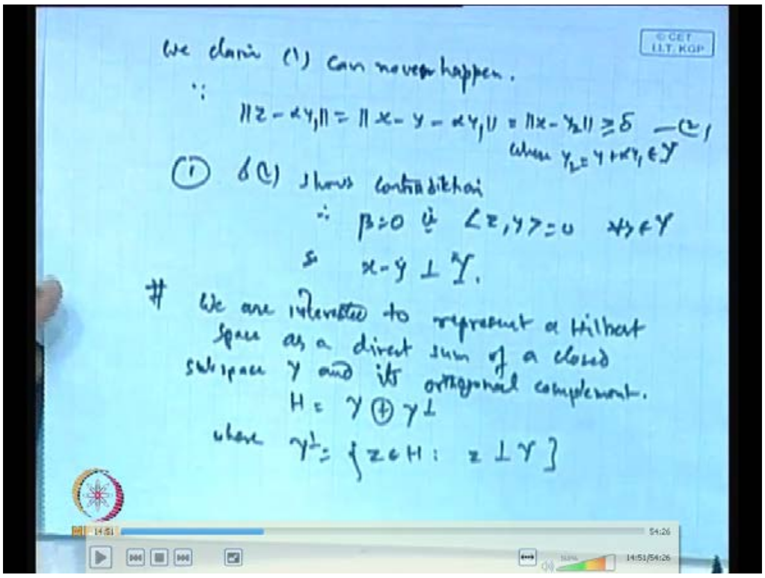
And then z z inner products of alpha y 1, is it or not? clear now this z, z and then this z we are putting here to take a minus sign outside and then minus alpha y 1 alpha outside y 1, this is it. So, i i have put it short, so z z minus alpha y 1 z minus minus plus alpha alpha bar y 1 y 1 inner product minus alpha bar inner product of z y 1 or this can be written as norm of z square minus alpha y 1, z.

So, y 1, z is it this is comma y 1, z plus if I take alpha outside, then what we get is alpha times norm y 1 square minus z y 1 inner product. Now, since alpha is our choice at our disposal, so choose alpha such that, this portion becomes 0 alpha norm of y 1 square minus inner product of z, y 1 is 0.

So, what we get alpha becomes inner product of z y 1 divided by norm of y 1 square substitute again (( )). So, we get norm of z minus alpha y 1 whole square is equal to the norm of z square minus, if I substitute this thing it becomes z y 1 into y 1 z divided by norm of y 1 square, is it not? (( ))

And this will be equal to norm of z whole square; now z y 1 is beta. So, we can write this beta and this is beta conjugate, so we are getting minus beta mod beta square over norm of y 1 square. Now, this is non negative this is non negative so; obviously, this is less than is strictly less than norm of z square is it not? But norm of z is what is our z, if you remember the previous result the previous theorem which we have had earlier. In this case this delta was this and in fact, it is nothing but the norm of z. So, this will be equal to delta square. So, this entire thing is less than delta a square.

(Refer Slide Time: 10:17)



But, what I claim is 1 is not possible, but we claim one can never happen why can never happen why? Because because the reason is, what is z minus alpha y 1 z is x minus y minus alpha y 1, so norm of this norm of this, now this is equal to norm of x minus say y 2, where y 2 is equal to y plus alpha y 1 belongs to capital Y, is it not?

Now, delta is this infimum value, this is our delta. So, if we replace y by any other point in this set then it will be greater than or equal to delta. So, basically this you are getting greater than or equal to delta.

So, norm of this will be greater than equal to delta therefore, square will be greater than equal to delta square, but here we are getting strictly less than delta strictly less than delta. So, this contradicts 1 and 2, so 1 and 2 shows contradiction therefore, our assumption is wrong therefore, norm beta must be 0, that is the inner product of z y is 0 for every y belongs to capital Y.

So, x minus y is perpendicular to capital Y, that source is it correct, so this gives the result. Now, we are interested this 2 lemma's will be helpful in giving the idea of the projection theorem. What we are interested is that we wanted to express the Hilbert space h as a direct sum of the two subspaces - one is closed, another one is the orthogonal complement of y and this is called the projection theorem.

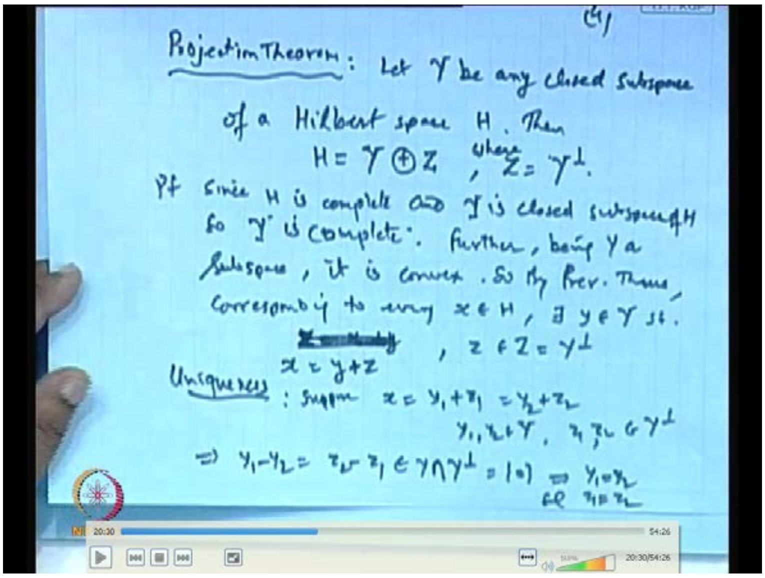
Any vector space if you look, the any vector space elements of the vector space can be expressed as a sum of the two subspaces, but this may not be unique the representation may not be unique, but if the representation is unique then we say this this is called the direct sum of the two space y and z and the intersection of y and z will be 0 theta that is what.

So, what we are interested is, we are interested to represent a Hilbert space H Hilbert space H as a direct sum as a direct sum of as a direct sum of a subspace of a closed sub space, I think of a closed sub space y and its orthogonal complements, orthogonal complement this is our interest.

That is we want H to be the direct sum of y and y perpendicular, where y perpendicular is the set of those vector in H such that, z is orthogonal to y orthogonal complement of y. Now, the idea of the direct sum you know, when any element of H can be expressed uniquely as a sum of the two elements while, first belongs to y and other belongs to y perpendicular.

And in particular when this is a 1 is the orthogonal complement of the other then we say it is a perpendicular orthogonal to y. So, we are interested, now this result is known as the projection theorem.

(Refer Slide Time: 15:06)



So, what is the projection theorem is? Let Y be any closed subspace of a subspace of a Hilbert space H then Y. H can be expressed as a direct sum of Y and Z, where Z is orthogonal complement of Y where, so (( )). Let us see the proof, now since H is complete, because it is a Hilbert space and y is closed; this is a closed subspace of H and we know the result every close subspace of a complete metric space or complete banach space is banach or Hilbert. So, y is complete, is it not? further being y, a subspace it is convex, every subspace is a convex set. So, it we have H as a vector space y is a subspace of a vector space, which is non empty convex sub set of H and complete.

So, by the uniqueness and existence theorem, we can always find a point in y, which is nearest to x by dropping the perpendicular from x to y, so by the previous result previous theorems is it not? By previous theorem (( )) for every x corresponding to every x belong to h, there is there exist an y belongs to capital Y such that, the z can be written as x plus y, where z belongs to capital z which is the y orthogonal, is it or not? z will be sorry I am sorry this is x x will be y plus z.

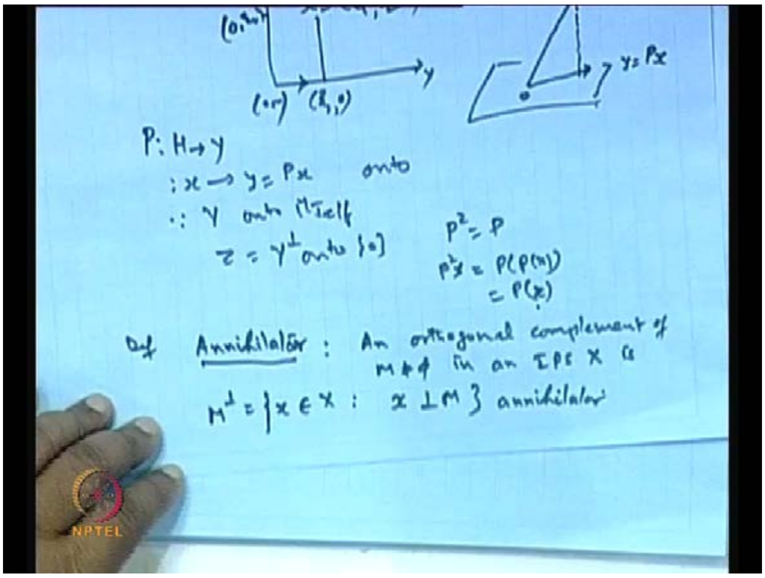
Because every x can be expressed in this form, y plus z where the y belongs to capital Y and z belongs to orthogonal (( )) or x minus y will be orthogonal to y, this is just, so by using the previous result we can say that every element x can be expressed as a direct sum of y n by orthogonal complement. Now, if this is unique, then it will be the direct sum because this representation is but if it is also unique then we say it is adirect sum. So, for the uniqueness suppose there are 2 representations.

Suppose x is equal to y 1 plus z 1 as well as y 2 plus z 2, where y 1 y 2 belongs to capital Y and z 1 z 2 belongs to capital Z that is by orthogonal. So, what will we, this will imply y 1 minus y 2 is the same as z 2 minus z 1, but y 1 and y 2 belongs to capital Y, y is a subspace, so y 1 minus y 2 will be the element of y. But z 2 minus z 1 is a point of y orthogonal complement. So, y 1 minus y 2 is also the point of y orthogonal complement clear.

So, it means they are the common element of y and y orthogonal complement is it not? So, so they will belong to the y intersection y orthogonal is it correct or not? So, once this is there, but y and y orthogonal complement its null. So, this is null because representation is unique and direct sum, so it is 0 therefore, y 1 is equal to y 2.

Similarly, z 1 is equal to z 2, so representation is unique hence we get this. So, projection theorem proof is very simple just a two line, now what is the consequence how to deduct get it that represent.

(Refer Slide Time: 20:37)



If we have this plain, suppose we are having this plain and here is this point x, then this is the vector x say here origin, then when you project it on this plain you are getting this vector y equal to p x, clear. Say for example, if I take x to be x i 1 x i 2 and then I want to project it here. So, this point is x i 1 0 which is y x.

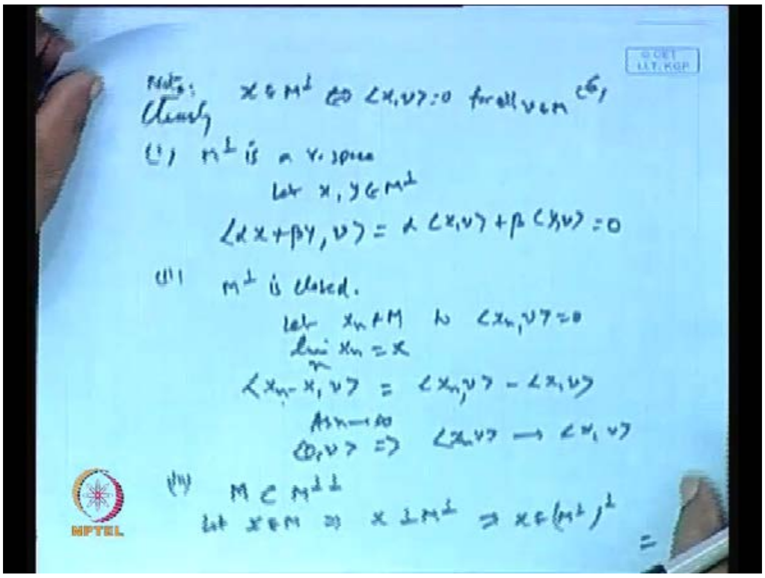
Now, if I project any point y on (( )) then it is nothing but, the same projection elements clear, any point on under this projection, where the second coordinate become 0, so every point of y will be projected to itself. So, we are getting a projection P x P from H to Y, where x point is projected to y which is P x.

And this projection has the following property, one is H p from this will be onto y and then this projection from y onto itself, any element x i 1 0 will be projected on its and z will be y orthogonal onto single (( )) 0, if I take the point here, then what is the projection here is 0 x i 2, so it will be 0 0.

So, this will project on this, it means the P square becomes P, because P square x is equal to p of P x and P x is equal to again x, so we are getting this P x P of P x becomes P x. So, this will be there. So, P square becomes P that is it is an (( )) potent operator bounded (( )) potent operator. So, this gives the idea of the projection that, if you want to project a one the closest element the shortest distance then from a set, then it can be obtained by dropping a perpendicular from H on this space and that projection will give clear.

Now, this also gives a concept of annihilator, this definition anything is annihilator, an orthogonal complement of m which is not is denoted by in an inner product space x is the set, x belonging to capital X, such that x is orthogonal to m, is it not? And this is known as an annihilator, a special name is given orthogonal complement inner product space y, we call it to a m perpendicular denoted by m equal to annihilator.

(Refer Slide Time: 24:48)



And this annihilator is obviously, at some vector space this is, so clearly m perpendicular is a vector space why it is vector space.

(( )).

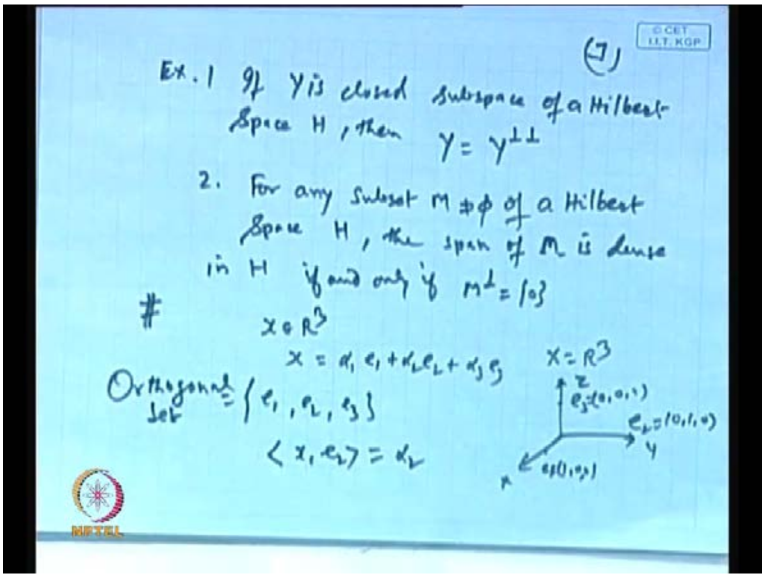
Yes. So, basically we are defining the annihilator as, because this as a note you can say that will be clearer M perpendicular is a vector space. So, x belongs to M perpendicular, if and only if the inner product of x v is 0 for all v belongs to M for all v belongs to m, is it or not?

So, using this we can say suppose, x and y are the two element belonging to m perpendicular. So, find out alpha x plus beta y and then v, then you will see alpha of x v plus beta of y v and each one will be 0, so this is a vector space. Now, second one is this m perpendicular is also closed, this is a closed subspace, why it will be closed? Let us take a sequence x n belongs to m, so we get inner product of x n v is 0, now let the limit of x n is n x.

If we show the x belongs to m perpendicular is fine. So, let us consider x n minus x v, now this will be equal to x n v minus x v, now take the limit as n tends to infinity, since inner produce is a continuous function, so this left hand side will give 0 into v, clear and that will give the corresponding v 0.

So, here we are getting x and v will tends to this entire thing will go to 0 is it not? So, this entire thing will go to 0 implies that, x n v inner product of this will go to x v. So, x must be a point in the m perpendicular. So, that will give the closeness for this is it clear or not? Then another exercise as in exercise you can say, always m is subset of m perpendicular perpendicular why? Because, if suppose x belongs to m then this implies that x is orthogonal to m perpendicular, if x is orthogonal to m perpendicular, then it must be the point of what? Orthogonal complement of this, so that is nothing but, the x belongs to the m perpendicular perpendicular. So, that is why it is sub space, so this (( )) things.

(Refer Slide Time: 28:14)



Now, as an exercise you can note and we will do it if it is not coming, but this is interesting exercise, the first is if Y is a closed subspace of a Hilbert space H, then Y is equal to Y orthogonal orthogonal, we have the equality sign provided the subspace y is closed in general y is subset of y perpendicular perpendicular, but equality holds if y becomes a closed subspace of x.

So, this another exercise which is that for any subset m, which is non empty of a Hilbert space H, the span of m the span of m is dense in H, if and only if, m orthogonal complement of m is a single (( )), so this two, I am just leaving and next time if you we will discuss the proof if it is not solution if it is not possible, so that (( )).

Now, after this concept as we have seen that orthogonal complements or orthogonality plays a vital role and it simplifies the many things, is it or not? Even for the over the linearly independent sequences, we have an advantage of the orthonormal sequences.

The reason is suppose we have a vector space say x equal to say R 3, then we say know, if x can can be expressed as alpha e 1 alpha e 2 alpha e 3, where what is e 1 e 2 e 3? These are the basic elements of this vector space R 3 where e 1 will be 1 0 0 e 2 will be 0 1 0 and e 3 will be 0 0 1, is it not?

So, basically this 3 which are the basic elements for R 3, they are (( )) to each other clear. So, this collection e 1 e 2 e 3 this is an orthogonal set. Orthogonal set means a set is said to be orthogonal, if we pick up any two elements then the inner product of this two must be 0 that is they are (( )) to each other, any pair will be orthogonal to other.

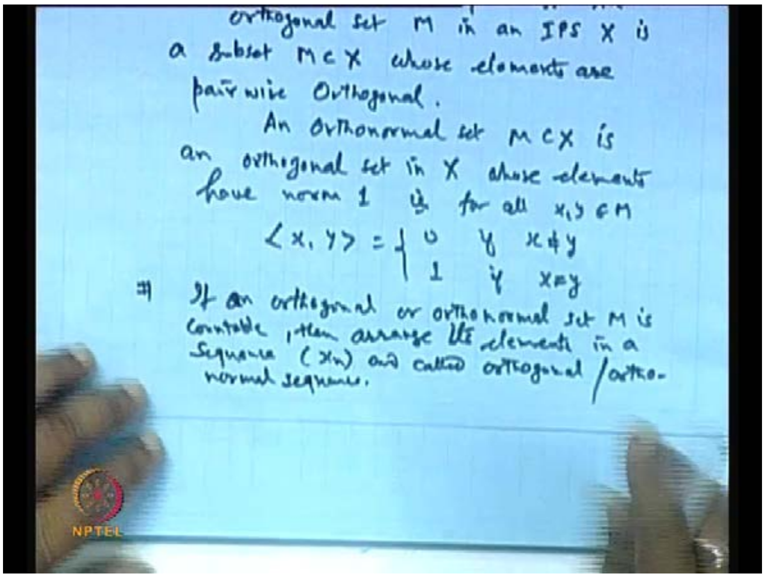
So, inner product of e 1 e 2 will be 0 inner product e 2 e 3 0, e 1 and e 3 will be 0, so they are orthogonal elements. In fact, the norm of each element is one, so we will call it as a orthonormal also, that we will come in (( )). So, any element x belongs to R 3 can be expressed in terms of the basis element. Now, here this basis element has a peculiar property they are at right angle to each other. But it never it is not always true, that all the basis element must be orthogonal to each other its not true, but what here is because it has a perpendicular property orthogonal property, the advantage is if we are interested in finding the constant alpha 1 alpha 2 alpha 3, then just take the inner product.

Suppose, I am interested in alpha 2 take the inner product, which e 2 what happen is alpha 1 inner product of e 1 e 2 will be 0, alpha 2 inner product e 2 e 2 will be 1 and rest will be 0. So, without any effort you can just find out the constant alpha 1 alpha 2, otherwise in a general example when the vector space is given and it is required to find the set of the vector are linearly independent, you what you require you go for the lengthy calculation and all the 5 4 5 equations solve it for alpha 1 alpha 2 alpha 3 etcetera.

And then if all alphas are coming to be 0, we say the set is linearly independent, if it is not then we say it is a dependent, similarly any vector space you wanted to find then this will be the problem. But in case of the orthogonal property that problem situation is must imply. Another advantage of suppose, I wanted to introduce 1 mod here say alpha for e 4 and I wanted to get this linear combination e 1 e 2 e 3 e 4 any elements belongs to that linear combination.

So, I required only one extra norm (( )) alpha 4 and that leads, that will be simpler if I deal with the orthogonal set of orthogonal elements. So, study of the orthogonal element or orthonormal elements, simplifies the situation simplifies the situation in case of the inner product space, clear.

(Refer Slide Time: 34:09)

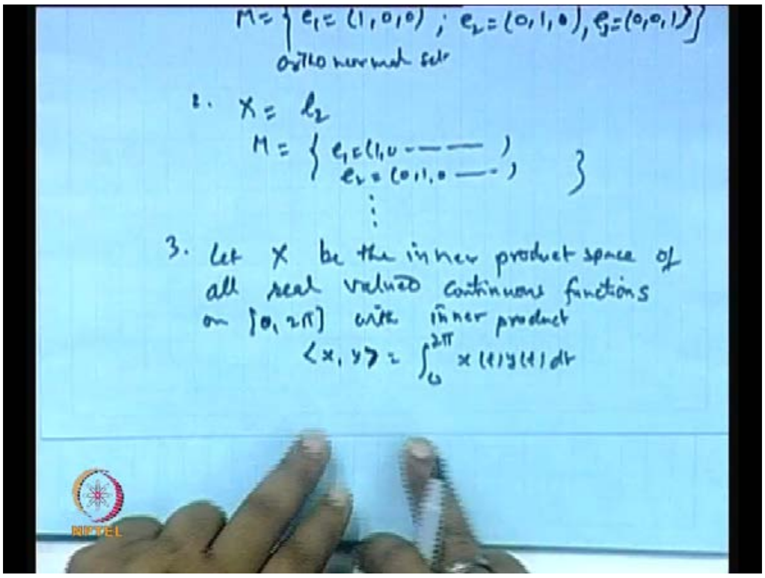


So, we will discuss what the orthonormal sequence is and orthonormal sets? So, let us say first definition of orthonormal sets and sequences. And orthogonal set M an orthogonal set M in an inner product space inner product space X is a subset is a subset M of X whose elements are pair wise orthogonal, pair wise orthogonal.

Then an orthonormal set M, which is a subset of X is an orthogonal set, set in X whose elements have norm 1, that is for all x y belongs to M, the inner product of x y is 0, if x is not equal to y and 1 if x is equal to y, clear. So, that is it.

Now, if a set m orthogonal sequence, if an ortho, if an ortho if an orthogonal or orthonormal set m is countable, then arrange this then arrange this elements arrange it is elements in a sequence, say x n and call it orthogonal or orthonormal sequence, orthonormal sequence is it ok? The same set if the set is countable, then arrange it in the form of the sequence x 1 x 2 x n and then this collect sequence will be called as a orthogonal sequence, if the elements are orthogonal means pair wise this joints only or orthonormal, if apart from the pair wise by design each one is having the norm 1, so that we can get the corresponding orthonormal sequence.

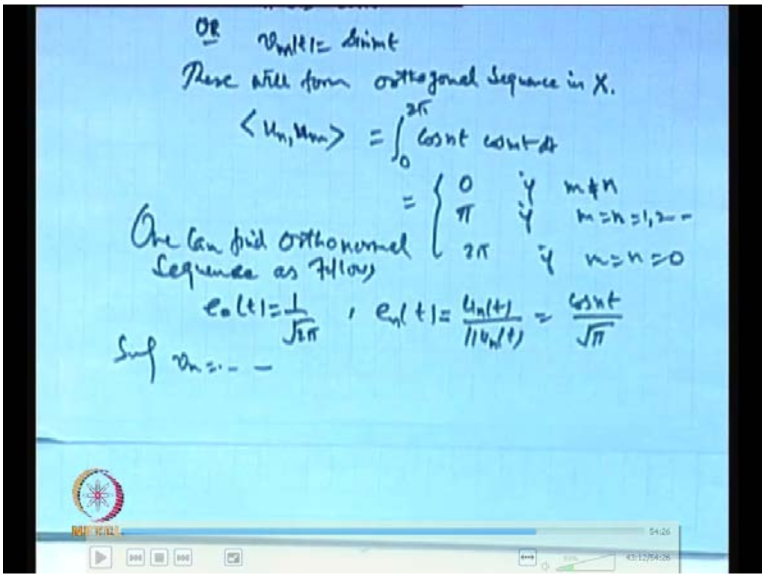
(Refer Slide Time: 37:57)



Now say for example, if we look this Euclidean space r n 3 that I have already discussed. The set M which is e 1 1 0 0, e 2 0 1 0, e 3 0 0 1, this is our orthonormal set, is it not? Each one is having norm 1 and they are at right angle to each other, clear. Then, if we take x to be say l 2, it is a inner product space, then what is the orthonormal sequence? That is also e 1 1 0 0 0, e 2 0 1 0 0 0 and continue up to infinity that will be the collection of the orthonormal sequence.

Now, suppose we have the set, another set c a b, set of all continuous functions- sorry let x c a b will not form a inner product space. So, let us take the another exercise say let x subset sub subclass, let x be the inner product space, inner product space of all real valued of all real valued continuous functions on the close interval say 0 to 2 pi. This inner product defined as inner product of x y as the integral 0 to 2 pi x t y t d t this will form an inner product we have seen already.

(Refer Slide Time: 40:18)



Now, an orthogonal sequence in this case, we claim that u n t, which is cos of n t or v n m t suppose sin of m t, these will form orthogonal sequence in X, because if we take the inner product of u n, u m, u n u m, just u n then what we get is 0 to 2 pi cos n t cos m t d t and we know the value of this will be equal to 0, if m is not equal to n, because when m is not equal to n then twice cos of cos v formula, so it become sin and then, when it integrate it we get the value 0 and equal to pi, if m is equal to n equal to 1 2 3 and equal to 2 pi if m is equal to n equal to 0, when both are 0 it is 2.

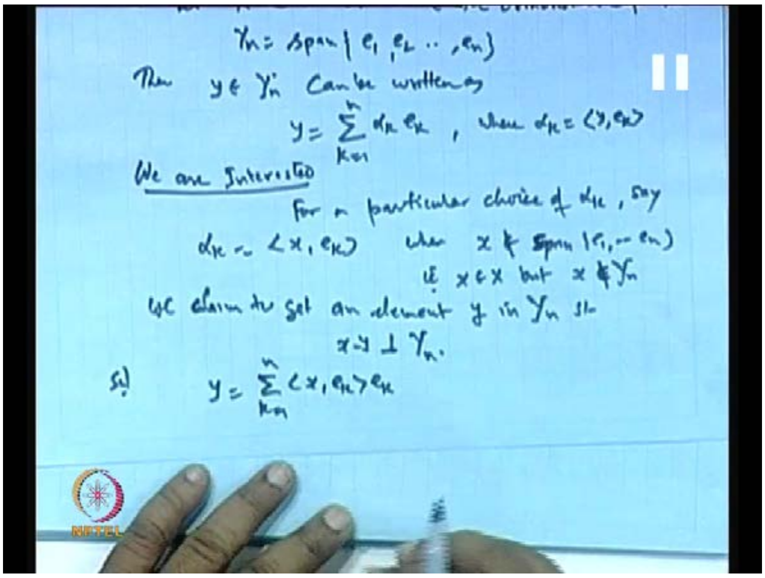
So, basically this u m, the inner product of this will be 0, when m is not equal to n, clear and when m is equal to n you are getting something else. So, we are not saying that this is a orthonormal sequence, but it is a orthogonal sequence and if you are interested in finding the orthonormal sequence from here, one can find orthonormal sequence as follows say the first sequence element of this e naught t, I am writing to be 1 upon root 2 pi. So, e 0 e 0 will be norm of e 0 e 0 squares will be 2 pi that will be get cancelled so 1, then e and t is u n t divided by norm of u n t and that is equal to cos n t over root pi. So, we are getting this sequence not only orthogonal, but each 1 norm of this will be 1, so it (( )).

Similarly, v n will form similarly v n will form the orthogonal sequence. So, that now, if we look this sin n cos n t is it not a terms for the Fourier series, because any Fourier series expansion x t can be written as a naught y 2 plus sigma n cos n t plus v n sin of n t.

So, this cos n t and sin n t, which are coming basically they are the orthogonal sequences and we are representing the x in terms of this orthogonal sequence.

So, as an inner product can be used or can be applied to compute this constant (( )) v n in case of the Fourier series is it. So, that is the advantage of this.

(Refer Slide Time: 44:02)



Now, another advantage of this orthogonal sequence, orthonormal sequence is as we have seen that suppose, we are dealing with this, let x be a inner product space, x be a inner product space and let us suppose that y n and e 1 e 2 e n be orthonormal sequence and e n be an orthonormal sequence or orthogonal sequence in x, orthonormal orthogonal sequence (( )).

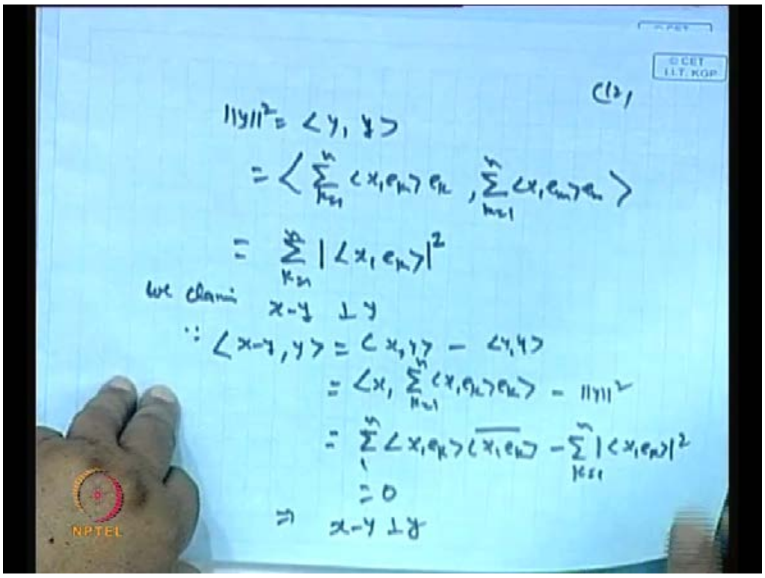
Now, y n is suppose a span of e 1 e 2 e n clear, then any element y belongs to capital Y n can be written as y as sigma, k is equal to 1 to n, alpha k e k, where what is alpha k? Is the inner product y e k, alpha k is the inner product. So, this is the general thing. Now, what we want it is let us suppose choose fix k sorry this alpha k we wanted to change. And so that a new term which you are getting y should come out such a way, so that we get the complement or orthogonal complement somewhere.

So, what we are interested is that, let us say for a particular choice of a we are interested in that, for a particular choice of alpha k say say alpha k is the inner product of x e k, where what is x? Where x is not the element of the span of e 1 e 2 e n means x does not belongs to y n x belongs to capital X that is x belongs to capital X, but x is not the elements of y n is it clear?

So, what we are a if I replace this alpha k by x another value constant depending on x x e k then can be get or v for a particular choice of this alpha k we claim, to get an element y an element y in y n such that, x minus y will be orthogonal to y orthogonal to y, n this is our claim.

Why let us see how, first thing is if I replace alpha k by this then, what will be our y the y becomes sigma k equal to 1 to n x e k e k is it not x e k e k this is our y.

(Refer Slide Time: 47:55)



Now, if I take norm of y square then we are getting this is norm of y square means inner product of Y y, so substitute this, so sigma k equal to 1 to n, x e k e k, sigma k equal to some other number let it be m equal to 1 to n, x e m e m is it not? Let it be on that number there is problem now you take it, K equal to 1 to n, so each term will be domain operated on this. So, e k when it is operated on e m when m is different from k it is 0.

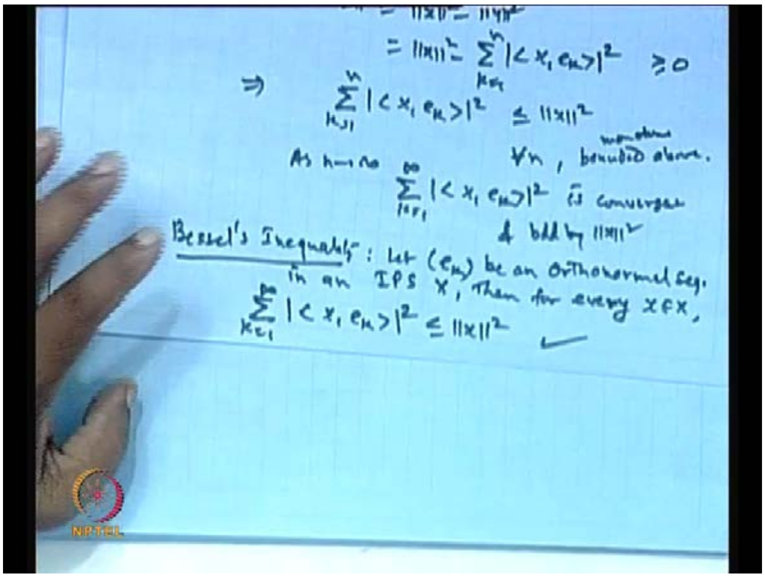
So, only the term will be the e k e k will be 1. So, basically what you are getting is sigma k equal to 1 to n x e k, x e k is outside and then another term which you are getting is x e m is it not? But conjugate, so we are we are getting mod of this x e k whole square is it or not? Did you get (( )).

(( ))

Because as soon as I open this thing will come here, so you are getting conjugate of it, so you are getting mod of this square. Now, now we claim, that x minus y will be orthogonal to y, why? Because inner product of x minus y, y this will be equal to inner product of x y minus y, y and x is y is this 1 sigma k equal to 1 to n x e k e k and minus norm of y square. So, this will be equal to sigma again if I take this thing x e k. So, this conjugate will come here.

So, it will be the same as x e k into x e k conjugate and what is norm y square y that already we have seen this is the modulus inner product of x e k whole square, k equal to 1 to n. So, 1 to, so basically both are 0 both are equivalent and we will get 0.

(Refer Slide Time: 50:40)



It means that x minus y is orthogonal to y, so let x minus y to be z let x minus y to be z. So, let z stands for x minus y, which is that is it clear or not? So, what will be the x therefore, x will be equal to y plus z. Now, once they are orthogonal y and z, so we get from (( )) norm x square will be norm y square plus norm z square by (( )) inner product.

Therefore, from here we get norm of z square, which is norm of x square minus norm of y square and this is equal to norm of x square minus norm y means sigma into modulus of x e k whole square k equal to 1 to n, clear.

Now, since the norm z can never be negative, so this is always be greater than equal to 0, this implies that sigma k is equal to 1, k equal to 1 to n modulus inner product x e k whole square is less than equal to norm x square, clear. Norm x square and this is nothing but, this is bounded for each n, this is true for each n bounded above y norm x. So, bounded above, so this series as n tends to infinity, the series sigma k equal to 1 to infinity, modulus of inner product x e square is convergent, is it correct or not?

This is bounded for each n and when the sequence is an increasing sequence, this is bounded above monotone sequence, a monotone sequence which is bounded above must be convergent, why monotone, because when you take increase n equal to n plus 1 another positive terms you are getting, so it is increasing sequence bounded above, so it must be a convergent and convergent and dominated or bounded by norm x square, is it correct? So, this leads to the inequality which is known as the Bessel's inequality.

What this Bessel's inequality says, let e k be an orthonormal sequence, orthonormal sequence in an inner product space capital X, then for every x belongs to capital X, this sigma k is equal to 1 to infinity, and modulus of inner product x e k whole square is less than equal to norm of x square and this inner product.

(( ))

This result is known as the Bessel's inequality it will be useful, Thank you, Thanks.