Functional Analysis

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Module No. # 01

Lecture No. # 17

Dual Basis and Algebraic Reflexive Space

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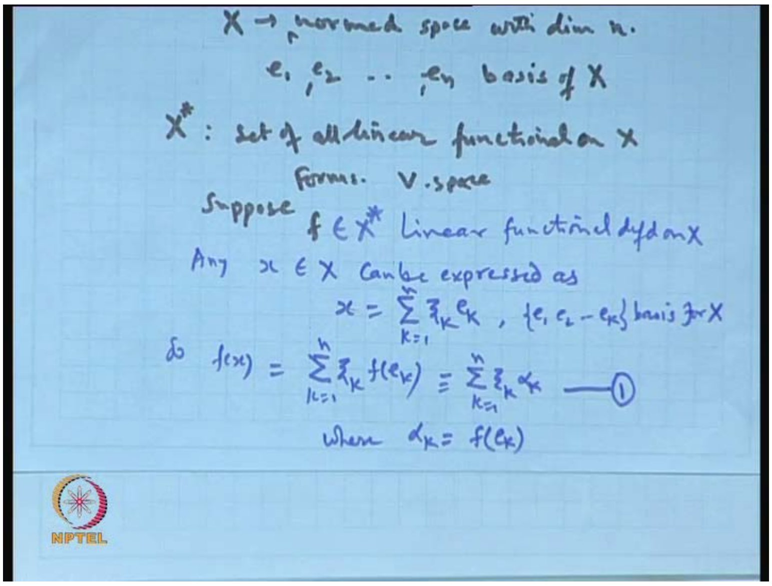

So, we will continue the same topic, that is, linear functionals on a finite dimensional case, first. And here, we will find out what is the dual basis for X, where X is our normed space, normed space and the dimension is, finite dimension, normed space with dimension n ok. So, dimension of this, well, normed space is n and let e 1, e 2, e n, these are the basis of X, basis of X. As we know that, all the linear functional, set of all linear functionals defined on X, this we denoted by X star and, in fact, it forms a vector space under the suitable addition and scalar multiplication; this we have discussed.

So, suppose, suppose f is, f is an element of, say X star, f is an element of this X star, that is a linear functional, linear functional defined on X, a linear function defined on X. Any element x belonging to capital X can be expressed as, in terms of this basis element xi k eta, x i k e k where e 1, e 2, e n are the basis for X. f is a linear functional defined on X. So, f of x, this will be equal to sigma k equal to 1 to n, x i k f of e k. Now, f is a functional. So, f of e k will be a, some scalar value. So, let it be, this equal to sigma x i k alpha k, k is 1 to n, where alpha k, where alpha k is the value of the e k under f; this much we can say; e k, these are the basis element where e ks are, these are the basis element, basis for X. So, alpha k is the value of the function at the point e ks. Now, if we look the equation this, say 1, then, f can be uniquely determined by the values of alpha k at the n basis element.

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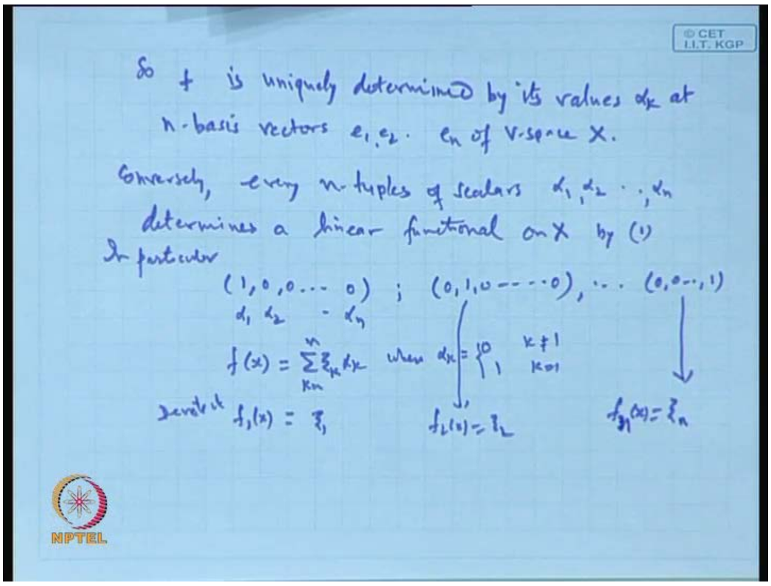

So, we can say from here, alpha. So, f is uniquely determined, determined by its values alpha k at the base n basis vectors, vectors e 1, e 2, e n of X, is it clear. It means, if this alpha 1, alpha 2, alpha n are given or the values are e 1, e 2, e ns are known, because it is a basis for this. So, any x can be expressed uniquely this; this representation is unique. Therefore, the value of f will also be unique and it will depends on alpha k, the value of e k, uniquely determined by this. If I think a converse direction, suppose, alpha 1, alpha 2, alpha n are given, then also, we can find the f, clear. So, for any choice of alpha 1, alpha 2, alpha n, one can obtain f. So, let us take the particular case of alpha 1, alpha 2. So, conversely, every n-tuples of scalars alpha 1, alpha 2, alpha n, determines a linear function, a linear functional, determines a linear functional on X by the equation 1, is it not; by 1, because this is the equation 1 and here, if we know alpha 1, alpha 2, alpha n, then, we can…

So, in particular, let us take this way; in particular, suppose, I take this scalars 1, 0, 0, 0; it means, this is one set of scalars, that is, this is alpha 1; this is alpha 2 and this is alpha n. We can get, similarly, if we take this scalar, what will be the, our f of x? f of x becomes sigma x i k, k equal to 1 to n and alpha k will be, this alpha k, where alpha k is 0 when k is equal to, k is not equal to 1 and 1, when k equal to 1, in this case. So, basically this is nothing, but what? x i 1 and this is equal to f 1 x; let it be denote this by, is it ok. Similarly, if suppose, I take another set of a scalar 0 1 0 0 0, then, correspondingly we get, f 2 x as x i 2 and continue this. So, we get from 0 0 0 1, the corresponding n x will be x i n, is it ok.

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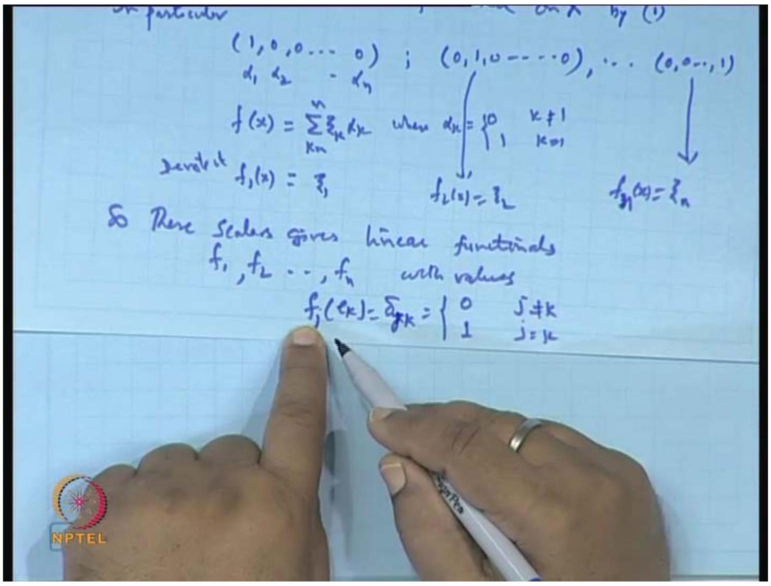

It means, these set of values, so, this set of scalars… So, we get… So, these scalars gives linear functionals f 1, f 2, f n; f 1, f 2, f n with the values like this, that f j e k is nothing, but the chronicle delta, delta j k which is equal to 0 and 1, if j is not equal to k and j equal to k, is it not. When we say f j e k, when f 1, f 1 means this, or 1, alpha 1. So, it is equal to 1; otherwise 0. So, this will give the corresponding… It means when e 1, e 2, e ns are given, we can construct this f 1, f 2, f n with the help of this scalars, ok.

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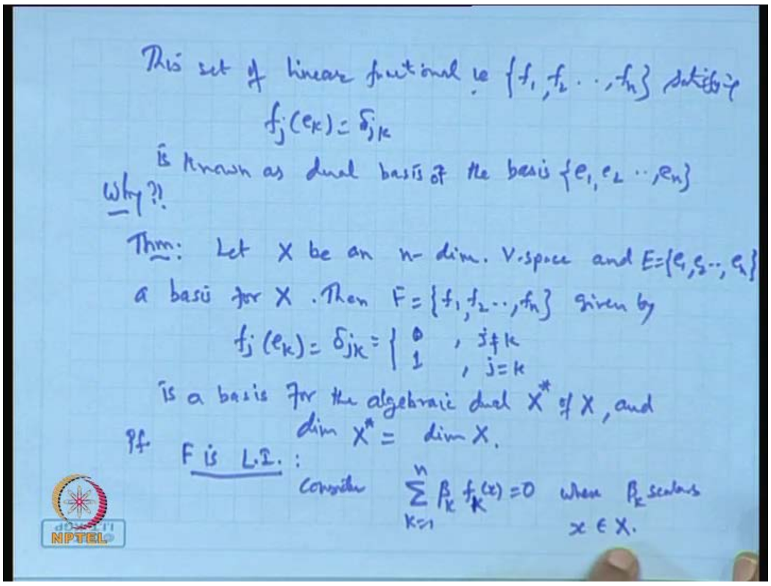

Now, this set of, this set of linear functionals, that is, this set f 1, f 2, f n satisfying that condition f j e k is the chronicle delta j k is known as the basis or dual basis of, dual basis of the basis e 1, e 2, e n, is it ok. So, this is the dual basis of n. But why it is known as the basis, because basis means, it should be linearly independent and it should respond the whole class. So, dual basis of this. So, in fact, we will show, this is why; the question is why. It means, unless we prove that, this collection f 1, f 2, f n is a linearly independent set and it ((responds)) any linear functional, then only, it will be the basis for X star.

So, we do this in the form of this theorem. What this theorem says that, let x be an n dimensional, let X be an n dimensional vector space and e, which is e 1, e 2, e n and e which is e 1, e 2, e n a basis for X; then, we claim that, collection of these element f 1, f 2, f n, say capital f, this set given by this way f j e k is the chronicle delta delta j k which is equal to 0, 1, when j is not equal to k and 1, when j is equal to k, given by this e j basis for the algebraic dual algebraic dual X star of X; and not only this, the dimension of X star will be the same as dimension of X, ok.

So, this is true in case of finite dimension. If X be a finite dimensional space, then, we can find out the algebraic dual for it and the dimension of X star and dimension of X will be same. Let us see the proof. So, we wanted to show that, this class F is a basis for dual X star; it means X star is the set of linear functional. So, this set must be linearly independent and it should be span X star. So, first is, F is linearly independent; this we wanted to show. So, let us consider this X, consider this expression, sigma beta k f k x equal to 0, k equal to 1 to n, where beta 1, beta k are scalars and f k 1, f k 2, etcetera defined like this and x is element of X, is an element of capital x, ok. Now, since x is an element of X, so, x can choose any value; e 1, e 2, e n, these are the basis elements of X. So, x can take e 1, e 2, etcetera.

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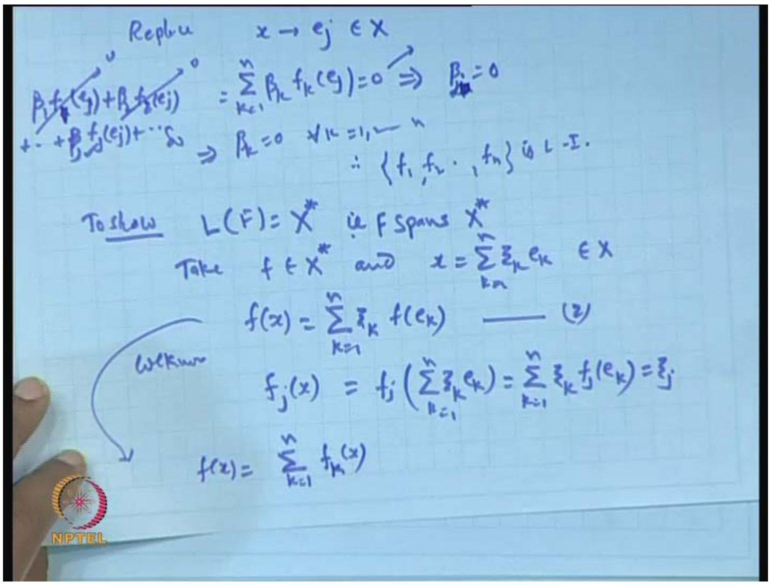

So, put x, replace x by x, replace x by e j, which is also an element of X. So, what we get? So, we get from here is that, sigma k equal to 1 to n, beta k f k e j equal to 0; this will implies that beta j must be 0; because f k e j will be 1 and rest will be 0; chronicle delta. So, because f k e j, it is satisfying this condition; f k e j is delta j k. So, it is 1, otherwise, it is 0. So, beta j is 0. It means that, if I keep on replacing x by e 1, e 2, e n, then, all these coefficient beta 1, beta 2, beta n are coming to be 0, is it ok.

Sir, K equal to then…

K is equal to 1 to n. So, when you take here, f k e j, when k equal to j, it is 1; otherwise, 0. So, it will be equal to beta j or beta k; any, beta k or beta j; when k is equal to j, this is 0; otherwise it is 0. So, beta k 0. So, So, we say all beta k will be 0, for all k, 1 to n. Therefore, this set f 1, f 2, f n is linearly independent.

Sir, beta k will be 0, provided, sir, it is not equal to, k is not equal to zero; if k is equal to j, then, it will not be equal to zero.

If, there are two; this case, how will you… Suppose, I expand this thing, what you get it or let it be expand this one, beta 1 f k e 1. Now, f 1e j, then, beta 2 f 2 e j; like this, beta j f j e j and so on, like this. So, it will be beta j. So, now, this part will be 0, this part will be 0, only this part will be 1. So, only beta j will be left out and rest will… So, beta j will be j, because this is equal to 0; total sum is 0, but here, only one part is left. So, beta j will be 0; it means that, beta k will be 0 for all case similarly. So, this is linearly independent, is it ok.

Now, next, to show, to show that, L of F is X star; that is, the set F spans, F spans X star; means, if we take any arbitrary element of X star, that can be expressed as a linear combination of the elements of F. So, let us take any f. So, take f belongs to X star, clear. So, and x which is sigma x i k e k, k is 1 to n is an element of X. So, f of x will be equal to what, sigma x i k, k equal to 1 to n and f of e k, is it ok or not, f of e k. Now, let it be, this one equation, let it be 2. Now, we know that, f j x, what is f j x, f j sigma x i k e k, k equal to 1 to n, which is equal to sigma k equal to 1 to n, x i k f j e k, because f is linear. Now, f j e k is chronicle delta. So, when j is equal to k, it is 1; otherwise 0. So, basically, this is equal to what, x i j, is it ok or not? Clear. Again, in a similar way, this rest of the part will be 0 and only one term will be left, which is x i j.

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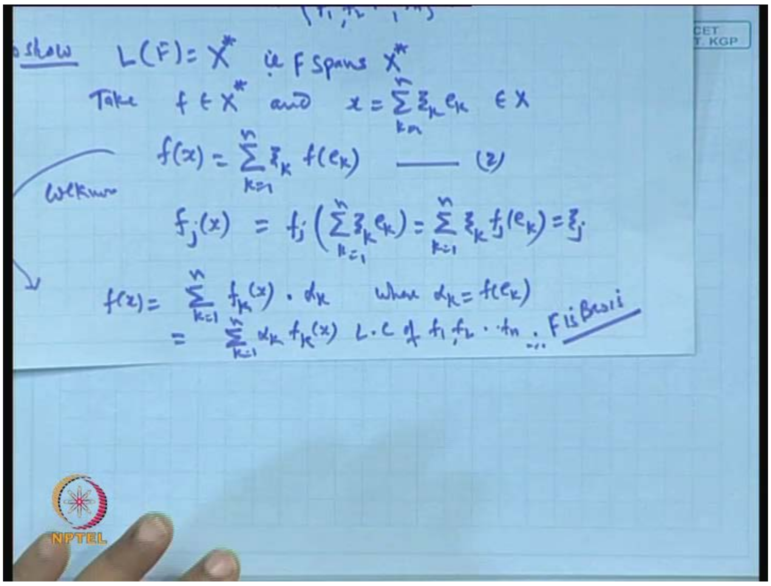

So, x i j can be replaced by f j x. So, in equation 2, if I take f x sigma k equal to 1 to n, x i k, can you not write this f k x; and, this f of e k is nothing, but a constant alpha k, a scalar quantity, clear. So, what you can get it, that f can be expressed as linear combination of the element of f k, a linear combination of f 1, f 2, f n. Clear? Therefore, F is a basis. Therefore, F is a basis, is it clear. So, nice. Now, this result, not only this, once it is basis, so, any element F can be expressed as a linear combination of f 1, f 2, f n and these are the basis. So, dimension of this will be n; only n terms. So, X star and dimension of X will be the identical, clear. So, that is all.

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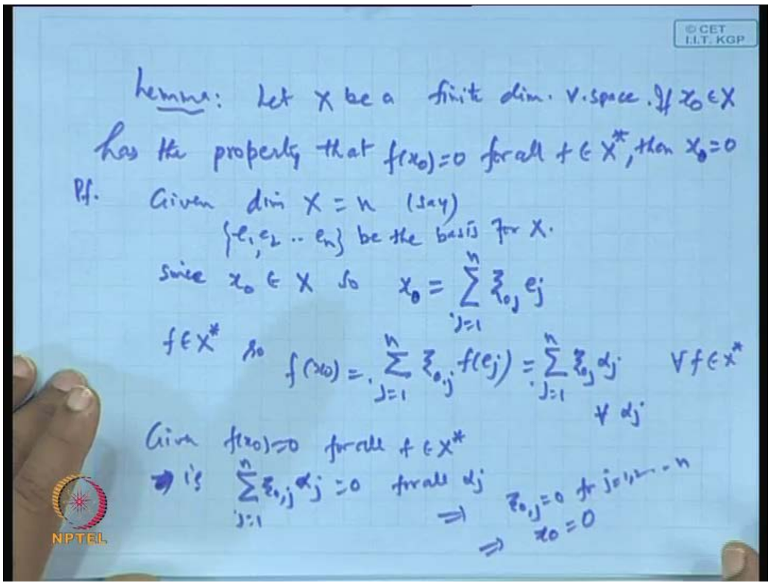

I think it is ok. Now, we have a very interesting result, that is in the form of lemma. What this says, let X be a finite dimensional, finite dimensional vector space. If x naught belongs to capital x, if x naught belongs to capital x, has, has the property that, that f of x naught is 0, for all f belonging to X star, then x naught must be 0, then x naught must be zero. So, what is this property? Suppose, X be a finite dimensional vector space and x naught be an arbitrary point on it; if the image of this point under f is 0 for all linear functional defined on X, then this is only possible, when the point x naught must be 0; means, there is no nonzero point, whose image will always be 0, for all f linear functional; it is not possible.

Only possible is that, if f is a 0 functional, 0 functional, then, image of any element will be 0; but here, x is fixed, point is fixed; we are choosing the images under different f. So, there is a possibility that, few image may go to 0, but all these, for all such f, if it is 0, it means, the point x naught has to be 0. So, that is the result, is it clear. The proof is again simple. Dimension of X is given. Given dimension of X, say n and e 1, e 2, e n be the basis for X. So, X can be greater and x naught is a point in, x naught is a point in X. So, x naught can be expressed as a linear combination of this basis element. So, we can write as x i naught j e j; why naught, because of this 0 and j is equal to 1 to n, ok.

Now, f is linear function; f belongs X star. So, f of x naught, this will be equal to sigma j equal to 1 to n, x i naught j and f of e j, but f of e j is alpha j, which is j equal to 1 to n x i naught j and alpha j. Now, as we have discussed already that, if e 1, e 2, e n are given, then, f can be uniquely determined by its value at these point; or if alpha 1, alpha 2, alpha n are known, then one can identify f; one can get the value of f. So, for each f means, if, if we keep on changing alpha j, this will correspondingly put any f, belongs to x naught, any f belongs to X star. So, if this is true for every f, belongs to X star, this is also true for every sequence of a scalars alpha j. You follow me? f determines alpha j. So, when you change f, alpha j will change; but f is any arbitrary f. So, it means you are getting a plenty numbers of alpha j.

So, if we say that, this result is, say, keep on changing alpha j, if this is true for this, all alphas; now, what is given is that, this f of x naught is 0, for all f belongs to X star; this is given. It means, that is, sigma of x i naught j, j equal to 1 to n, e alpha j equal to 0 is given, for all alpha j, is it not. For any f means, for all alpha j also; for any alpha you put. So, if a linear expression is 0 for any values of alpha j, this is only possible, when x i naught j must, must be 0. So, this implies that, x i naught j must be 0, for j equal to; otherwise, it is not possible. This combination, total sum cannot be 0, for any range of alpha j, alpha 1, alpha 2, alpha n. So, this only, this implies our x naught will be 0 and that is completes the result; I think it is ok.

Sir, this is from the first equation.

Yes, from the first equation, this is, yes.

As, because x of zero j is zero for all j, so, the x zero…

Earlier one?

Earlier one, that will be zero.

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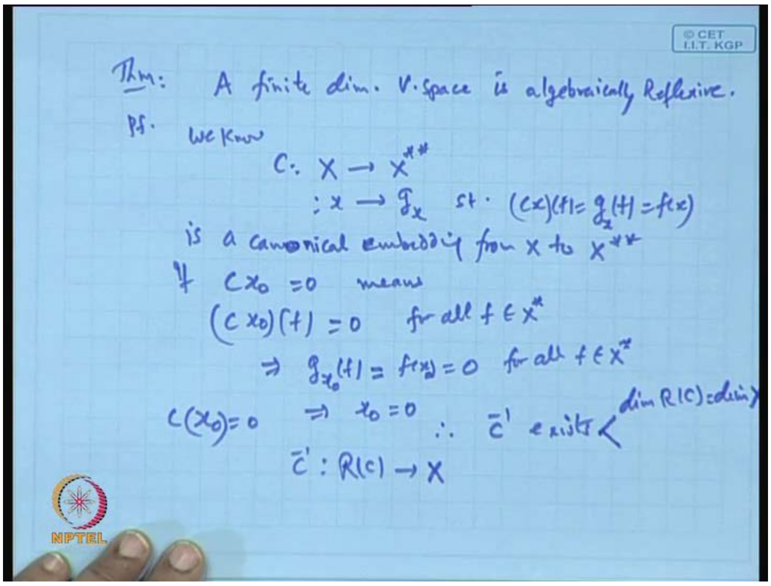

Clear. So, this is another, now, interesting is that, in case of the finite dimensional, a finite dimensional vector space, a finite dimensional vector space is algebraically reflexive, a finite dimensional vector space is algebraically reflexive. It means, if the second dual of this coincide with itself, or there is a mapping, canonical mapping from X to X double star, which is one-one onto and preserved all the operation, and then, images will be the identical; then, we say, it is a reflexive mapping. So, in case of the finite dimensional, if X be a finite dimensional vector space, then, it should be a reflexive space. We know that, a mapping C from X to X double star, this transfers x to g of x, such that, C x f equal to, c x f equal to g of x f equal to f x, is it not; that way we have defined, canonical embedding is an…Canonical embedding, canonical embedding embedding, this we know, is it not. So, if we take C x naught and it is one-one mapping and R C is a subset of this, canonical embedding, from X to X double star, from X to X double star; this I, already we had discussed.

Now, if C of x naught f is 0 means, means C of x naught f is 0 for all f, for all f belongs to X star, clear. So, this implies that, g of x naught f, which is equal to f x is, f x naught is 0 for all f belongs to X star; but we have seen that, f of x naught is 0 for all f, implies x naught must be 0. So, if C x naught is 0 means, C x naught f is 0. It means, for all f, this is true. So, f of x naught 0. So, C x naught equal to 0 implies x naught is 0. So, therefore, C inverse exists; that result is, C inverse will exist. Once c inverse exists, C inverse is a mapping from the range set of C to X, range set C to X; and not only this, when it exists, so, we get from here, the dimension of R C is the same as dimension of X. As soon as inverse exists, both will have the same dimension; but here…

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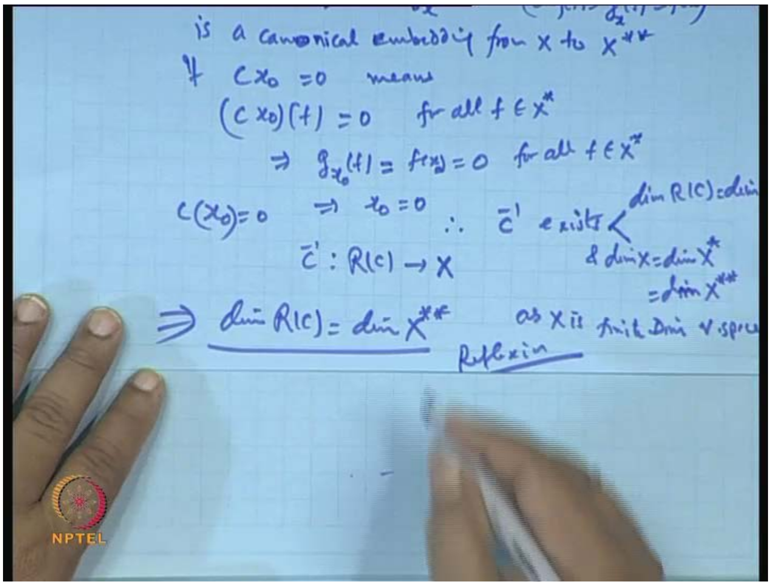

But, if e 1, e 2, e n is a finite dimensional case, so, in case of the finite dimensional, dimension of X and X double star is the same. So, since X is finite dimensional, dimension of this equal to dimension of X, which is equal to dimension of X double star, as X is finite dimensional vector space. So, once it is finite dimensional vector space, this will also be true. Dimension of X equal to dimension R C. Therefore, from here, dimension of R C is the same as dimension of X double star, clear. So, this shows that, the C, which is a mapping from X to X double star, is a one by one to mapping and it preserves the operation as well as dimension of R C, and dimension of X star is the same, which is the same as the dimension of X. So, this shows that, our space, finite dimensional space is reflexive. So, this shows the reflexive.

Sir, X is reflexive.

X is reflexive. X is reflexive. I think so…

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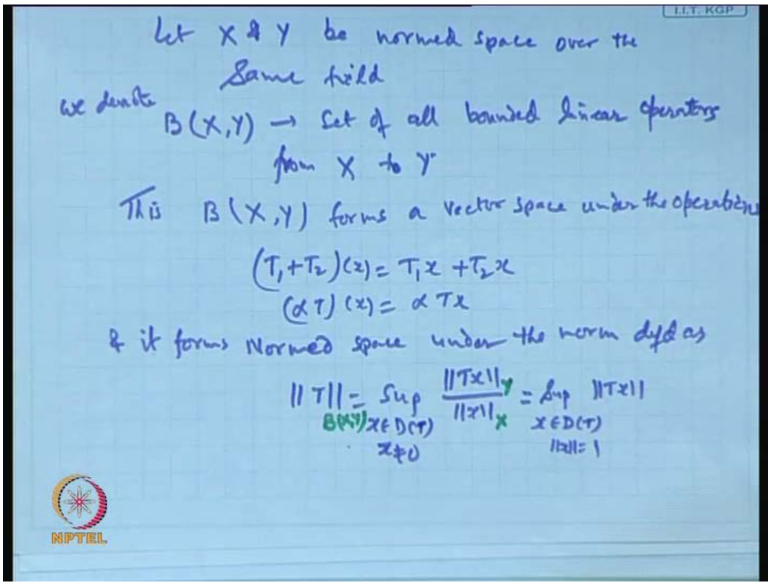

Now, so far, we have seen the linear functional on the finite dimensional case. Now, we are switching off to a arbitrary case in finite dimensional normed space. So, let X and Y be any infinite dimensional space, be a normed space; need not be a finite dimensional over the same field, over the same field. We denote B X Y, we denote B X Y as the set of all bounded linear operators, bounded linear operators from X to Y, bounded linear operator from X to Y. Now, this B X Y forms a vector space under the operations, under the operations T 1 plus T 2 x is T 1 x plus T 2 x and alpha time T x equal to alpha of T, x; that usual, just like a linear functional; similarly, here the operators.

So, in a similar way we can show, this is a… And, it is a normed space and it forms normed space under the norm, defined as norm of T equal to supremum norm of T x over norm x, x belongs to the domain of T, where x is not equal to 0. Under this norm, it will be, we can prove this is a norm, clear. And, this same definition, if you remember, this is also equivalent to, say, supremum of norm T x, where x belongs to D T and norm of x is 1; both are equivalent definition. So, this gives you the norm. It means, B X Y becomes a norm, under this.

Now, let me just clarify one thing. This is the norm on B X Y. So, we can say, it is, the norm of T is on B X Y. This is the norm on Y; this is the norm on X; but writing always this way, is a very clumsy. So, we try to avoid it; but understanding is, whenever the T x is given, it means, it is the point in Y. So, the norm of Y is used; here, the norm of X is used and this is the norm of the operator from X to Y, a bounded linear operator from X to Y, clear. Now, question arises, under what condition this B X Y becomes a Banach space? Because it is a normed space, we can prove all the condition of norms are satisfying. So, B X Y is a normed space, fine. B X Y is a norm, but whether it is a Banach space or not and what should be the restriction on X and Y, so that, B X Y, under the same norm, becomes a Banach, that is a complete normed space.

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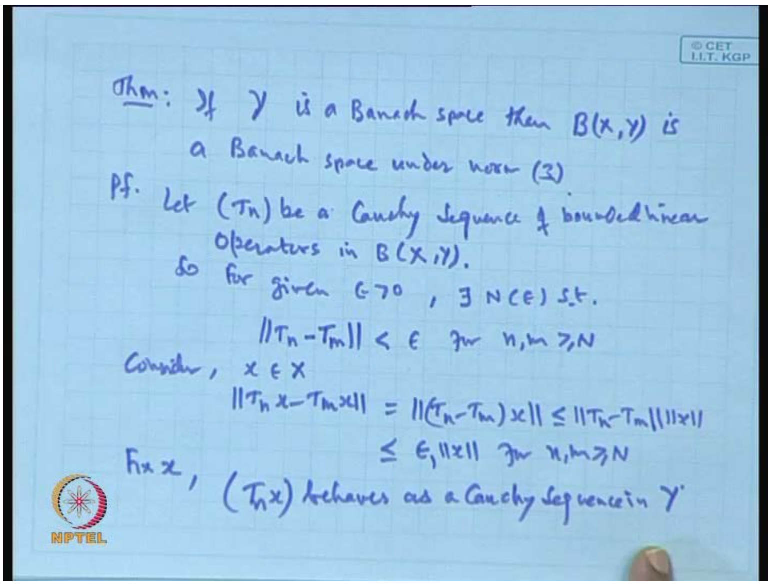

So, we will show that, when Y is Banach, then, B X Y is a Banach space. So, if capital Y is a Banach space, is a Banach space, then, the set of linear operator from X to Y is a Banach space, under the same norm Y, under this norm, say 3, under 3; this is 3, under the norm 3. So, what is required to show here, if Y is a Banach means, you have to choose; B X Y you wanted to be a Banach. So, it means, you consider a Cauchy sequence of the operators, bounded linear operator on B X Y and if that Cauchy sequence converges to an operator, which is also bounded and linear, and convergence is in the norm of B X Y, then, it is a complete normed space.

So, let us choose T n be a, be a Cauchy sequence of bounded linear operators of B X Y, of bounded linear operators in B X Y, bounded linear operator in B X Y. So, T n is Cauchy means, so, for given epsilon greater than 0, there exists an capital N, depending on epsilon such that, norm of T n minus, norm of T n minus T m, this will be less than epsilon, for n m greater than equal to capital N, by definition, clear. So, consider now, norm of T n x minus T m x, where x belongs to capital X. Consider this. This can be written as norm of T n minus T m x. Now, T n is a sequence of bounded linear operator. So, T n minus T m will also be bounded linear operator. So, it will be less than equal to norm of T n minus T m into norm of x.

But, as n m goes to infinity, the norm of T n minus T m goes to 0. So, basically, this is less than equal to, say, epsilon 1 into norm x, for n m greater than equal to capital N; is it ok now? So, suppose I fix x. Once you fix x, this quantity is fixed. So, we can write epsilon 1 into norm x as epsilon 2. So, T n x minus T m x is less than epsilon 2 for n m greater than equal to N; it means, for fixed x, this sequence T n x behaves as a Cauchy sequence in Y. Is it not?

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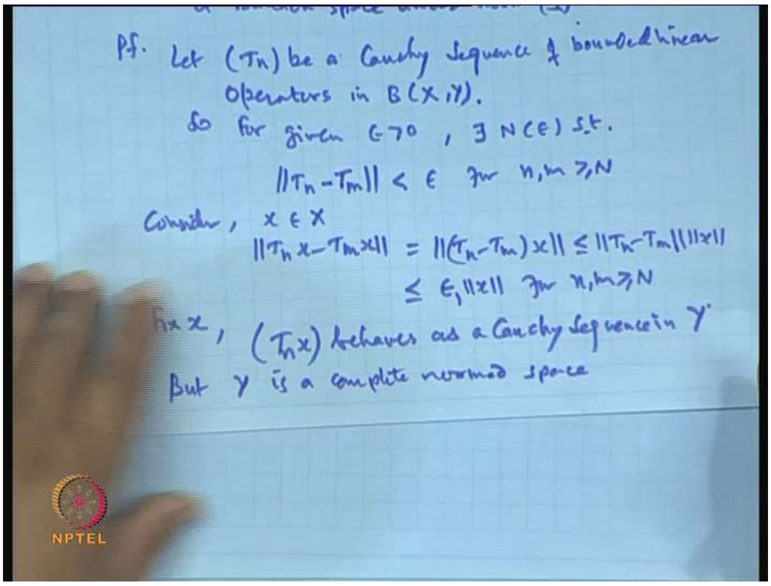

Because, T n x is the image of x under T n, so, it is a point of Y; and difference, any arbitrary difference is less than epsilon. Therefore, for fixed x, it is a Cauchy sequence in Y; but Y is complete, but Y is complete, but Y is a complete normed space, because it is Banach. So, every Cauchy sequence must be convergent.

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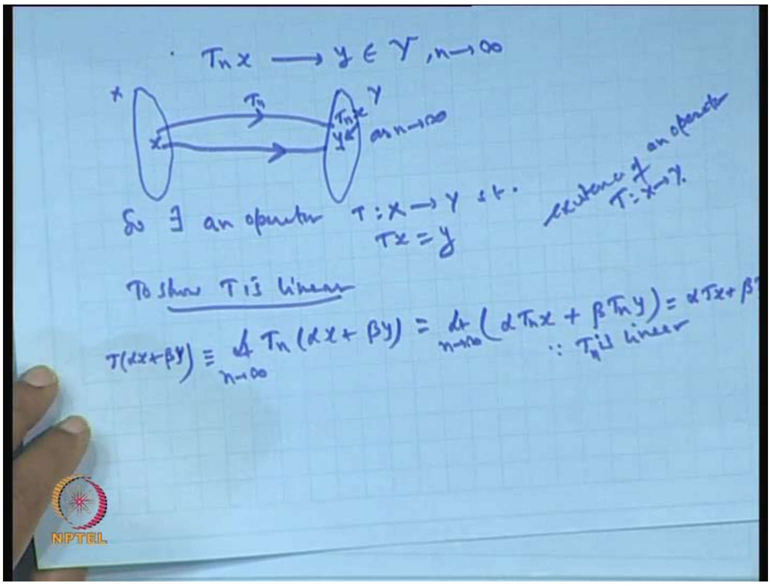

So, this Cauchy sequence, this, note it. So, the sequence T n x will converge to a point y in capital Y, is it ok or not, as n tends to, this sequence will converge to N. So, now, what we get. So, it means, that is, this is our X; this is our Y; T is an operate; this is an operator T n. Now, here, you are taking x. So, image of this T n x is coming to be y. Then, n tends to, this is equal to T n x. So, this image is going to y as n tends to infinity, is it not. Do you follow me? It means that, this y is obtained because of x fixed. So, there must be an operator, which rate directly to x to y. So, there exist an operator T, which transfer from x to y, such that T of x equal to y, is it clear? y.

As n tends to infinity.

As n tends to infinity…

We are taking so many operators...

Yes, limiting of this values T 1, T 2, T n, this x is coming to be y.

(( ))

So, it means, there must be an operator which can transfer x directly to y. So, an operator exist, T and y. Now, this operator, it may not be bounded, may not be linear, is it not. So, we have to show, if it is bounded and linear, then only, it will belongs to the class B X Y, clear. So, now, to show T is linear, and upto here, we have shown the existence of T. This shows the existence of an, of an operator T from X to Y. This is the only thing we have shown so far. Now, if I further prove that, T is linear as well as bounded, then, it must be a point in B X Y. Now, T is linear can be proved like this. Limit of this T n alpha x plus beta y as n tends to infinity.

Now, since T n is linear, so, because of the property of the linearity, this can go, alpha T n x plus beta T n y; because T is linear; T n is linear. And then, limit, when you take the limit here, so, finally, you get, alpha T x plus beta T y, because limit of T n is T y, is it not. We are identifying limiting of T n is T y. So, here is also, T of, this can also be written as T of alpha x plus beta y, is it ok or not; because limiting value of T n, we are denoting by T.

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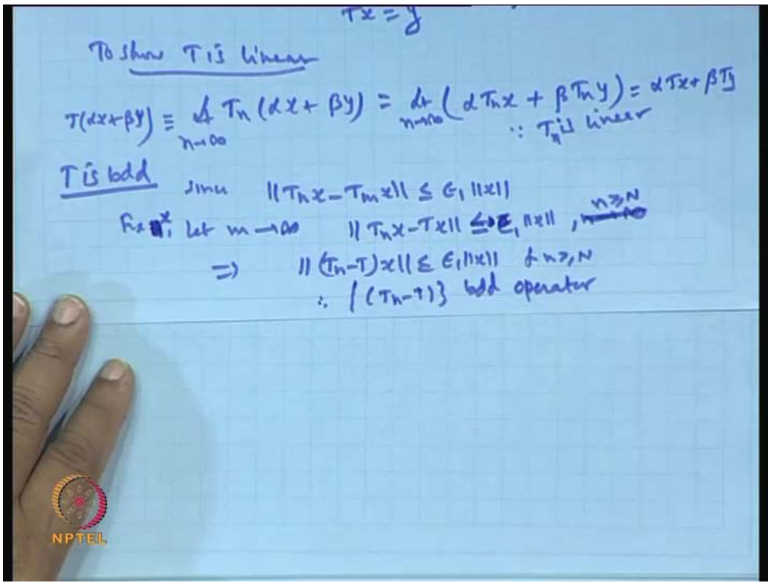

So, basically, T satisfy this condition. So, T is linear, is it clear. Now, T is bounded; this I will do it from here; since norm of T n x minus T m x is less than equal to epsilon 1 into norm x, this we have shown, ok. So, now let us take, fixed x and let m tends to infinity. So, if m tends to infinity, then, this implies, T n x minus T x will tends to 0, is it not. Now, sorry, this is less than equal to, I am sorry, this is less than equal to epsilon 1 into norm x, norm x, is it correct or not. By this, only m tends to infinity; limiting, limit is a continuous function. So, it will go inside; this will not be affected; it will affect, this T m will go to T, so, as n tends to infinity, like this.

Now, from here, can you say T n minus T x is less than equal to some epsilon 1 norm x, for n greater than equal to capital N, is it ok. Now, this shows…No, here fix n; n and x both and m tends to infinity; then only fix, no fix x, fix x and m tends to infinity; this I am not touching. So, here, you can say, n is greater than equal to capital M; it is ok, now. Now, this is… So, can you say, this sequence T n minus T, this is a bounded operator, by definition. Norm x; so, some constant. So, it is a bounded operator.

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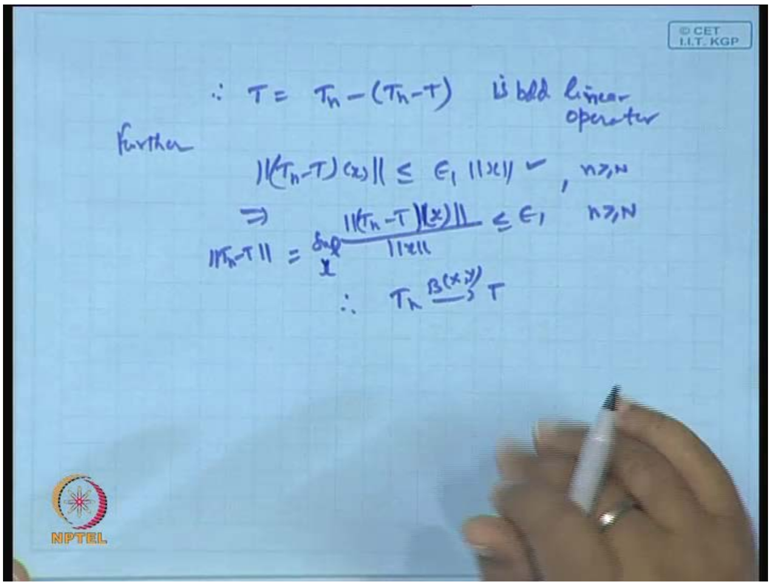

So, once it is a bounded operator, then, you can say, therefore, T which can be expressed as T n minus T n minus T is a bounded operator. This is a bounded linear operator, because this bounded linear, this is bounded linear. So, combination of the two bounded linear operator; so, T will be bounded linear operator. Is it clear or not? So, T is not only linear, it is also bounded. Now, further, norm of T n minus T x, this is less than equal to epsilon times norm x. So, from here, we get norm of T n minus T.

Which one?

T n minus T, we have shown bounded; T n is also given to be bounded. So, two operator is the sum of the bounded operator are bounded. It is given; T n is the element of B X Y. So, it is bounded linear operator. So, these two are bounded linear; therefore, T is bounded linear. Now, this will be equal to, if I…Norm of t n x, what is the norm of t n x? This is the supremum or take the supremum over all x. So, it is bounded. So, we get the supremum of norm of t n x is finite; take this norm of… I think, I will just write. This is divided by norm x; take supremum over it; is it not less than this, which is equal to norm of T n minus T, as n greater than equal to capital N.

Sir, we are taking supremum over x.

Over x. Over x.

Are these condition is valid for all x?

For all x; supremum is taken for all x. And, this is true for n greater than equal to N. So, this is also true, for n greater. Therefore, T n converges to T in the norm of B X Y, because, norm of T n minus T goes to 0. So, this. So, this completes the proof. That, supremum of this divided by x; you divide this by x. So, norm of T n minus T x over norm x; supremum is taken over x. Now, this is less than equal to epsilon, when n is greater than equal to N.

Sir, this is there by definition.

By definition, which one? This equal to this, by definition, because, norm of T is the supremum norm T x over norm x; so, by definition. So, this is, is equal to this, which is less than. So, this shows, when n is sufficiently large, T n will go to T. Therefore, T n converges in the norm of this, clear. So, this completes the proof.

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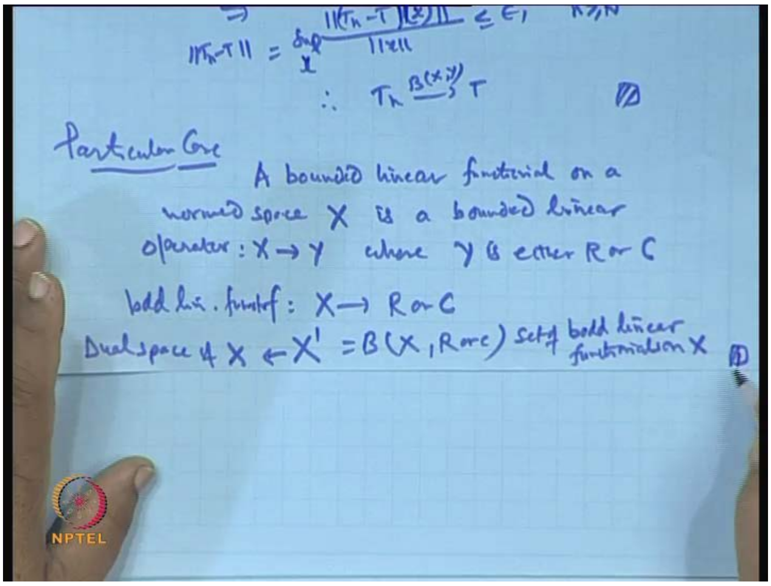

Now, as a…So, this is complete. Now, as particular case, if I take the functional, a bounded linear functional on a normed space X, is a bounded linear operator from X to Y, where Y is either R or C; a bounded linear functional is a bounded linear operator, where the range set becomes the field of a scalar, either R or C. So, basically, f is a mapping from x to R or C. This is a bounded linear functional, bounded linear functional. So, bounded linear functional is a particular case of the bounded linear operator, where the domain remains the same; range becomes either R or C; but R and C are complete, ok.

Now, this is, we denoted by, so, B X R or C, this we denoted by X dash, set of all bounded linear functional, functionals on X; we denote it by X dash and is called dual space of X, is called the dual space of X, is this clear, denote it by X dash. So, first we have defined the bounded linear operator; then, replacing Y by R or C, we are getting a bounded linear functional and that collection is called the dual space. And, it will be complete, because R and C will always be complete. So, in the next class, we will see the examples of the dual spaces. Thank you, thank you.