Functional Analysis

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Module No. # 01

Lecture No. # 02

Holder Inequality and Minkowski Inequality

In the last lecture, we have discussed the concept of metric space, the definition and some of the examples of the metric space. Today, we will do few more problems on the metric space and also, an important inequality, which is known as the Minkowski inequality, as well as, the Holders' inequality will be derived. Now, these inequalities are used basically, to establish the fourth property of the, fourth condition of the metric space; that is triangular inequalities. And, as we have seen in case of R 2 or may be R n, we have left that part to verify, whether the triangle inequality is satisfied or not. Now, those things can be easily proved or verified, with the help of Minkowski inequality. So, today, our concentration will be basically, on the Holders' and Minkowski inequality. Now, before going to the Holders' and Minkowski inequality, we will give one example, which is also an interesting and important, the set of all bounded functions, which is defined over the set A; because, we have taken the l infinity, which is the set of bounded sequences.

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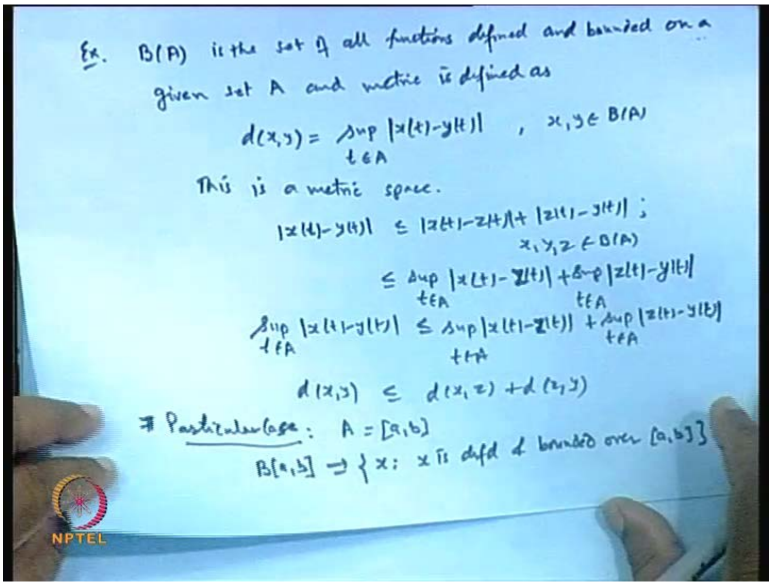

Now, bounded functions, we denote it by B of A, is the set of all functions, set of all functions, defined and bounded, bounded on a given set, on a given set A. And, the metric, and metric, we defined, is defined as d of x y at the supremum of mod x t minus y t, t belonging to A, ok, where x and y are the elements in B A and supremum is taken over all ts belonging to A. Now, this set, B A, together with this metric concept d, forms a metric space. So, this is a metric space, which can be verified easily, because d, which is defined on this, is a well defined thing, because function is defined and bounded over that closed, over this set A. So, the supremum value will be attained and we get d to be a finite, real, non negative thing.

Then, d of x y, if it is 0, then, supremum of this thing will be 0. It means, for all t, x t must be equal to y t; and vice versa, if x, if x t equal to y t, for all t, that is, x equal to y, then, in that case, the supremum will be 0 and d of x y will be 0. Third, d of x y is d of y x is obviously true and the fourth one, can easily be seen; if I break up this part, mod of x t minus y t, this is less than equal to mod of x t minus z t plus mod of z t minus y t, where z is any other… x, y, z, these are the points in B A, set of all functions. These are the functions, defined and bounded over A. Now, take the supremum over t on the right hand side. So, it is less than equal to supremum over t, belonging to A of this difference, plus supremum of z t minus y t over t, belonging to A.

So, this part is less than equal to supremum of this, for every t belonging to A. So, left hand side is independent or left hand side depends on t, but the right hand side is independent of t, because once you take the supremum, this is a finite thing. So, take the supremum on the left hand side over t, we get, this part is less than equal to supremum of x t minus y t, t belongs to A, plus supremum of this thing, mod z t minus y, this is z t, this is also z t, sorry; so, so, x t minus z t plus supremum of z t minus… So, this is this one; correction is here, this one. Now, this is basically, the distance between x and y, and this is the distance between x and z, this is the distance between z and y. So, triangular inequality is satisfied for this function d. Therefore, B A will be a metric space. So, this will be the class of all functions, which are bounded.

Now, as a particular case, if A is replaced by the closed interval a b, then, this space we denote B of a b, and we say, it is a set of all bounded functions, defined and over the closed interval a b. So, we will denote, this is, B a b, the set of all bounded functions x, where the x is defined and bounded over the interval a b. And, the metric is defined in terms of the sup of this. So, this is another example of a metric space. Now, this, we have discussed the example of a sequence space; we have discuss the example of the function space; we will now go to a, another example of a sequence space, which requires the concepts of the Minkowski, which require the, the use of Minkowski and Holders' inequality.

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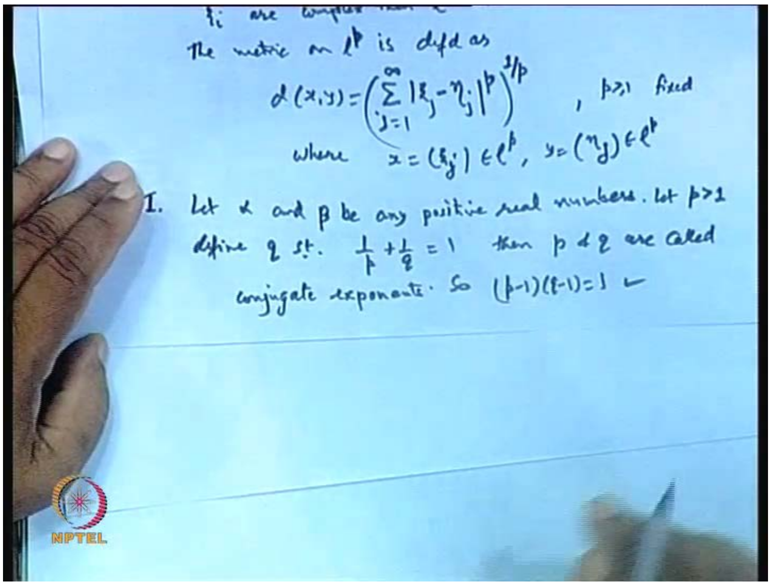

And, that space is l p space, which is also a metric space under a suitable metric d, where p is greater than equal to 1, p is greater than equal to 1. What is this l p space? l p space, l p space is the set of those infinite sequences x i 1, x i 2, x i n and so on, such that, sigma of mod x i i power p, i is 1 to infinity, is finite, where this x i 1, x i 2, x i n, these are scalar quantity. So, where x i's are scalars. If xi's are real, if xi's are real, then, the l p space is said to be real space, real l p space; and if xi's are complex numbers, then, we say, call l p as complex l p space. So, l p is the class of those sequences x, infinite sequence x i 1, x i 2, x i n, such that, sigma of x i i power p is finite; that is, those infinite sequence which are p (( )) sequence. So, we get this one.

The metric on l p is defined as d of x y as sigma j equal to 1 to infinity, mod of x i j minus eta j power p power 1 by p, where p is greater than equal to 1 is a fixed number. Now, the question arise, whether this definition of the metric is well defined; whether d is well defined on l p or not; whether this series converges or not? Second one is, if it is well defined, whether this satisfy all the condition of the metric axiom or not. So, if we look this metric, the first three condition follows immediately. d is greater than equal to 0; obviously, because the mod is there. It is a real value; finiteness will be tested; finite, unless it is convergent, we cannot say this thing is finite, ok.

The finite and real, finite part will be taken care by the Minkowski inequality. Then, d of x y equal to d of y x, if I interchange the position, this will not change. d x y is equal to 0; then, obviously, individual term will be 0 and x i j will be equal to eta j. So, x equal to y and this, vice versa is also true. The fourth property, that is a triangle inequality, again requires the use of Minkowski inequality. So, basically, before going to test, whether this is a metric or not, we first do derive the result for these two inequalities, which are known as the Holders' inequality and Minkowski inequality. Now, here, x and y, these are the points in l p space; x i j and eta j, these are the point in l p space, ok.

So, before going this, let us see the proof. So, first, we will derive an inequality. What is the result is, let alpha and beta be any positive real number, real numbers, numbers and p and q, and let p is greater than 1; define q, such that, 1 by p plus 1 by q is 1; then p and q are called conjugate exponents, exponents. And, obviously, when this condition is satisfied, so, we get, p minus 1 into q minus 1 is 1, which can easily be seen. p q, so, q plus p is p q and immediately, we can get this result, clear. Now, based on it, we, in order inequality is...

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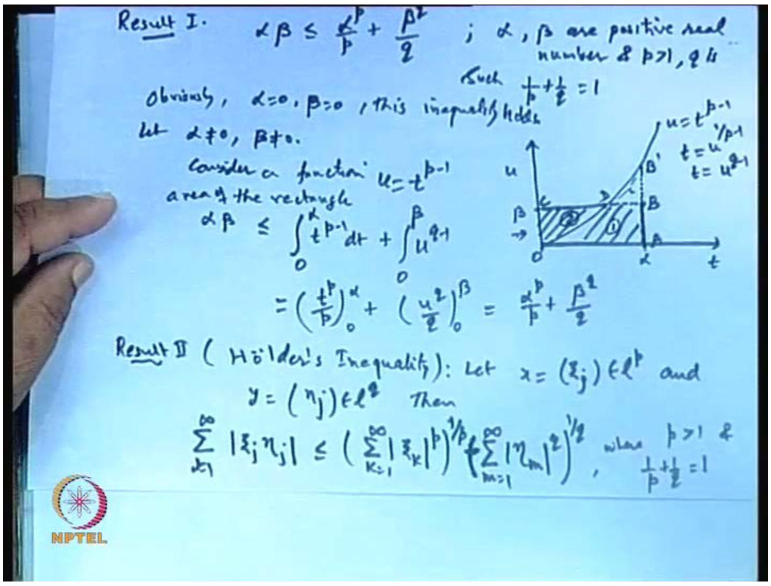

Result. Alpha and beta be any two positive number, then, alpha into beta is less than equal to alpha to the power p by p plus beta to the power q by q, where alpha, beta are positive real numbers and p is greater than 1; q is such that, 1 by p 1 by q is 1. Now, obviously, for alpha is 0 and beta is 0, this inequality hold, holds; it is true, obviously. So, let alpha is not equal to 0, beta is not equal to 0. So, for general, we wanted to... So, suppose, we have a function, say, this is t, this is, say u. So, consider a function u as t to the power p minus 1. So, if we have this function, means, this is a continuous curve, may be like this, u is equal t to the power p minus 1.

Alpha and beta, this is alpha and here is, say beta. So, alpha, beta are the two positive real number. Complete the rectangle. Then, this alpha, beta denotes the area of this portion, like this, area of this portion; that is, the area 1 plus 2, 1 plus 2; these are the two areas. Now, this, alpha beta area will be… So, area of the rectangle, whose sides are alpha and beta, means, this sides are alpha and beta, O alpha, is less than equal to t to the power p minus 1 d t, 0 to alpha, 0 to, say here, alpha, plus q to the power, u to the power q minus 1, 0 to beta. Why, because, this area, alpha beta is the area of the rectangle; rectangle is, if I take this as A, this as B and this as C and this is, say D.

Then, this alpha beta is the area of the rectangle OABC. And, this area is less than equal to the area of this portion, say B dash, OAB dash, which is the area of this part, because the curve is, u equal to t to the power p minus 1, bounded between 0 to alpha. So, the area will be 0 to alpha, t to the power p minus 1 and if I look the function through the axis of x, in this direction, then, this portion, this portion of the curve is nothing, but t to the power, if I take here, then, t is equal to u to the power 1 by p minus 1 which is as good as q minus 1. So, this is the curve, if I look along this direction. So, this curve will be u to the power q minus 1 and bound is 0 to beta; it varies from 0 to beta. So, this area. So, 1 area plus 2 area, this one is less than equal to basically, this portion is extra, so, this one. Now, it is simple integration and when integrate and substitute the value, you get t to the power p by p under the limit 0 to alpha plus u to the power q by q under the limit 0 to beta and that gives you alpha to the power p by p plus beta to the power q by q, ok.

So, this gives you the result and which is valid for any positive real number, for beta and p is greater than 1 and q is the conjugate exponent of p. The second result, which we also called, the Holders' inequality, inequality. What is this, Holders' inequality says, let x, which is x i j belongs to l p and y, which is eta j belongs to l q, where p and q they are the conjugate exponents; then, the product of these two sequence is x i eta i will be in l 1 and we have this inequality, j equal to 1 to infinity mod of x i j eta j is less than equal to sigma, sigma k is 1 to infinity, mod of x i k power p power 1 by p plus, sorry, not plus, multiplied by, multiplied by sigma, sigma m is 1 to infinity, mod of eta m power q power 1 by q, where p and q are conjugate; p is greater than 1 and q is, such that, 1 by p plus 1 by q is 1. Now, this inequality is known as the Holders' inequality and this is valid for all sequences, which are in l p, X and another sequence Y, which are in l q.

So, the product will be in l 1. This is a series, if this right hand side convergent, means, this will be finite. So, it basically, the product of these two sequences x i by a coordinate y, that product will be in l 1. So, that is one, clear. Now, let us see the proof for this first result. Proof of this result, ok. So, in order to prove this result, we will make use of this inequality first, that, the inequality which we have derived, alpha beta is less than equal for alpha p by p plus beta q by q, for any alpha beta are positively n number. So, we will make use of this inequality here.

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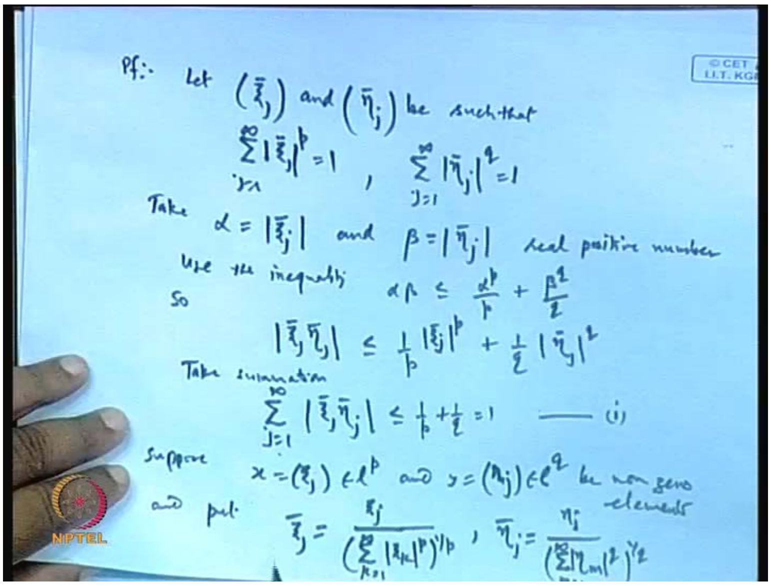

So, let us start. Let x i j bar and eta j bar and eta j bar be the two sequences, such that, the sigma of this mod x i j bar power p, j is 1 to infinity, is 1 and mod sigma of mod eta j bar power q, j is 1 to infinity, is say, 1. Let us take these two sequences and which can be easy one. Suppose, I take x i j to be 1 by 2 to the power 1 by p and p, say, yes, n, p n or p j, then, it will be 1. Sigma 1 upon 2 to the power j by p; so, j will get, p gets cancelled and we get sigma j, sigma j, 1 upon 2 j will be 1. So, such type of sequences are available and we get cancelled. Now, let us take alpha to be mod of x i j bar and beta to be mod of eta j bar, ok. Now, alpha and beta we have chosen. So, both are real, positive number. Hence, we can use this inequality, inequality, that is, alpha beta is less than equal to alpha to the power p by p plus beta to the power, beta to the power q by q. So, substitute alpha beta here. So, we get, mod of x i j bar eta j bar is less than equal to mod of x i j power p, power p by p plus mod of eta j bar power q, ok.

Now, take the summation, take summation. So, sigma of j, j is 1 to infinity, mod of x i j bar eta j bar is less than equal to… Now, when you take the summation, then, sigma of mod x i j bar is 1; sigma of this is 1. So, it will be the same as 1 plus 1 by p plus 1 by q, that is 1. So, what we get it that, sigma j equal to 1 to infinity, x i j bar eta j bar is 1. Let it be say 1. It means, if I choose x i j bar and eta bar, for which this is true, then, it will satisfy the condition 1. Now, taking the advantage of this inequality, we are now in a position to derive the Holders' inequality. So, what we do is, suppose x, which is x i j is in l p and y, which is eta j, eta j is in l q, be the non-zero, be non-zero elements, be non-zero elements in these spaces. And, let us say, and put x i j bar as x i j over sigma, sigma mod x i k power p power 1 by p, where k is 1 to infinity; and, eta j bar to be eta j by sigma eta m mod of this, power q, m is 1 to infinity, power 1 by q. So, if I choose the x i j bar and eta j bar in such a way, then, obviously, this sequence x i j bar eta j bar satisfy the condition that, sigma of this thing is 1; because as soon as see mod, and then, sigma, you will get power p; then, it will comes out to be 1. It means, it satisfy the condition of, earlier condition. Hence, the sequence will be satisfy this condition, must satisfy 1 also. So, we can say this type of sequence x i j bar eta j bar must satisfy 1.

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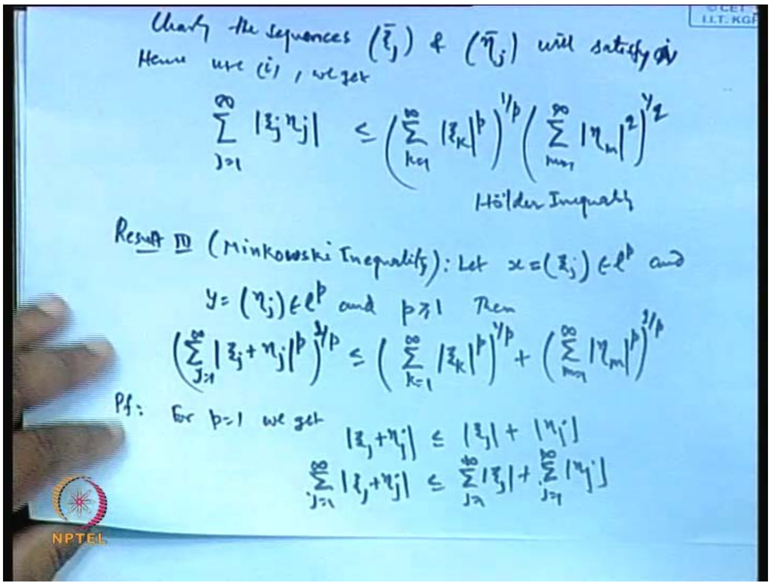

So, clearly, the sequences, clearly, the sequences, x i j bar and eta j bar will satisfy 1, will satisfy 1, or will satisfy, if this to be 1, let it be, say, here is A. So, it will satisfy A. Hence, use 1, i, because they satisfy i, so, we get, we get from here, when you substitute this values in i, what we get it, i is this, sigma of this is one, so, write down this mod of this in sigma. So, sigma of this mod x i j eta j, j equal to 1 to infinity is less than equal to, is it not, 1, sigma k equal to 1 to infinity, mod x i k power p power 1 by p into, is it not, this into this, so, into sigma m is 1 to infinity, mod of eta m power q power 1 by q; and, that gives you the Holders' inequality. Now, one thing we observe here that, when we say the Holders' inequality, then, left hand side we are taking the summation from j equal to 1, whereas, the right hand side the summations are taken from k and from m, and k equal to 1 to m; so, k, with respect to, k equal to 1 to m and with respect to m, 1 to m.

In fact, because the series, x i, sigma of x i k mod x i k power p is convergent (( )), because the sequence x is in l p; this is the sequence x in l p. So, this part will be finite; that is, the sequence x i k will be pth, absolutely pth summable sequence. So, this will be the absolute sum, will be finite. Similarly, eta I, that sequence eta m or eta i, will be in l q, is in l q. So, this will be finite. Therefore, when we interchange the terms, it will not affect the summation, because in case of the absolute series is, converges absolutely, summation remains the same, whether we interchange the positions of the terms. So, we get this. Therefore, it hardly matters, we, whether we take j equal to 1 to infinity, or j equal to 1 to infinity here, or maybe k is 1to infinity or m is 1 to infinity, clear. So, this is correct. In some, one may write also, j equal to 1 to infinity, mod x i j eta j is less than equal to j equal to 1 to infinity of j, x i j power p power 1 by p into sigma j equal to 1 to infinity mod eta j power q power 1 by q. So, that is one, clear. Now, next result, which, third is the Minkowski inequality, Minkowski inequality, w, Minkowski inequality.

So, the Minkowski say, let x, this is x i j, a sequence in l p and y, which is eta j in l, in an l p and p is greater than equal to 1; then, the Minkowski inequality says that, sigma of mod x i j plus eta j power p power 1 by p, where summation is taken over j from 1 to infinity, is less than equal to sigma k equal to 1to infinity mod of x i k power p power 1 by p plus sigma m is 1 to infinity mod of eta m power p power 1 by p, ok. So, basically, this inequalities says that, we can write down the sum of the two sequence in terms of this inequality; x i j is one sequence; y j is another; then, sum of the two sequence have this relation, which is valid for all p greater than 1, equal to 1; this and x and y are in… Now, this is a well defined thing; first, because, when x is in l p, this sum will be finite; when y is in l p, this sum will be finite.

So, the total of this sum will be finite. Therefore, sum of these series will be finite. So, if x is in l p, y is in l p, then, the addition of the two sequence x plus y will be in l p. Hence, as a result, we can say that, l p is a linear space; means, alpha time series also will be in l p, if alpha is finite. So, it becomes a linear space and that is also a justification from this; this one thing. Second part is that, here, we again take k is 1 to infinity, m is (( )), as I told you earlier, that we are free to choose the terms in any fashion; still, the sum will remain the same. So, that is why, there is no loss of generity, even if I take k equa1 1 to infinity or j is 1 to infinity, like this. The proof of this.

For p is equal to 1, the inequalities follows immediately, just using the triangular inequality. For p equal to 1, we get mod x i j eta j is less than equal to x i j plus mod eta j, because of the triangular inequality, and then, take the summation sigma j equal to 1 to infinity mod of x i j plus eta j is less than equal to sigma j equal to 1 to infinity or k is 1 to infinity, mod of x i j plus sigma j equal to 1 to infinity mod of eta j. So, for p equal to 1, the result follows immediately. Then, now…So, let us take the p greater than 1. So, take p greater than 1; p is greater than 1.

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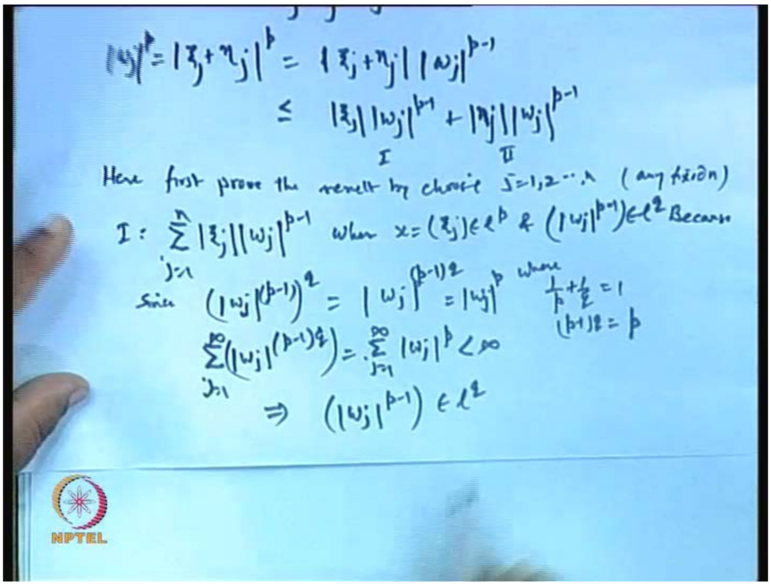

Now, to simplify the formula, let us take x i j plus eta j as omega j. So, mod of x i j plus eta j power p, this will be equal to, same as mod omega j power p; and, this can be written as x i j plus eta j into mod of omega j power p minus 1; take one of the term outside and get. Now, apply the triangular inequality. So, we get mod x i j mod omega j power p minus 1 plus mod eta j into mod omega j power p minus 1, clear. Now, this we get it. So, this is the first sum and second. Now, let it be 1 and 2; separate out this. Now, here, we will take, first for this result, here, first prove the result, (( )) prove the result by choosing j from 1 to n; that is the Minkowski inequality, we will just restrict upto j equal to 1 to n. And then, for n is sufficiently large, we can take it. So, where n is any fixed n, any fixed n, this is our… So, we will prove. So, let us take the, first is, the first part is, sigma mod x i j mod of w j power p minus 1, j is 1 to n, ok.

Now, apply that Holders' inequality. Holders' inequality, j equal to 1 to n, as a particular, when j is 1 to n, rest will be, say after n, is 0. So, we can use the Holders' inequality without any problem, but only thing is, Holders' inequality requires the product of the two sequence, where one of the sequence is in l p, other sequence should be in l q. So, whether this part, is it in l q? That is… Now, obviously, mod of w j power p minus 1; then we raise the power q, then, this becomes mod j power p minus 1 into q; p and q are the ((raised to it)). So, 1 by p plus 1 by q is 1. Therefore, when we take the p minus 1, p minus 1 into q, then, what we get? Just see here, when you take 1 by p here, then, p minus 1 by p, equal to 1 by q.

So, p minus 1 into q becomes p. So, this is basically, equal to mod j power p. Now, sigma of this thing, sigma of this part, p minus 1 into q, j is 1 to infinity, this is the sigma j equal to 1 to infinity, mod w j power p; but w j is a x i j plus eta j, which is in l p, because l p is a linear space. So, this will be in l p. So, the sum will be finite. Therefore, this sum is finite. So, this implies that, the sequence w j mod w j power p minus 1 is in l q. This implies that, mod of w j power p minus 1 is in l q. So, here, we take the sequence, one sequence is in l p; other sequence here, where x in l p and this sequence, power p minus 1 is in l q, because of this; because of the following reason. So, we can, without any problem, we can apply the Holders' inequality to this. So, use the Holders' inequality.

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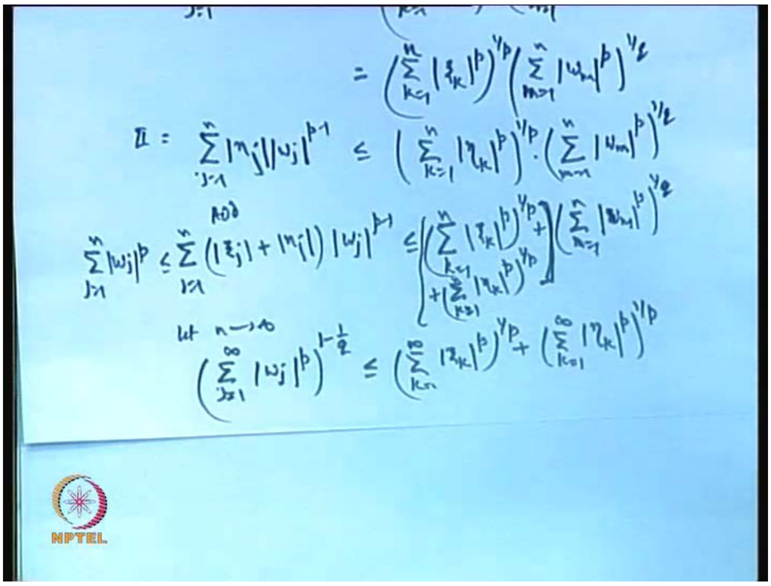

Use Holders' inequality.

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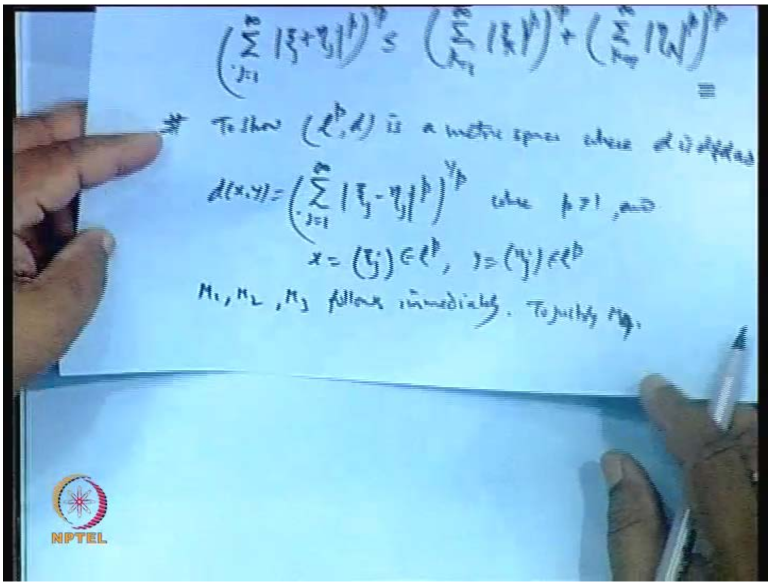

So, we get from here is sigma j is 1 to n, mod x i j mod w j power p minus 1. If I use the Holders' inequality, then, it will be less than equal to sigma mod x i k power p, say, k is 1 to n, I am taking, power 1 by p into sigma mod w j, here, w m, let us be change w m power p minus 1 power q, m is 1 to n and power 1 by q, ok. And, this will be equal to, if I take this is sigma k equal to 1 to n mod x i k power p power 1 by p and we have seen that, p minus 1 into q is p, so, this is the sigma mod w omega m power p, m is 1 to n, power 1 by q. So, this is the first part. Now, second part. In a similar way, for the second, we can use this sigma mod of eta j mod of w j power p minus 1, j is 1 to n. If I apply again, the Holders' inequality with the sum j equal to 1 to n, then, what we get is, this is less than equal to sigma k equal to, sigma k equal to 1 to n mod of eta k power p power 1 by p into sigma m is 1 to n mod of w omega m power p power 1 by q, power 1 by q.

So, if I add them, then, we get these two term; addition is here. So, what we get is, sigma sigma x i j plus eta j, this is x i j plus eta j into power mod w j power p minus 1, j equal to 1 to n, is less than equal to sigma sigma k is 1 to n, mod of x i k power p power 1 by p plus sigma m is 1 to n, power, omega m power p power 1 by, sorry, this is x i k plus eta k, is plus eta k. So, this will be eta k, eta k, I am putting here, ok. So, plus sigma k, k is 1 to n mod of eta k power p power 1 by p and this entire thing is multiplied by sigma omega m is 1 to n omega m power p power 1 by q, because this is common. Now, this part, if you go through the previous slides, then, basically, this is the w j power p. So, we can say, this entire thing is a greater than equal to sigma w j power p or omega j power p, j is 1 to n, ok.

So, take this towards this side. So, take the summation and we get from here is, divide, let n tends to infinity; let n is tending to infinity. So, when we take n tends to infinity, summation will not differ and we get sigma j is 1 to infinity mod of omega j power p and here, when you divide, 1 minus 1 by q is less than equal to sigma over k 1 to infinity mod x i k power p power 1 by p plus sigma k is 1 to infinity mod of eta k power p power 1 by p. ok. But 1 minus 1 by p is 1 by q; 1 minus 1by p, whenever 1 minus 1 by q is 1 by p. So, we get from here is… So, sigma j is 1 to infinity mod of omega j power p power 1 by p is less than equal to sigma k is 1 to infinity mod x i k power p power 1 by p plus sigma k is 1 to infinity, 1 to infinity, mod of eta k power p power 1 by p, power 1 by p, ok.

And, this will be nothing, but that omega j x i j plus eta k. So, we get the inequality, j is 1 to infinity mod of x i j plus eta j power p power 1 by p is less than equal to sigma k is 1 to infinity mod x i k power p power 1 by p plus sigma k is 1 to infinity mod eta k power p power 1 by p; and, that is nothing, but the Minkowski inequality, clear. Now, this inequality also gives the guarantee that, this series is convergent. Because this is convergent, this is convergent and this is convergent. And further, it is also used to justify that l p is a metric space under this. So, now, to show, to show l p under that metric d is a metric space, where d is defined as, defined as d of x y as sigma j is 1 to infinity mod x i j minus eta j power p power 1 by p, where, where p is greater than 1 or equal to 1; and x equal to x i j belongs to l p; y, which is eta j belongs to l p. Now, how this follows is… So, first three, M 1, M 2, M 3 follows; immediately, 4, M 4, to, to justify M 4, what we do is, we consider this.

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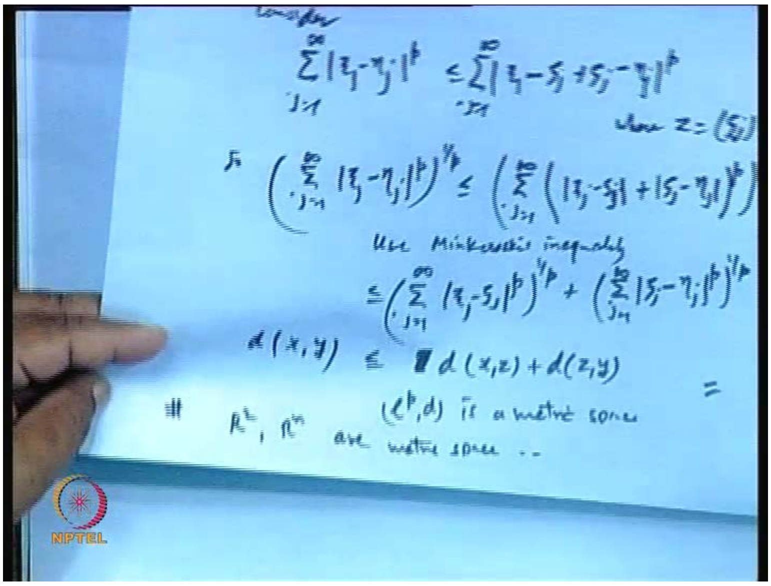

So, consider, mod of x i j minus eta j. Now, this will be equal to, power p; then, this is less than equal to mod x i j minus, say zeta j, plus zeta j minus eta j, ok. Then, power p, p is greater than 1; p is greater than 1. Now, this will be… Now, take the sigma of this, j equal to 1 to infinity; take the sigma j equal to 1to infinity. Now, take the power, where z, which is zeta j belongs to l p. Now, take the power 1 by p; power p power 1 by p. Now, this is less than equal to sigma j equal to 1 to infinity mod of x i j minus zeta j plus mod of zeta j minus eta j power p power 1 by p; that is right. Now, apply the Minkowski inequality. This is as good as that two sequence x i, x i plus y i, mod of x i plus y i power p power 1 by p. So, you can use the Minkowski inequality, inequality. We get, this is less than equal to sigma j is 1 to infinity mod of x i j minus zeta j power p power 1 by p plus sigma j is 1 to infinity mod of zeta j minus eta j power p power 1 by p, ok.

And, this will be equal to, this is, the left hand side, this is the metric d of x y; this is less than equal to metric, metric d of x z plus metric d of z y; that is, the triangle inequality is followed. So, this proves the l p space under d is a metric space. Now, as a consequence, we can also say, as a consequence, R 2, R n, these are all metric spaces, under the metric defined earlier; under the metric defined earlier. So, we need not to go in detail, but this will follow. So, this will be.

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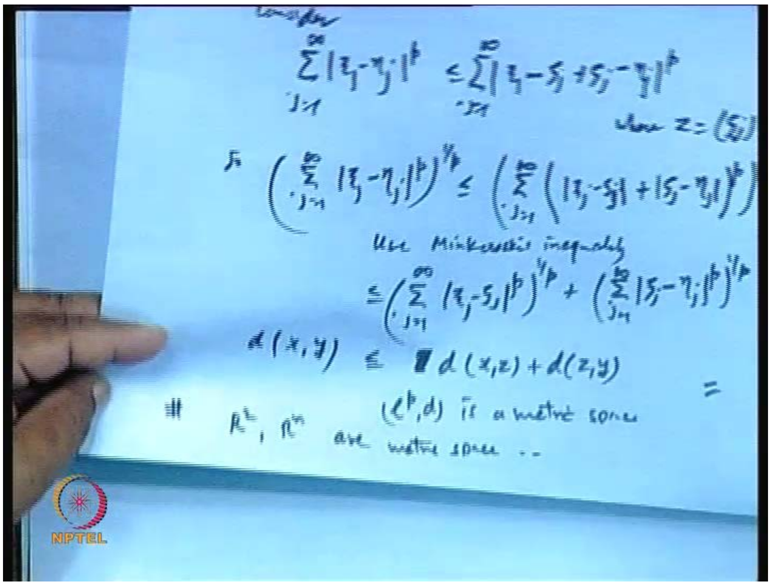

Now, we have one more concept here, that is, the concept of the sub spaces. So, let us say, define subspace. First, let X d be a metric space, metric space and Y is a non empty subset of X, is a non empty subset of X. And, suppose, suppose, d bar is the restriction of d on Y cross Y; what is the meaning of restriction? This is our metric space X d, d is a distance function and Y is a non empty subset of this. So, here, the elements of Y is, say y 1, y 2, like this. Now, when we choose the points in Y and if the distance between y 1 and y 2 under the metric d bar, is the same as the distance under d, then, we say, d bar is the restriction of d; that is, d bar y 1, y 2 is the same as d of y 1, y 2, for y 1, y 2 belongs to capital Y; then, this is the restriction of this, clear.

So, we have this. For example, if we take this sequence. Suppose, A is the subset of, l infinity, we have seen, l infinity is the set of all bounded sequences and then, if we take this example, let us say, if A is the subspace of l infinity, consisting of all sequences of 0s and 1s, ok. Then, the induced metric on A is nothing, but the discreet metric; because, what is the discreet metric? Discreet metric, if we take the discreet metric, say d, d d, x y is 0, if x is equal to y, and 1, when x is not equal to y; it is as good as the supremum of mod x, mod of x i i minus y i, oh, sorry, eta i, where x is x i i and y is eta i, y is eta i, ok. So, because this, these two are 0s, equal, then, it will 0; otherwise (( )). So, discreet metric becomes the induced metric point. Thank you. That is all.