Functional Analysis

Prof. P. D. Srivastava

Department of Mathematics

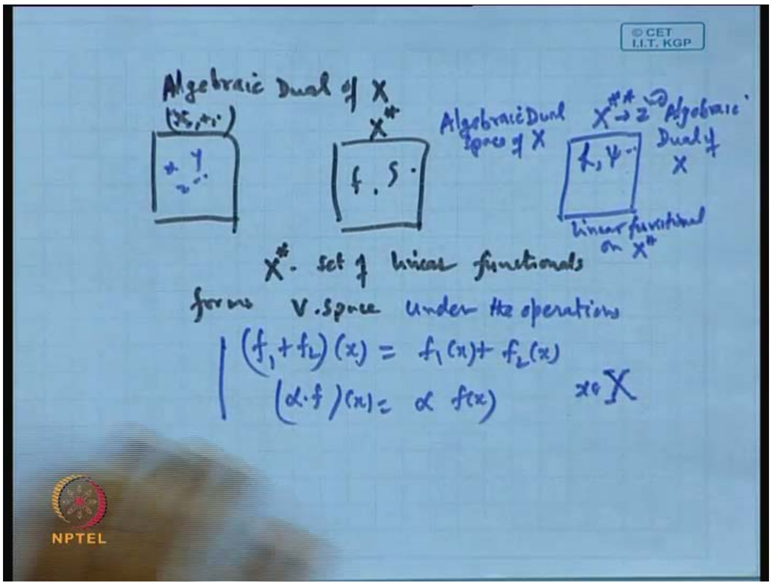
Indian Institute of Technology, Kharagpur

Module No. # 01

Lecture No. # 16

Concept of Algebraic Dual and Reflexive Space

(Refer Slide Time: 00:16)



Algebraic dual of a vector space X and in fact, what we have defined like this, that, if X be a vector space, then, set of all linear functional defined on X, we denote this X star. And, this set of all linear functional, linear functional it forms a vector space, vector space under the operation, under the operation, addition and scalar multiplication, defined as f 1 X plus f 2 X, f 1 plus f 2 X is f 1 X plus f 2 X and alpha times f X is alpha of f x. So, with respect to the addition and scalar multiplication, it again forms a vector space. All the properties, all the conditions of the vector space are satisfied, with respect to these operations. Hence, this space will be vector space.

f 1 plus f 2 X is f 1 X plus f 2 X and alpha dot f means alpha into f X. Now, this space, which is a vector space, we call it as a algebraic dual space of X, or the algebraic dual of X. Now, since, it is again a vector space, so, we can further think of, we can further think of a set of all linear functional defined on X star. So, let it be that, linear functional is h, psi and so on. These are the linear functional, linear functional on X star. And again, in a similar way, we can show that, this class of all linear function defined on X star, again forms a vector space. So, this is also a vector space and we call the second algebraic dual of X, algebraic dual of X, clear.

The question arise, what is the need of considering these second dual or first dual at all? Is there any purpose of writing this thing or just simply taking the linear functional and making a vector space, then, again I will write, linear function making vector space is no use, unless, they have a certain value. The relation between X and X double dash star, X double is an important part. So, we would like to establish certain relation between X and X double star and that will tries the vector space X, ok. So, our aim is to how to relate, how to find the relation between X and X double star; that is the our aim, first thing.

The elements of this small x.

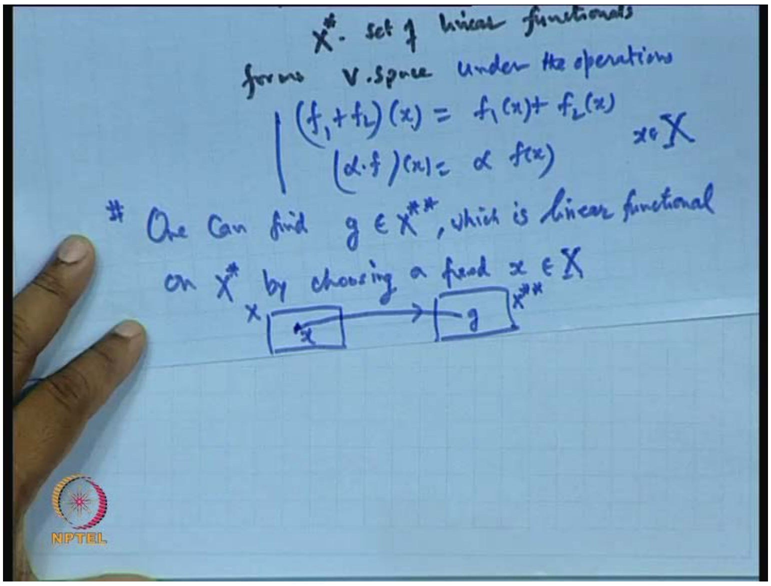
What is the elements of this capital X, sir.

This capital X, this capital X, is it not. You are talking about this.

Small x.

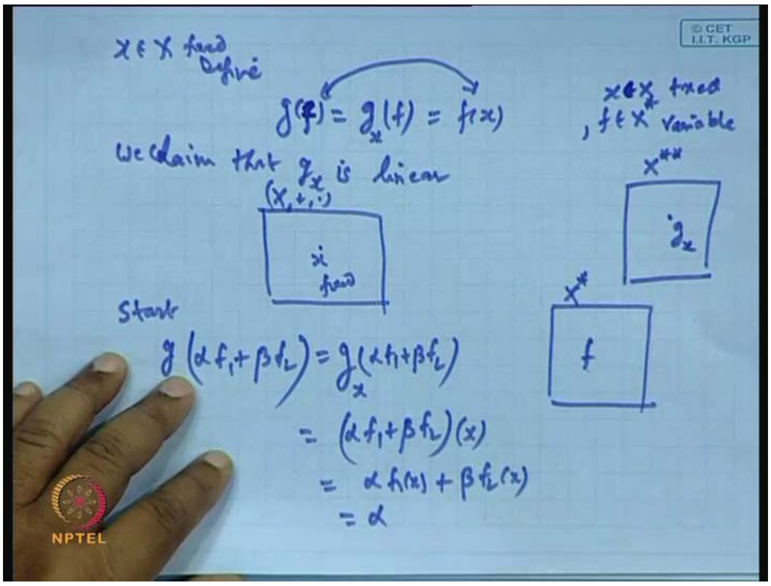
This small X is an element of X. These are the points x, y, z, these are the points in this space. In fact, I have, taking the same operation here; here also, I am taking the same operation; here also, I am taking the same, but it may be different, difference on this, ok.

(Refer Slide Time: 00:16)



So, we are interested in finding the relation between the original vector space X and the second dual X double star, so obtained, ok. Now, one thing is clear that, one can find, one can find an element g, one can find an element g, belonging to say, double star which is, which is linear functional on X star, by choosing a fixed point x belonging to capital x. So, what is this claim is that, suppose, I fixed x here; this is our X and I fix a point x here. Then, corresponding to this, one can obtain an element g, belonging to X double star, we can find a g corresponding to this x. How?

(Refer Slide Time: 06:33)



Let us see, how does it hold. We define, let us define g of x as, g of f, sorry, g of f as g x f, because x is fixed; x is fixed, is it not? x is fixed. We are defining g of f equal to g x f, which is equal to f x. x is given; x is fixed; we are interested in finding the g. So, let us say, g is an element of X double star. So, domain of g will be the elements in X star. So, f is an element in, f belongs to X star. And, this will be a variable one, while the x belongs to capital X, it is a fixed element, is it clear now. We are choosing a fixed element and then, we wanted to take an element, define a, get an element g in X double star.

So, the domain of g will be X star. So, let us say, f be an element belonging to X star which is variable one; because g can take any value at any point. So, g of f and this we are writing as g x, g suffix x f which f x, is it ok. Now, we claim that, this g x is linear.

That f belongs to X star.

f belongs to X star.

g belongs to.

X double star; that we are showing; that we wanted to show.

Sir, how can that g f be all time be equal to f x?

No, this is our definition. this is our definition. In fact, this will be a point, here we are taking x; this is and here is X double star, this, like this; this is our X double star. So, if I take x here, we wanted to find the point g here, related to x. So, it means g will depend on X, clear. How to get this g? We have to define in such a way function, so that, it becomes… Because, in order to get the g in X double star, g x must be a linear functional; because x, on this, must be linear functional on X star. This f belongs to X star.

But, how can this g f be again equal to f.

That is why I am saying, x is fixed. We wanted an element g in f star; we do not know whether g is linear or not; just any arbitrary function, I am just taking any g; we define the g as g f equal to f of x. This is our definition, one can define, why; because, x is given and f is a point in the X star, here all f. So, what is the domain of g? Domain of g will be the X star or lies in the X star, clear. Is it ok?

g f, it lies in X double star.

g f is a value. g f is a value at the point f. It is simply a value; it is, maybe scalar quantity, because it is a functional.

g f of x will be f x, clear. This g f has no value, unless it is given at the point x, because, here it is g f, but at what point x. So, you have to get the, starting with the x; we do not know this part; this is not known, ok. But since, we wanted the g to be in X double star, there must be a domain of g must be X star. So, dummy points are f, variable value; g will range over any arbitrary point on X star. So, we are taking g of f, ok. Now, g of f, this will come over here.

g of f means, again f x.

So, f x means, which belong to X star.

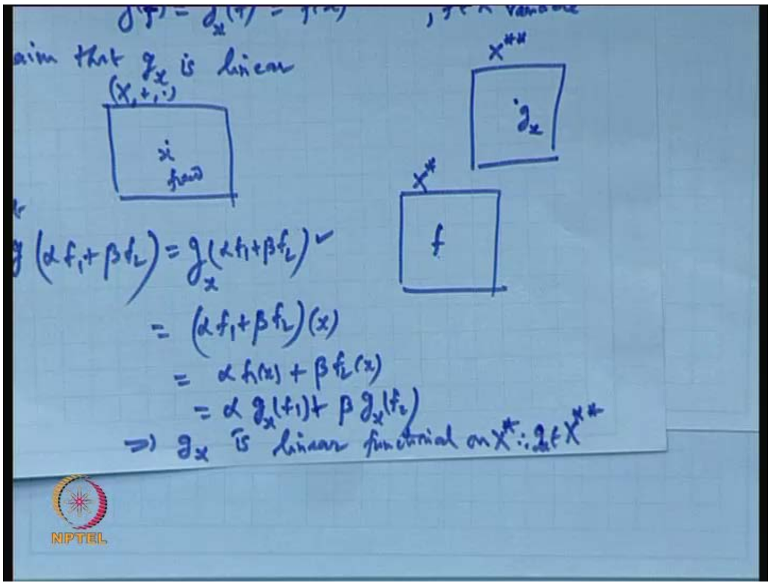
f x will be the X star; that will be the point; we will see. Let us see, what, where does it go? Where does it go? g x f. So, if we take this one, suppose I define this way. Basically, this is our definition; g f equal to f of x, because f is known, x is known, ok. Now, we claim that, g x is a linear or g is linear. So, what is to be done is, g is linear means, you start with this alpha f 1 plus beta f 2, is it not. If g of this equal to alpha times g f 1 plus beta times g f 2, then, it is ok; then, is it ok or not. Now, this will be equal to, this is g x alpha f 1 plus beta f 2 and this, by definition, it will be nothing, but the alpha f 1 beta f 2 and x; is it clear now.

So, this will be a scalar; remember, this is f 1, f 2, f n, what; the linear functional on x. So, f 1 x, f 2 x, these are the scalar. So, this will be a scalar quantity, ok. Then, this will be alpha f 1 x plus beta f 2 x; is it ok? This completely, this will be a scalar and this is equal to alpha f 1 x plus beta f 2 x, clear. Now, by definition, this can be written alpha g.

Sir, this scalar belong to…

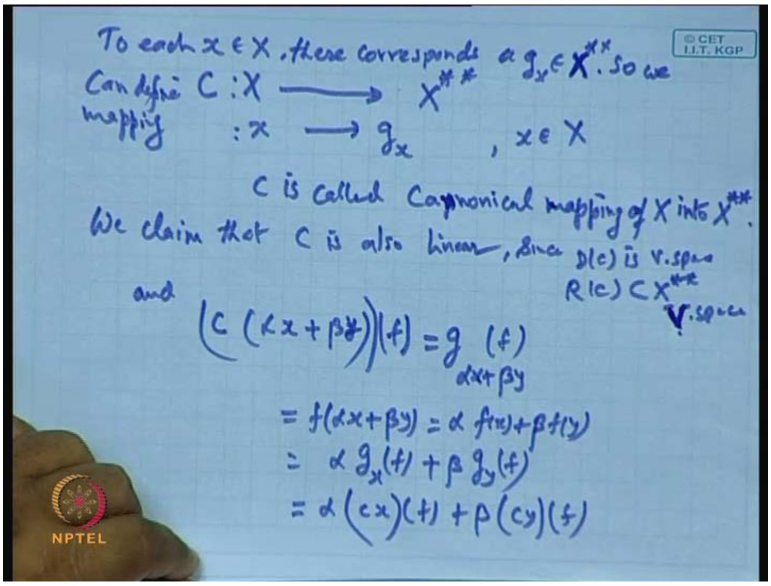
x, scalar though is a field, in the field of this vector space. Scalar is a, vector space is the set of vectors, whose components are a scalars. So, they are the field, either real or complex. They do not belongs to a… They are the elements of the field of the vector space.

(Refer Slide Time: 06:33)



So, alpha g of, this, we can write like this, plus beta g of x f 2, is it not. So, we are getting this; clear. And, that is nothing, but what? This shows that, g x is linear functional, g x is linear functional, on which, on which space; because that elements are f 1 and f 2 belongs to class X star. So, on X star, ok. So, it means, the g, which I have defined in this fashion, basically, comes out to be point of X double star. So, this belongs to X double star. Therefore, g x belongs to X double star, is it not? Clear?

(Refer Slide Time: 15:08)



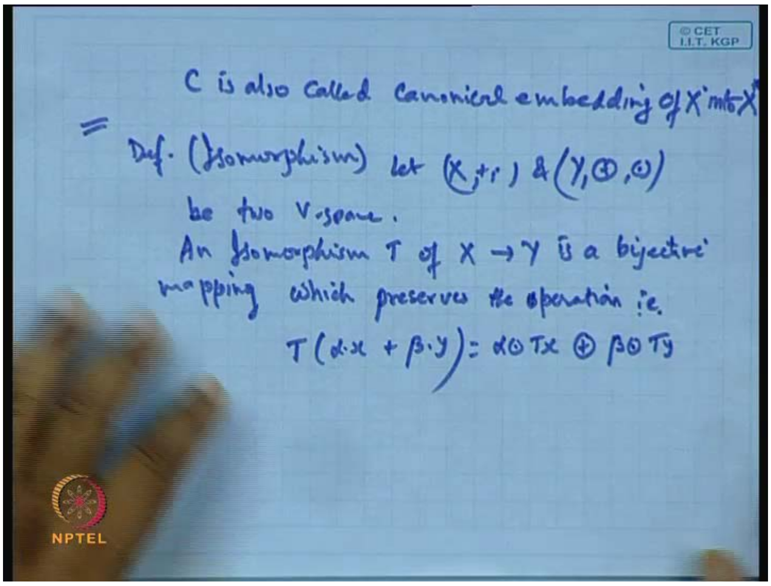
So, what, now, the picture is like this that, we are having X; we are having X double star and a mapping one can define, which transfer x to g x, is it not. So, to each x belonging to capital X, there corresponds a g x, belonging to X double star, such that, this happen, corresponding to each x, there exist a g x belonging to X double star. So, so, we can define a mapping C, from X to X double star, which carries the point x to g x, where x is an element of X, is it not. It means, from X, element of X, we are getting directly to the element of X double star; clear? And, this mapping is called canonical; capital X is a vector; small x is also is an element of X. So, it is a vector quantity.

Vector quantity.

Yes, but when you write the f of x, it becomes a scalar. So, it is called Canonical, Canonical; this is called Canonical a mapping of X; Canonical mapping of X into X double star, is it clear. So, this is…Now, this mapping C, so obtained, is also a linear mapping. The C, we claim that, this C is also linear, since, its domain is a vector space and the range lies in the vector space; domain is vector space, because X and length range lies in X double star, which is also vector space. So, since domain of C is a vector space and the range of C lies in a vector space and, and if we take the condition of linearity, alpha x plus beta y…

So, take any two point in x, capital X, x and y; find their linear combination. So, C of alpha x plus beta y, clear. Subset R C is a subset of X double, here. V vector space; this is a vector space, clear. Now, we wanted to show the C is linear. So, take the two point x and y, belonging to capital X and take the linear combination; the image of this under C; but it cannot go directly to X double star. So, first, it will transfer to where; it will take the help of X star; elements of the X star. So, f. Now, this by definition, this comes what, x image, because C x goes to g of x. So, this will be equal to g of alpha x beta y f, is it clear. And then, by definition, it will be equal to f of alpha x plus beta y, which is alpha of f x plus beta of f y and which can be written as, alpha g x f plus beta g y f and that will be equal to alpha c x f plus beta c y f; check it, ok.

(Refer Slide Time: 21:14)



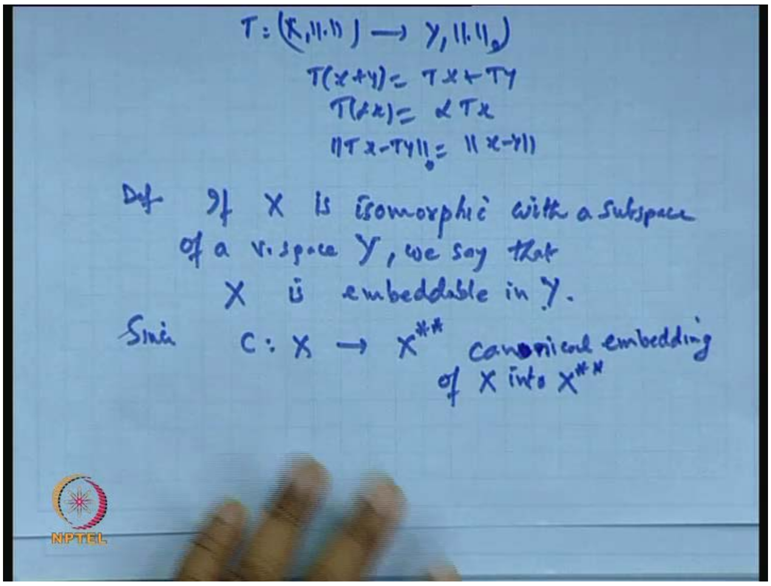
So, this one. So, what this shows, this shows the C is linear; this C, which we have called the canonical mapping of this, we also used another word a Canonical embedding. C is also called the Canonical embedding. C is also called Canonical embedding, Canonical embedding of X into X double star; X to X double star, clear; X to X double star ok. So, we have got this thing from this one; from X to X double star, we are able to introduce a mapping, which transfers every point x to the point of X double star and this mapping is a into mapping; so, range of R C lies in X double star. Now, what is the relation between R C and X double star? That simple; if R C equal to X double star, then, we call it the space to be a reflexive space; that we will come back; that is our purpose.

Now, in this entire process, we require some definition of the isomorphic, isometric and so on. Because we are dealing with the spaces, suppose two spaces are discussed, if their metric structure is the same or their, the operations are scaling the same way, then, only the points are changed; then we consider, these two space metrically equivalent metrics, equivalent spaces. So, ((where)) the concept of isomorphism… So, we require the definition of isomorphism and this definition, of course, it will vary from structure to structure; if we take this vector space, suppose x and y are the two vector space, then we say, let X and Y, say, this be the two vector spaces; then, isomorphism, an isomorphism T of X to Y is a bijective mapping, which preserves the operations.

That is, the T of alpha x plus beta y. Here, this is the addition for vectors and dot is the corresponding scalar operation, scalar multiplication and, if it is equal to alpha dot T x plus beta dot T y; if this is the same as this, which preserve the bijective mapping, which preserve the operation T x plus T y equal to T x plus and alpha dot x is equal to this one, ok, then we say, it is an isomorphism, one on one to, ok. x belongs to X, y belongs to Y. Similarly, in case of this operation, similarly, if X, d and X bar, d bar be the two metric spaces, then, then, an isomorphism, an, an isomorphism T of a metric space X d to X delta d delta, is a bijective mapping; bijective means, one-one onto mapping, bijective one-one mapping; bijective mapping; subjective and injective, one-one onto mapping, mapping which preserves the distances, distance; that is d bar T x, T y is the same as d of x, y where x and y belongs to capital X, ok.

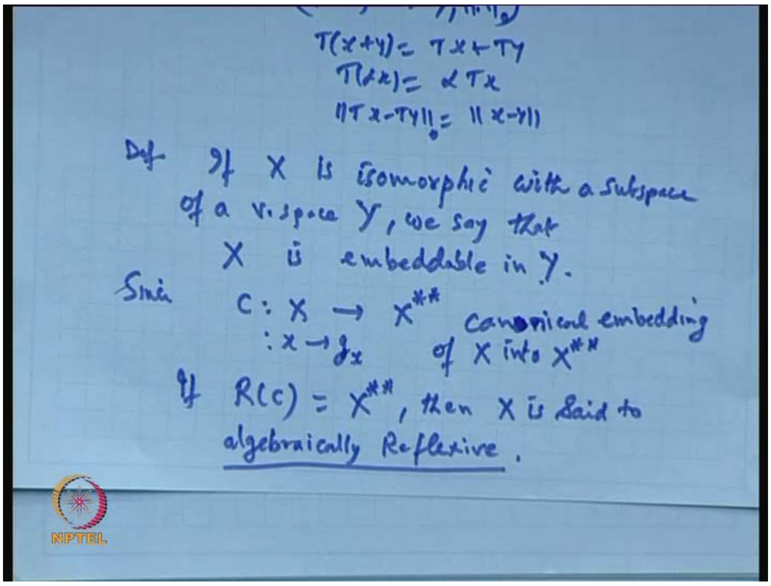
If it is a normed space, it means, it is a vector space together with the norm. So, in case of the normed space, this condition must be satisfied as well as the distance or the norm is preserved; norm of X T x, T y, T x minus T y is the same as the norm of x minus y. So, in case of the normed space, the isomorphism defined in a similar way; means, all these is operations, the algebraic structure with the operation, whatever it maybe, if they are preserved under the metric T, then, we say T is a isomorphism. x and y both are the elements of X; because, this T is a mapping from X to Y. So, both are the element of X; image is all in Y; T x, T y they are in Y; here also same x and y, in case of the normed space we say, is it not.

(Refer Slide Time: 28:15)



So, suppose T is a mapping from normed space, then, what happen is, X norm to Y norm dash 0, then, it is a preserve the operation, the T of x plus y equal to T x plus T y; T of alpha x equal to alpha of T x and norm of T x minus T y is the same as norm of x minus y, 0, where norm is also written, clear; is it clear. Now, we take another definition of embeddable. If X is isomorphic, if X is isomorphic with a subspace, with a subspace of a vector space Y, if X is isomorphic with a subspace of a vector space y, then, we say that, X is embeddable in Y, embeddable in Y, clear. So, in the above, since C, which we defined as a mapping from X to X double star is a canonical mapping, canonical mapping from X to X double star, we also called this as canonical embedding. We also called this is as canonical embedding of X into X double star, in a similar way. Canonical embedding of this, clear. Because, we are mapping C, which is a linear mapping; preserves the operations; and, they get Canonical embedding.

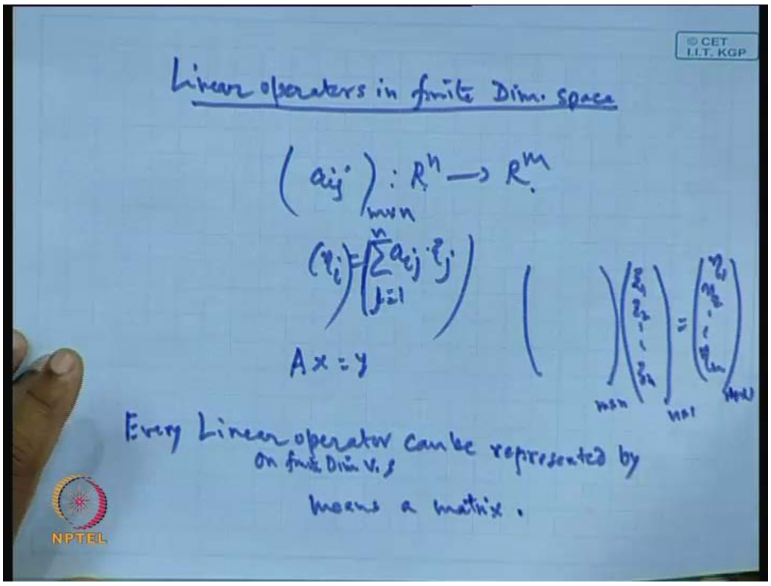
(Refer Slide Time: 28:15)



This is a canonical; C we have defined earlier, x to g x, x to g x and this was a canonical mapping from X to and in fact, this preserves all the operations. We calculate canonical embedding of X into X dash. And, if the range set of C, if the range set of C is coinciding with X double star, then, X is said to be algebraically reflexive, reflexive. So, this is the important. So, we have defined, introduced the algebraic reflexive space, clear. It means, algebraically reflexive means that, a original space X is given; with the help of the duals, you are constructing a second dual of this space; then, if the range and mapping is defined from original space to the range space, to the second dual of this, a canonical embedding is defined.

If the range of that mapping C coincide with the second dual, then, we say the X and X double star, they are equal, is it not. And, they are algebraically reflexive space. It means, their elements may be different, but, so far, the algebraic, the properties are concerned, the metric properties are concerned, norm properties are concerned, they will behave as if they are the same carbon copy of the, carbon copy of the same thing. So, they are algebraically reflexive. What is the inverse is defined? Inverse is something different; inverse is yes, if it is one-one onto, inverse will be different for it.

(Refer Slide Time: 33:25)



So, this is what…Now, we come to the linear operator in a finite dimensional case. Finite dimensional space. As we have seen earlier, in case of the normed space or metric space, when we define the spaces, the finite dimensional spaces are simpler than the infinite dimensional. In fact, there are many results, which are true in general, for a finite dimensional case, but may not be true in, in a infinite dimensional case. For example, the finite dimensional case, any norm, all the norms are equivalent norms; they give the same topology, clear. In a finite dimensional case, one can also say, compact set is a close and bounded and vice versa, but in case of infinite dimensional, this is not true.

So, there are so many things are there, which are common to all the spaces, if the space is finite dimensional, ok. So, here also, we wanted to know, what is the benefit of constructing, of finding the linear operator, or studying the linear operator on a finite dimensional case? Can we get some results, which are common to, for a linear operator, for any finite dimensional space, ok. Let us see, first, the linear operators, can it be represented in terms of some operators always; I mean, can we represent in terms of the matrices; because, we have seen in the last case example, when we taken the matrix, a matrix A of order m cross…

This is a matrix A of order m cross n, then, it defines a mapping from R n to R m, is it not. And in fact, this is a i j is a matrix; then, what we get is a i j, x i j, sigma j is equal to 1 to n, and that will be the point of eta i, is it not. That was the point in eta i. So, it means, this operator, matrix behavior in an operator from R n to R m, clear. So, conversely, also, can be proved that, every linear operator on a finite dimensional space can be represented by means of a matrix, clear; and that is the m of this discussion. We will show that, any linear operator on a finite dimensional vector space, finite dimensional space can be represented by means of a matrix.

Sir, matrix is an operator or a matrix is a matrix?

Matrix is an operator, when this R m, R n to R m, because this R m is a vector. What is this? They say, this is the matrix A; A x is equal to y, you write; y…

This is the R n to R m.

This is the matrix. Here is, x i 1, x i 2, x i n, is it not; and what you get; you are getting to be what, eta 1, eta 2, eta n. Now, this is a m dimensional vector space; this is n dimensional and this is m cross n dimensional. So, basically, transfer this vector to this vector. So, it is an operator.

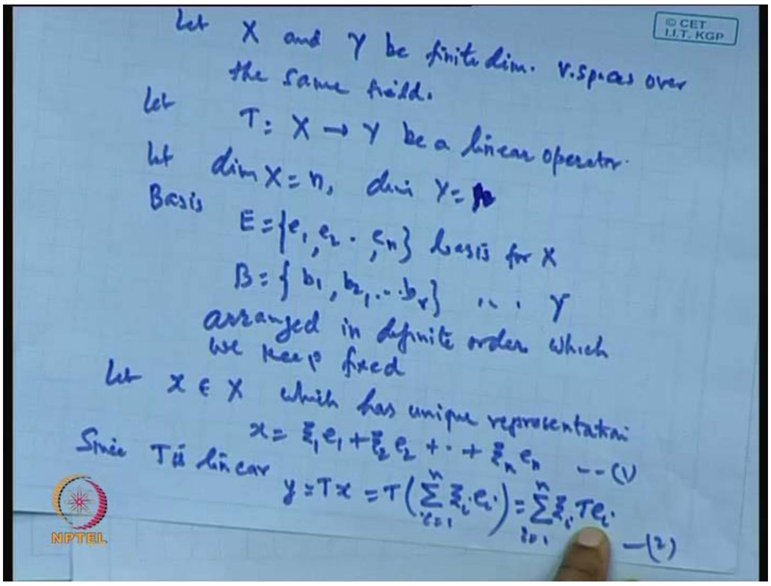
Sir, what is the difference between mapping and suppose, we are writing like this C, X to X star.

Mapping, mapping is a general word. Mapping is a general word, but when the domain range are the vector spaces or lies in the vector space, then, we say this is an operator; either this vector space may be normed space or something; but if the range set, is comes out to be a either real set of real number or set of complex number or set k, which is the field of the vector space, then, it is called a functional; but functional operator, they are all mappings, clear.

Dot operator.

Dot operator is a scalar quantity. So, it is a functional; but closed product is a vector. So, it becomes an operator, clear. Whenever you get image of your vector, when you get the vector, then it is an operator. But image of the vector, if you get an scalar, it is a functional. And, mapping can, mapping is the rule, which assign every element of x to a unique element of y; that is all. So, it is a rule. So, operator also satisfy that rule; functional also satisfy that rule. So, mapping is a general word, clear; is it clear. So, that…So, now, we will try to find that, the representation of the operator in a finite dimensional case. So, we claim, every linear operator, every operator can be represented in terms of the matrix. So, every linear operator can be, every linear operator on the finite dimensional, on finite dimensional vector space or finite dimensional, say vector spaces, can be represented by means of a matrix. This is our theory.

(Refer Slide Time: 39:36)

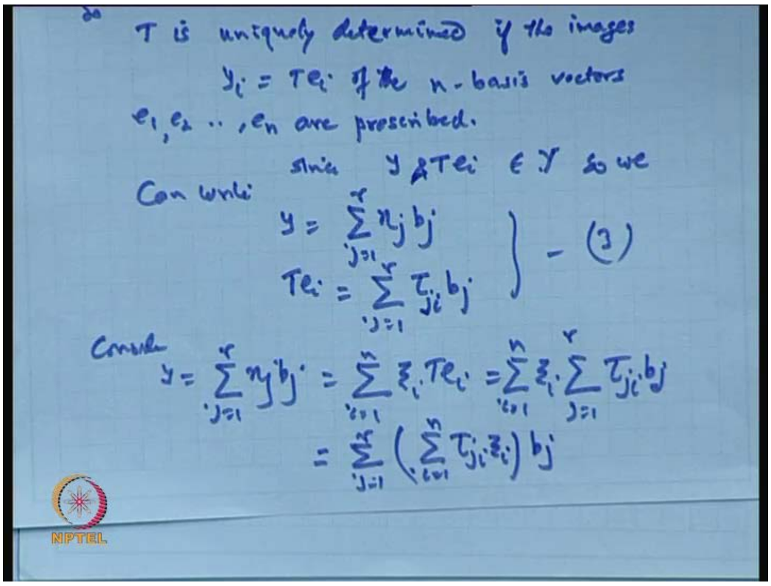


So, let us see, suppose X and Y be finite dimensional vector space, vector space over the same field, over the same field; field of scalar; means, if X is a real vector space and then, Y should also be real vector space; if X is a complex vector space, then, Y should also be a complex vector, over the same field. And let us suppose, T is an operator from X to Y, be a linear operator, linear operator. Since X and Y be the finite dimensional vector spaces, let the dimension of this X and Y, dimension of X is suppose, n, dimension of Y be suppose, m, or let this be r, r, dimension of this is, say r, and the basis element, corresponding basis be, suppose E; elements are e 1,e 2, e n; this is the basis for X; and B, which is a b 1, b 2, say b r, be the basis for Y, clear. And, in order to get the unique things, we assume or we fix the position of the elements of this; so, we arrange the elements of basis E and B in a fixed order.

So, we say, be the base, let E and B, basis for a, arranged in some fixed, in definite order, which we keep fixed. What do you mean by this? Say, e 1, e 2, e n are the basis elements. So, some can, someone say, no, I take e 2 as a first element, e 1as the second element, or e 5 as the first element and e 7 as the second, like that. So, let him decide first; first, he should, because basis elements, n elements we have to picked up; those are the basis elements. Then, arrange it in a, some definite order and throughout our calculation, in this section, we retain this order as it is, ok. Similarly, b 1, b 2, b n also, we retain the same order, so that, there will not be, uniqueness will not be a problem. Then, since these are the basis elements, so, any element x belongs to capital X, will be represented uniquely in terms of basis element.

So, let x belongs to capital X, which can, which has a unique representation in terms of the basis as x is x i 1e 1, x i 2 e 2, x i n e n. Let it be equation 1, clear. Now, since, T is linear and T is mapping from X to Y, so, it will transfer the image T x, it will transfer the image T x to its element y in capital y and that will be equal to T of sigma x i i e i, i is 1to n, but because of the linearity, this will give the sigma i equal to 1to n, x i i T of e i, is it not. i is 1 to, is it ok or not; i is 1to n like this, T. Let it be equation 2, ok. Now, since this T, which we have defined, T x is coming as a summation of x i i eta i, x i i are known; e 1, e 2, e n are known; it means the T x or T can be obtained, if we know that T e i, is it not; if T e i are known, then, T can be represented in this.

(Refer Slide Time: 45:16)



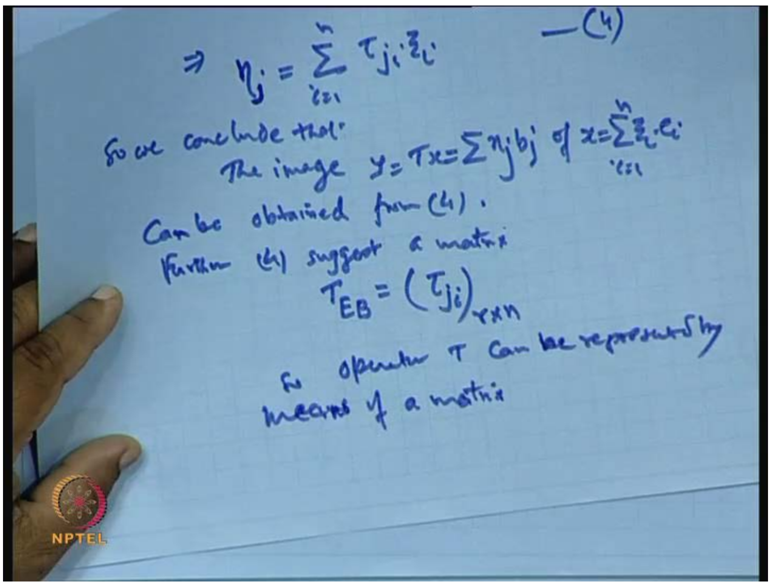
So, we can say T is uniquely determined, T is uniquely determined, if the images y i which is equal to T e i of the n basis vectors, vectors, that is e 1, e 2, e n are prescribed, is it not. If these vectors are known, then, one can obtain that T uniquely. Now, since y and

T of e i, they, y and this, belongs to capital Y, is it not; y is equal to T x and this T i e i which is y i, is also an element of Y. The basis element of y is b 1, b 2, b j, b r. So, this can be expressed in terms of the basis element. So, we can write y in the form of sigma eta j b j and j is 1 to r and similarly, y i or T e i sigma j equal to 1 to r.

Now, here b j, I am using double notation suffix as j i; though, normally what happen is, that i j should be written, is it not; but here, I am writing this thing j i, so that, in the final thing, it will be a proper way for, because here, writing this will not affect; because, this is simply an scalar quantity and T e i is an element of Y. So, it can be expressed in terms of the basis element; that is all. So, we do not have any problem. So, let this be 3. Now, start with y. Consider y, which is equal to sigma eta j b j, j is 1 to r, clear; but this y, is also the same as this from 1; this is also the same as sigma i is 1 to n, x i i T of e i, is it? So, we get this also. Substitute the value of T i from 3.

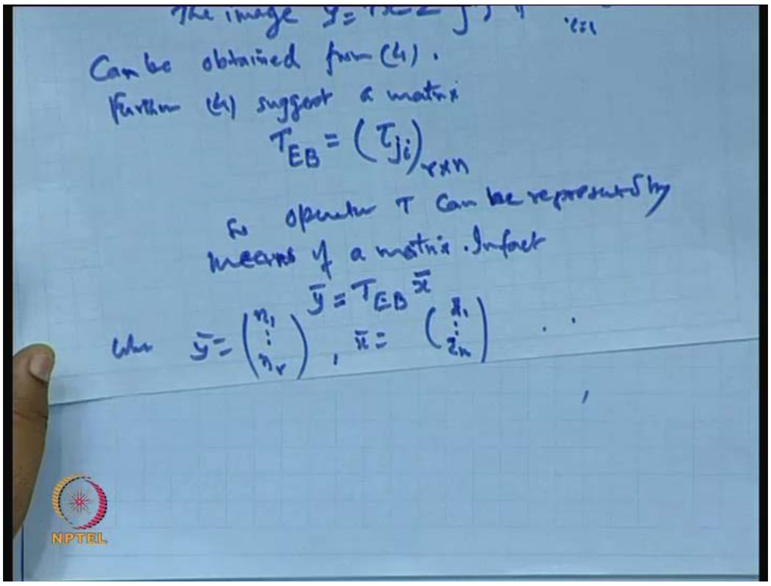
So, what we get is, sigma i equal to 1 to n x i i summation j equal to 1 to r tau of j i b j; substitute this thing. This can be written as sigma j equal to 1 to r, within bracket, sigma i equal to 1 to n, tau of j i x i i into b j, ok. Is it clear now, this one? Just I have interchanged this all, because they are finite in number. So, there no problem and this, you can write it here. Now, if we look this thing and this thing, b 1, b 2, b n, b r, these are the basis element, linearly independent vectors. So, linear combination of these two are identical; it means the corresponding coefficient must be the same.

(Refer Slide Time: 49:57)



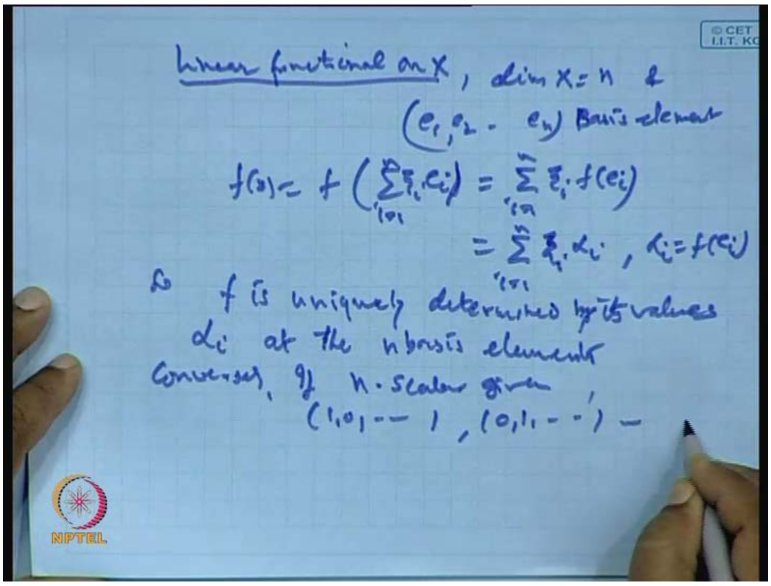
So, this implies that, b j, eta j equal to sigma i equal to 1 to n tau of j i x i i, is it ok. Let it be 4, is it ok. So, we conclude that, the image y, which is T x which is equal to sigma eta j b j of x, which is sigma x i i e i, i is 1 to n, can be obtained, can be obtained from the 4. If you want to find the image of this, you require eta j and eta j can be computed from, with the help of 4. And, not only this; this also gives, further, 4 suggest a matrix T of EB, whose elements are tau of j i, with j varies from what, 1 to r. So, r cross n, ok. It means, an operator, linear operator can be represented by means of the matrix, finite matrix of order, r cross n, ok. So, this proves that, clear. So, that means… Is it ok, now, clear. So, we can have this representation unique and this, respect to the basis of this, ok.

(Refer Slide Time: 49:57)



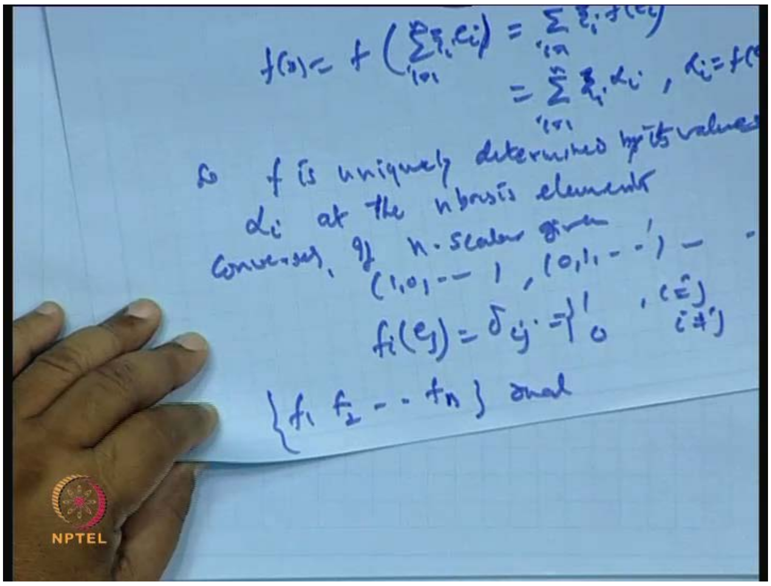
Now, if we look this thing, again, further, say, suppose, so, operator T can be represented by means of, by means of a matrix; fine, by means of the matrix. In fact, column, if you start with the column matrix, etcetera, we can get that, and we get, in fact, we can say, y bar is equal to T of E B x bar, where y bar and e bar is, where y bar is this, what is that, eta 1, eta 2, is it not. eta 1, eta 2, eta r. x bar is x i 1, x i 2, x i n and this T of E B, we can write it as the t 1, t 2, t s, that we can… So, this is nothing, ok.

(Refer Slide Time: 54:00)



Now, let us come to another result, which I am not going to in detail is that, linear functional of this, in case of the linear functional on x, when the dimension of this x is, say n and the basis elements, and e 1, e 2, e n be the basis element, basis elements. So, we can write it, f of x equal to f of sigma x i i e i, i is 1 to n and that can be written as, sigma i is 1 to n, x i i f of e i and that will be equal to i equal to 1 to n x i i alpha i, where alpha i is f of e i. It means, if we know the alpha 1, alpha 2, alpha n, then, one can find out the f. So, f can be obtained there. So, f is uniquely determined by its value, determined by its values, uniquely determined by its values, alpha i at the n basis element, elements. Similarly, conversely, if n basis elements are given, if n scalars are given, that is, 1 0 0 0 0 1 0 0 and so on, 0 0 1, then one can find the n functional, like this.

(Refer Slide Time: 54:00)



f i is a, as a ((chronical)) delta i j, which is 1, when i is equal to j; otherwise 0. And, this functional f 1, f 2, f n, this is called the dual basis of, this all called the dual basis of, of the basis e 1, e 2, e n. So, that is all.

I will explain this part next, in the next class. Thank you. Clear?