Functional Analysis

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Module No. # 01

Lecture No. # 15

Bounded Linear Functionals in a Normed Space

We have discussed the linear operators, and we have seen the property also; one result that, if, T be a linear operator from D T to y, where x and y are normed spaces, are normed spaces over the same field, then, T is continuous if and only if, T is bounded.

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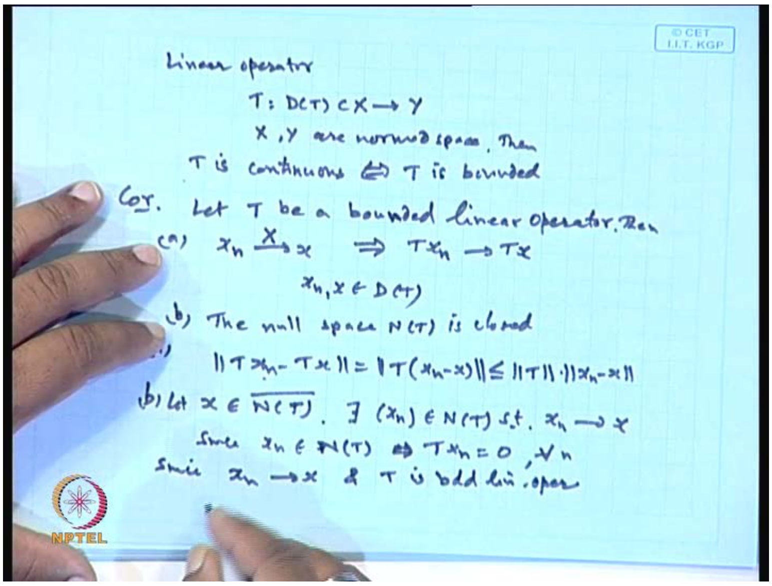

So, this is very interesting result that, in case of linear operator, the continuity and the boundedness are the same thing. So, you have to only prove one side, either T is continuous, then, automatically it comes out to be bounded or if T is bounded, then, it will be a linear operator; then, this will be a continuous operator, provided T is linear. So, as a corollary of this result, we can further derive this result; let T be a bounded linear operator, bounded linear operator, then, x n converges to x, in the metric of, in the norm of x, where x n and x belongs to the domain of T. Then, it will imply that, T of x n will go to T of x.

And second result, which can be derived from here; the null space, null space, that is n T is closed. The first result follows immediately, because T is linear, bounded linear operator. So, it will be a continuous and because of the continuity, it will transform the convergent sequence to a convergent sequence. Otherwise, also, one can prove it in other way, say, T is given to be a bounded linear operator. So, we can say T of x n minus T X, this is equal to norm T of x n minus x, because T is linear. Then, further T is bounded. So, we can say, this is norm of T into norm of x n minus x. Now, it is given that, x n converges to x in the norm of x. So, this part will go to 0, as n tends to infinity. Therefore, this will go to 0, hence, the result follows. So, nothing to be. Similarly, conversely, one, sorry, similarly, we can prove the second part also.

The null space n T is closed. What we have required is that, all the limits, points of this any sequence, which is in n T, must be the point in n T. So, let us take a, an, any arbitrary element x belongs to the closure of the null space. Now, if I prove that, this x belongs to n T, then n T will be closed. So, since it belongs to the closure of n T, closure of n T means, either it will be a point of n T or may be the limit point of n T. So, if it is point of n T, nothing to prove. So, if it is a limit point of n T, then, there exist a sequence x n in n T, such that, x n goes to x, is it not; x n goes to x in that.

Now, if x n belongs to n T, since x n is an element of n T, therefore, T of x n will be 0, for all n; because null space means, set of those point, where the images are 0s. And, since x n converges to x and T is, x n converges to x and T is bounded linear operator, so, by the part first, T of x n will go to T X. So, we can say from here that, T of x n will tends to T of x by first; but T of x n is always be 0, because this is in n T. Therefore, it will implies that, T of x must be 0. So, this shows, x must be a point of n T. So, n T will be closed, clear. So, as a corollary, we can derive these two results that, the bounded linear transform operator, transform the convergence sequence to the convergence sequence and the null space of that bounded linear operator will be a closed set.

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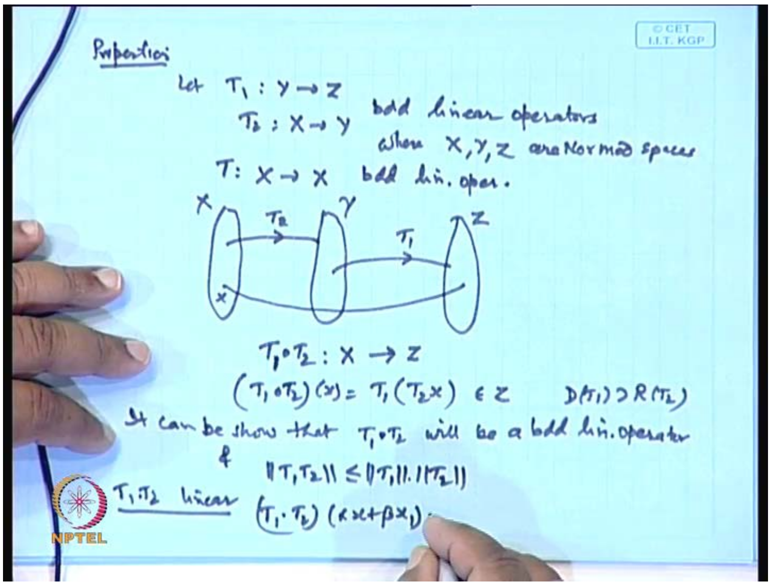

We can further go deeper, to the properties of this bounded linear operator. The properties that, just like a functions, if f and g are the two functions, one can introduce the concept of composite functions f and g, provided the domain and range suitably satisfy the condition. So, let us suppose, T 1 is an operator from Y to Z; T 2 is an operator from X to Y, and both are, say, bounded linear operators, bounded linear operators, where X, Y, Z are normed spaces, are normed spaces, ok. And, let us suppose, T be an operator from X to x, is also be a bounded linear operator. So, here, it is our x; this is y; this one is z. We assume suitably, the norm on this. X to Y, the operator is defined as T 2; from Y to Z the operator is defined as T 1. So, if we take any point x, we want to send directly to z, then, we get a composition operator as T 1 composition T 2 is an operator, which can send, transfer x to z; that is T 1 composition T 2 x, that can be written as T 1 T 2 x.

Now, T 2 is a mapping from X to Y. So, the T 2 x will be point of y. And, if the domain of T 2, range of T 2 x lies in the domain of T 1, then, this is well defined and we can say, this will be a point in z. Because, T 1 image of this, if the domain of, say T 1, covers the range of T 2. So, we can say like this. Now, once we have T 1 and T 2, both are bounded linear operator, then, it can be shown that, T 1 composition T 2 will be a bounded linear operator, a bounded linear operator from x to z, and not only this, the norm of T 1 T 2 can be shown as less than equal to norm of T 1 into norm of T 2.

The T 1, T 2 is linear operator can easily be proved, because, if we replace x by a combination alpha x plus beta y, the correspondingly, the change will come over here; that is, alpha x plus beta y or we can say, T 1 T 2 linear. So, what we do is, let us start with T 1 composition T 2, alpha x plus beta, say x 1, where x and x 1 are the point of capital x. Then, according to this, T 1 T 2 alpha x plus beta x one, but T 2 is linear from X to Y.

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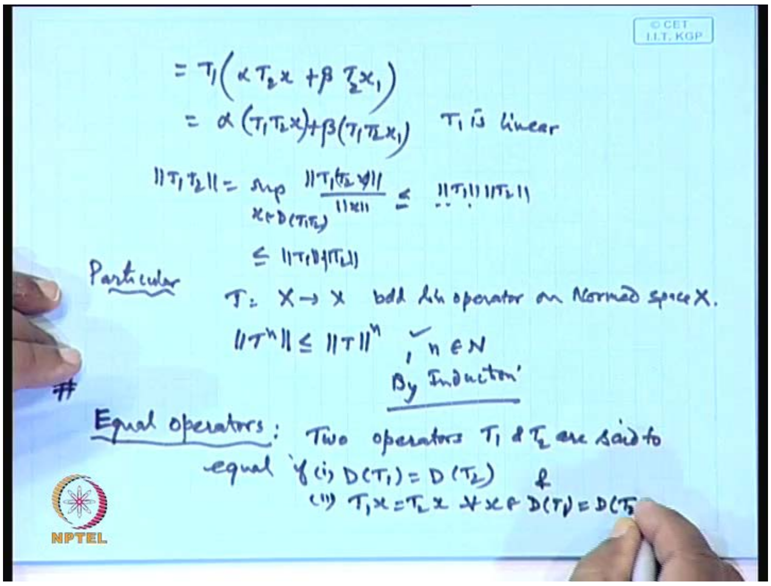

So, it will convert into the, by property of the linearity, we can get from here is, T 1 alpha times of T 2 x plus beta times T 2 x 1. Again, T 1 is linear; again T 1 is linear. So, we can further write, alpha T 1 T 2 x plus beta T 1 T 2 x 1 and that shows, the T 1T 2 is linear operator. And, boundedness also, it can be proved. The norm of T 1 T 2 is basically the supremum of norm T 1 T 2 x and divided by norm x, because the norm of T 1 T 2, this is the supremum of norm of T 1 T 2 x divided by norm of x and then, x belongs to the domain of T 1 T 2, like this, ok. So, similarly, we can show like this and finally, it can come out to be the, less than equal to norm of T 1 into norm of T 2, is it not? Because, this is again, T 1 is bounded. So, we can write, this is less than equal to norm of T 1 into, norm of T 1 into supremum of this thing, that is norm of T 2, clear.

So, we get from here, this is… Now, as a particular case, if T is an operator from x to x, say, be a bounded linear operator on the normed space x, normed space x, then, we can say that, norm of T to the power n, this is less than equal to norm of T to the power n, where the n is a positive integer. Just by induction, this we can show. By induction, we can prove this thing. T 1, T 2. So, let T 1 is equal 2, T 2 is equal to T and we get this result; continue this for n equal to k, then, k plus 1, one can show this result holds. So, this is also…Now, there are another concepts, just like in mapping, we have a similar concept here, the equal operators. We define the two operator T 1 and T 2, operators T 1 and T 2 are said to be equal, are said to be equal, if their domains are the same, if domain of T 1 is the same as the domain of T 2 and the corresponding range set is also there; that T 1 x equal to T 2 x, for all x belonging to the domain of T 1, which is the same as domain of T 2. So, just like a mapping, two functions are said to be equal, if the corresponding domains are same and the range set are equal. They take the same values for each x, for x belonging to that common domains. So, here also the equality has a...

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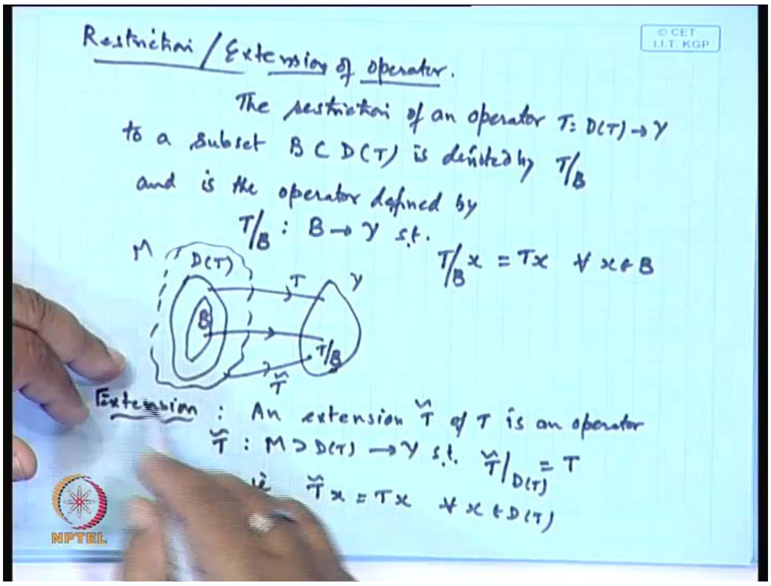

Now, a new concept, which we introduce one to a restriction, and restriction and extension of an operator, of the operator. What is that? The restriction of the, the restriction of an operator T from domain D T to Y, to a subset B of D T is defined, is denoted by T restricted on B and is the operator defined by the restricted operator T on B is a mapping from B to Y, such that, the images of any element x under T B will be the same as the image of x under T, for all x belonging to B. The meaning of this is, say, suppose this is our D T. And, here it is B, which is a subset of D T. t is an mapping from D T to certain range set say Y. All the elements of B, need not be the elements of D T, because, B is a subset of T. We are interested to define an operator on B, such that, that the image of this operator, the image of any element of B, under this operator, will give the same values, or gives the same value as the T of x, then, we say T restricted B is the restriction of T on B.

That is, we are reducing the domain of T B to a subclass or we wanted to redefine the operator T on a subclass, in such way that, images are same, clear. So, such an operator T B, we called a restricted operator or the restriction of the operator T. The reverse of this process, we called the extension. That is, we say extension. An extension T delta, say, of T is an operator from M, which is a superset of D T to, say Y, such that, when we restrict this T delta on D T, then, it should amount the T; that is, the image of any element under T delta, that is the image of T delta x should be the same as T x for x belongs to D T.

So, just like a, here, here we are reducing the domain for T delta. T delta is overdefined over the bigger class, where T is defined on the smaller class. So, T delta we will call the extension of t. So, we wanted to extend it; this is our, say, this is our, say M. So, here, when we are defining, then, this definition will give the T delta. Now, when T is given to be a bounded linear operator on a domain T; and we wanted to extend it, its domain of definition. So, it means, we want, we have to define some operator T delta, which will be a, an extension of the operator T.

Now, when we define, or when you extend the operator T to T delta, on a bigger class M, then, we may or may not be able to retain the properties of the operator T. For example, the T, if it is a linear operator, the T delta may not be linear; if T is bounded, T delta may not be a bounded linear. So, there are infinitely many ways of extending the definition of T, for over a bigger class. But what we are interested in, we wanted to give an extension of T to T delta, over a bigger class M, under the restriction that, a minimum property, minimum property, that is, linear property, boundedness property are retained.

So, that extension will be useful, because we are extending the domain; we are enhancing the domain. Instead of D T, we are considering the bigger domain and the operator we so defined, is not losing the property of boundedness and linearity. So, this will be a useful extension and under what condition this useful extension is possible, that we will see in the next result. So, the next result tells, how one can retain the boundedness and linear properties of an, a bounded linear operator, when it is extended to a bigger class. So, this result is given in the form of theorem. What theorem says, let T be a operator from D T to Y and be a bounded linear of, be a bounded linear operator.

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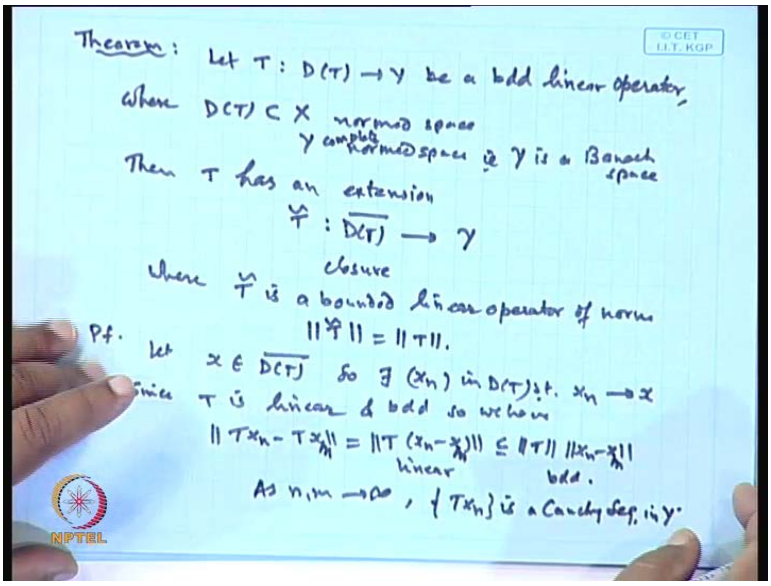

Where D T lies in the normed space, lies in the normed space and Y is also a normed space. In particular, complete normed space; Y is a complete normed space. That is, Y is a Banach space. So, suppose T is a bounded linear operator from D T to Y, where D T lies in the normed space and Y is a Banach space, ok. Then, T has, then, T has an extension, T delta defined from DT closure; this is the closure of D T to Y, where T delta is a bounded linear operator, operator of norm, same as the norm of T. So, this result says that, if we extend the domain of D T to its closure and replace Y by a, the range set to be a subset of a Banach space, that is Y is a Banach space, then, this extended operator T delta will remain bounded, linear and the norm will be the same as the norm of T, ok.

So, this extends, this theorem says, how to, how one can extend the bounded linear operator to a bigger class, under the restriction that, the boundedness, linear property and the norms are retained, ok. Let us see the proof. So, in order to prove this, first we will show the existence of T delta; because T is given, is a bounded linear operator is given. So, we have to first justify that, such a T delta will exist. And, second one, if it exists, then, it should be bounded and linear, second part; and finally, we will show that, both are having the same norm. So, to show the existence of T delta, we should see, that T delta is defined over the entire class x D T, D T closure. So, let us take a point x in the D T closure, which is an arbitrary point, in the D T closure.

Now, since it is a closure point, x belongs to the closure of D T. So, by definition there exist a sequence x n in D T, such that, x n converges to x, clear; by, because, x is the limit point. So, there must be a sequence of the points in x n available, which converges to x. Now, since T is linear as well as bounded, both, so, we have norm of T X n minus T of x, this is equal to norm of T X n minus x, by linear property; and then, because of the boundedness, we can say this is less than equal to norm; this is because of the boundedness of this.

Now, since x n converges to x, so, this norm of x n minus x will go to 0. Therefore, this x n, let us take M here; T of x n x m x n x m; since x n converges to x, is a convergence sequence, so, it is must be a Cauchy sequence. So, this Cauchy sequence, this will tends to 0, when n m goes to infinity. So, as n m tends to infinity, this implies that, sequence T of x n is a Cauchy sequence, Cauchy sequence in Y. But Y is complete, because this x n converges to x is a convergence sequence. Every convergence sequence is Cauchy sequence. So, this will go to 0, when n m tends to; therefore, this will go to 0, when n m goes to infinity. So, T of x n behaves as a Cauchy sequence in Y. But y is given to be Banach; so, every Cauchy sequence must be convergent.

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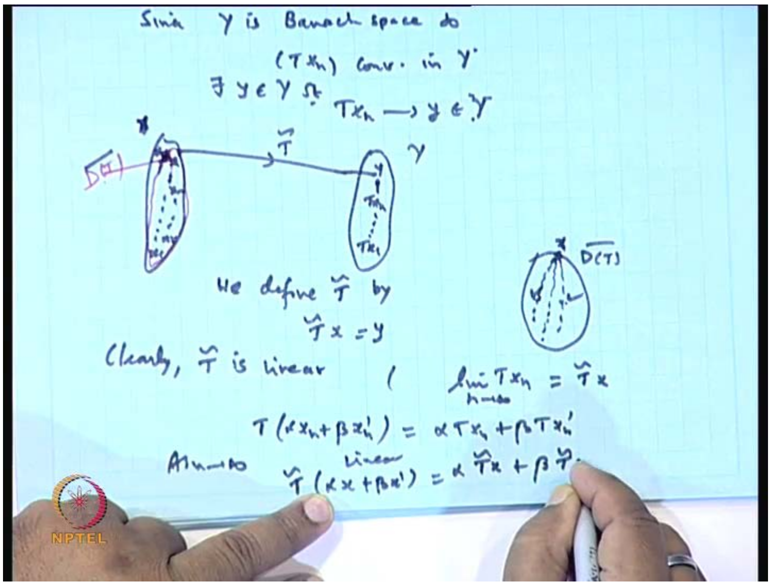

So, since Y, since Y is Banach space, so, this Cauchy sequence convergent. So, the sequence T of x n converges in Y. Therefore, there will be a point in Y, belongs to capital Y, such that, T of x n will go to y in capital y. So, now, what we get, this is our x; this one is y. Here, we are getting the sequence x 1, x 2, x n which goes to say, x. The corresponding sequence T X 1, T X 2, T X n, this converges to y. So, this x may be the point, this is the D T; I am taking this is, say, this one is D T. Let me just point out. Say, this is our D T. So, this point x, may be a point on the boundary; may be the point on the boundary. So, this is the point x. Here, we have x, which belongs to the closure of D T, closure of D T. So, we are able, able to define an operator T delta, such that, that image of this T X becomes y. So, we define, we define T delta by T delta x is equal to y.

So, existence of T delta is (( )), that we can get that image of x belongs to delta T to y, an element of this. Now, this definition of T delta is, should be independent of the choice of the sequence, which converges to x. Suppose, there is another sequence, which goes to, this is our D T closure and here, it is a x. Now, this is as one sequence; this is another sequence. Suppose, I take a sequence, which is combined to this and it converges to x, then, since it has a subsequence which converges to x, this has converge. So, this sequence will also converge to x. So, if this x limit, the definition of T delta which you define, is independent of the choice of the sequence, which goes to x. So, nothing.

Now, clearly, T delta is linear; why, because the T delta x is defined as basically, the limit of this T X n; basically, what we are doing, the limit of T X n, as n tends to infinity; this is, is defined as the T delta x, is it not; because this is equal to y and y is equal to T delta x. So, T is linear. So, if we take the T of alpha x n plus beta, say x n dash, then, because T is linear, we get alpha T X n plus beta T of x n dash and then, taking the limit as n tends to infinity, we get from here is, T delta alpha x plus beta x dash, becomes alpha of T delta x beta of T delta x prime.

I hope this will be clear, because, it is direct; there is nothing to it; it is very simple. To show T star is linear, take the help of this definition; the T star x is equal to y and y is obtained as limit of the T X n. So, we can say, the limit of T X n, which is equal to, say y, y is equal to T star x. So, taking the help of this T as a bounded linear operator, we can show immediately, the T delta is a linear operator. Similarly, one can show that, T is also a bounded operator.

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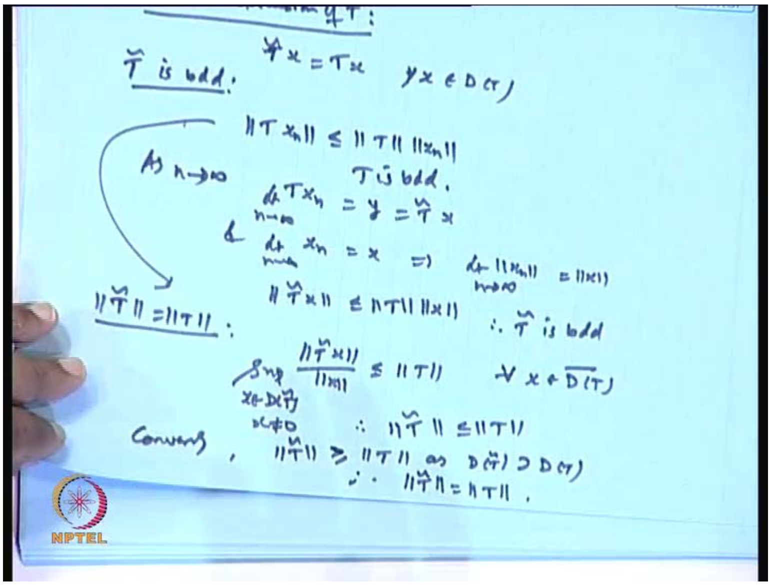

Let us see, how, T is linear, clear. Now, T star is an extension of T; T star is an extension of T, because for every x belongs to t, because, T delta x coincide with T X, when x belongs to D T, for every x belongs to D T. It means, the T is delta and t are equal on the subset D T; T delta is defined on closure of D T. So, it is an extension of t. Now, to show T star is bounded, what we do is, we start with the norm of T X n. Now, this is less than equal to norm T into norm x n, because T is bounded. Now, as n tends to infinity, as I told earlier that, this T of x n, when you take the limit as n tends to infinity, is y, which is equal to T delta x, y, which is the T delta x. And, this x n, limit of this x n, when n tends to is x; but norm is a continuous function. So, limit of the norm x n as n tends to infinity, is norm of x.

So, from here, if we take this as n term, we are getting this is norm T delta x is less than equal to norm of T into norm of x. So, there exist a constant c, such that, norm T delta x is less than equal to c times norm x. Therefore, T delta is bounded. Then, to show the T delta norm is the same as norm of T, this to show. From here, we get norm of T delta x divided by norm x is less than equal to norm T. This is true for every x belongs to the domain of D T closure. Hence, take the supremum. So, supremum of for x belongs to D T delta and x is not equal to 0, is further less than. So, this shows, the norm of T delta is less than equal to norm T, clear.

The converse is automatically true. Conversely, norm of T delta will always be greater than equal to norm T, as the domain of T delta is bigger than the domain of t. So, when we extend it, the definitions are extending; the, this will, the length will be enhanced; not, it will not be reduced. So, we always get, this is greater than equal to this. So, combined these two, we get norm of T delta and norm of T is this. And, that is proves the results. So, this result says that, we are able to get an extension of a bounded linear operator to over a class, which is the closure of the earlier class, (( )) and the properties of the boundedness and closed and linearity are retained; that is all.

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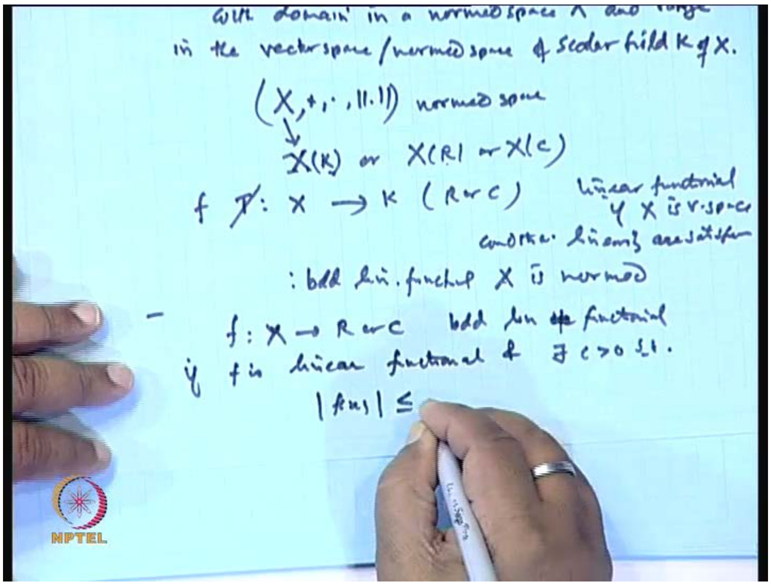

Now, so far, we have considered the bounded linear operator in which the domain and range, both were the normed spaces or say, vector space, when we take the linear operators. Now, when the range of the set, operator is replaced by the field of a scalar, that is either a real number or a complex number or a field, in general, a scalar field k of the normed space x, then such a operator, we call it as a functional. So, a bounded linear functional is basically, a bounded, is a bounded linear operator; is a bounded linear operator, where the range lies in the boundary linearity with, with domain in a vector space, in a normed space x and range and range in the vector space or normed space, you can say, k vector space or n space of a scalars field k, K of x.

That is, the meaning of this is, say x be a vector space, suppose, then, there are two operation, addition, multiplication and if it is a normed space, then we say, this is a normed space; but when you take x to be a vector space, basically, we are not writing k. This is a field of a scalars on, on which, the vectors are defined. So, x is a normed space or x is a vector space over the field k. This k may be a R or it may be a C; if it is R, we say the real vector space or a complex vector space. So, what we see here that, when an operator T from x to k, k is either R or C, then, this T be denoted by f and it is called a linear functional, if x is a vector space; provided it satisfy the condition of the linearity; provided it is a linear, condition of linearity are satisfied, are satisfied, ok.

But if x is a normed space, then, we call it this as a bounded linear functional. So, we define the bounded linear functional as a bounded linear operator, where the domains remains same in the normed space x, but the range, in place of the y, we are taking a, in the field of a scalars k of x. So, if x is a real vector space, real normed space, the field k comes out to be r. If x is a complex normed space, k comes out to be the C, ok. So, we define the bounded linear. Now, once it is a bounded linear, the norm is defined like this. So, we say, a bounded linear functional f is a bounded linear operator with the scalars field of norm x as a range set. So, boundedness, we say, let f is a mapping from x to R or C, is a bounded linear operate functional, if f is linear functional and there exists a C greater than 0, such that, mod of f x is less than equal to C times norm of x, norm of x. So, that will be equal to…

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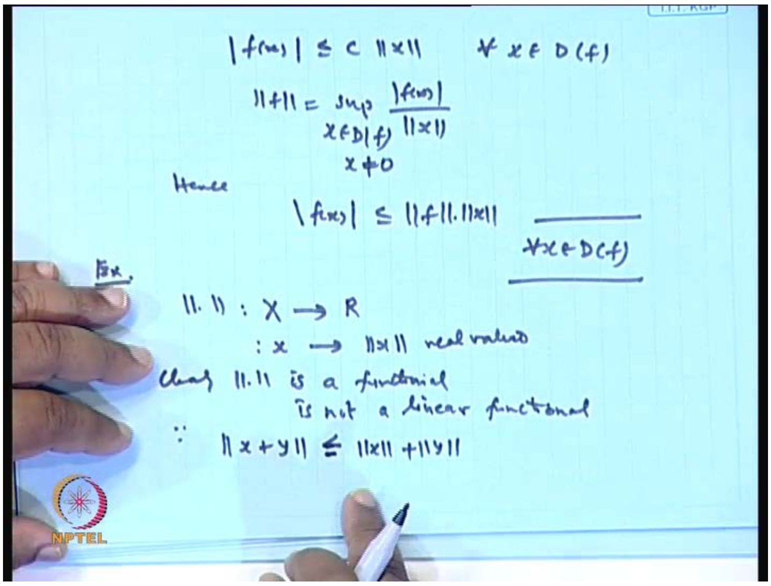

So, if there exists a number C, such that, for all x, that is, for all x, this is true, this is true for all x, belongs to the domain of… Again, the minimum value of C is called the norm of f. So, the norm of f is the supremum of mod f x over norm x, where the x belongs to the domain of f; x is not equal to 0; this is called the norm. So, just like a previous case, we can define the norm of the bounded linear function as this. Hence, the relation we can say, the mod of f x is less than equal to norm f norm x, holds for every x belongs to the domain of f. So, this result must be true.

The same results, which we have proved earlier, in case of the bounded linear operator, continue to hold good in case of the bounded linear functional. For example, if f is a bounded, f is a linear functional, then, continuity and the bounded, boundedness will remain the same and one can find out the corresponding results, which we have proved, in case of the bounded linear operator holds good for the bounded linear functional. Now, there are certain examples of bounded linear functional. Let us see the norm. The norm is a mapping, which vary from a vector space x to R. If it such that, image of x takes the value norm of x; it is a non-negative real functional, real valued function and satisfy those condition norm x is greater than equal to 0, norm of alpha is mod alpha into norm x, norm x plus y is less than equal to norm x plus norm y and so on and so forth, ok.

Now, this norm, we claim, it is a boundary, this is a functional; is a functional, because this is a real valued thing. This is the real valued number. So, it is a scalar quantity. So, mapping from normed space to a field of scalars, so, it is a functional; but it is not a linear functional, why. Why it is not a linear functional, because, if we look this simple property x plus y norm, it is not equal to the norm of x plus norm of y; rather than, it is less than equal to. It means, the linear properties is not retained. So, that is why, norm is not a linear functional; it is a non-linear functional, but it has a lot of importance.

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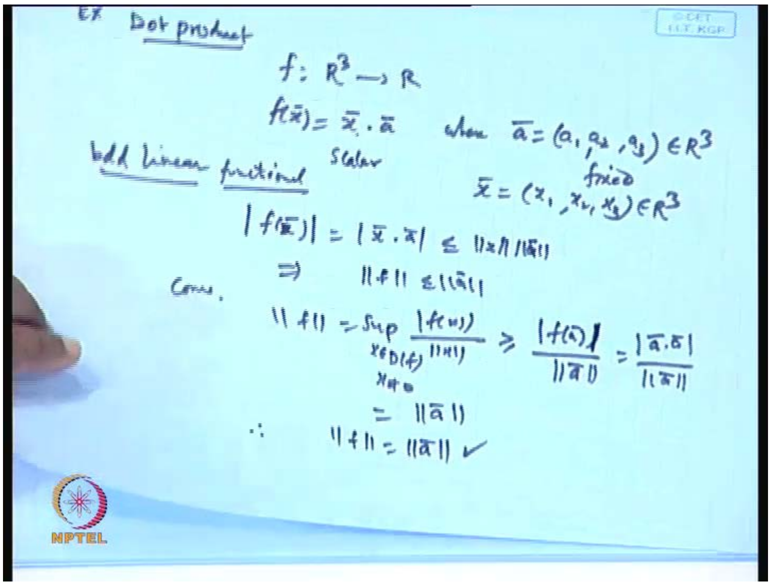

Another example is our dot product. This we have already seen, if dot product is a functional. In fact, a mapping f from R 3 to R, if we define as f of x equal to, x bar equal to x bar dot a bar, where a bar is a fixed vector in R 3; this is a fixed vector, while x bar is a variable one. But the dot products comes out to be a scalar quantity. So, this is a functional; it is a linear functional. So, it is a linear functional. In fact, we will show it is a bounded linear functional. Linearity follows immediately, replace x by alpha x bar plus beta x 1 bar, we can see and the boundedness is shown like this; mod of this equal to mod of x bar dot a bar, which is less than equal to norm of x into norm of a. And then, divide by norm, so, we get norm of f is less than equal to norm a, ok.

And conversely, norm of f, because it is the supremum of mod f x over norm x, over all x belongs to the domain of f, x is not equal to 0. So, we can say, this is greater than equal to a particular value, say a, a bar, this is a bar, a bar divided by mod of a bar. And then, you say, apply this condition; this is equal to mod of a bar dot a bar divided by norm a bar. So, it is equal to basically, norm a bar; because, this will be norm a bar square and we get this. So, from here we get that, norm of f equal to norm a bar; that is, this dot product comes out to be a bounded linear operator, with the norm equal to the norm of the fixed vector, a bar which we have chosen.

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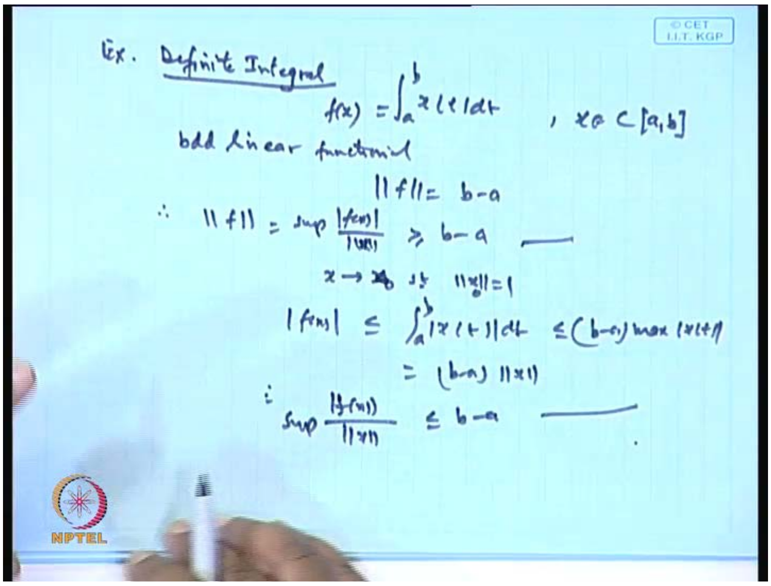

Another generalized result, example, is our definite integral. Though, when we say the definite integral, a to b, x T D T, when x is a fixed continuous function, then, these gives a number simply. But when we say, x is a variable one, belonging to the, say, set of all continuous function defined over the interval a b, then, we denote this by f x and it becomes a functional defined on x. Now, this functional is a linear functional. So, it is a bounded linear functional and in fact, we can show that, norm of f equal to b minus a. I think, this one way it is clear, because norm of f, if I look, this is the supremum of mod f x over norm x. So, replace x by a, where, such that, norm of x equal to norm of a, one, norm of x is equal to 1; replace x by such x naught, where this is 1. So, if we replace with this by norm, then, we will see it is greater than equal to b minus a, ok.

And conversely, mod of f x we can write it, mod of f x is less than equal to integral a to b mod of x T D T, which is less than equal to maximum value of this x t; but this is, the maximum is norm. So, basically, this is b minus a into norm x, divide by this. So, we get mod f x over norm x, take the supremum, is less than equal to this. So, combine these two, we get norm f equal to b minus a.

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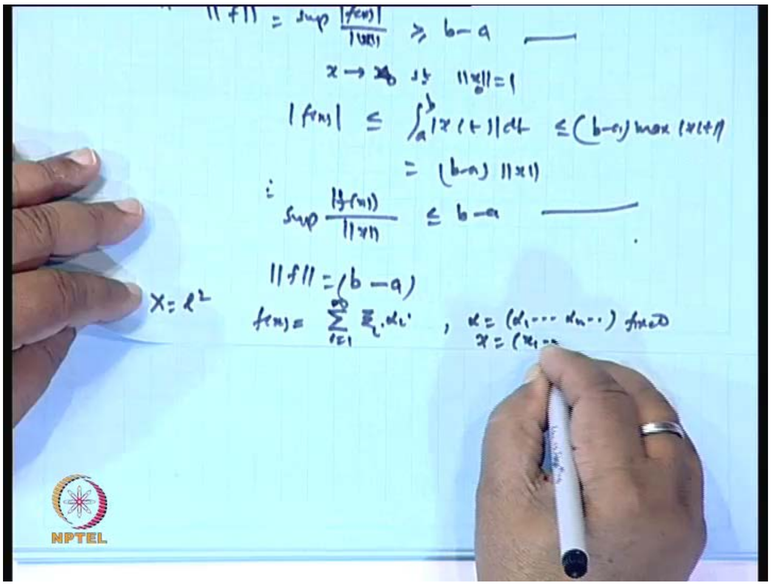

So, definite integrals, they are also behaves as a bounded linear functional, having the norm equal to the length of the interval on which it is defined. Then, we can also think that, example of l 2 space. If we take l 2, space this is a Banach space, normed space and if we fix that one element and define f x as sigma x i i into alpha i, i is 1 to infinity, where alpha is alpha 1, alpha 2, alpha n, so on, is a fixed sequence and x, which is x 1, x 2, x n is a variable one; then, you can see that, this will be a bounded linear functional defined on l 2, I think you can just go.

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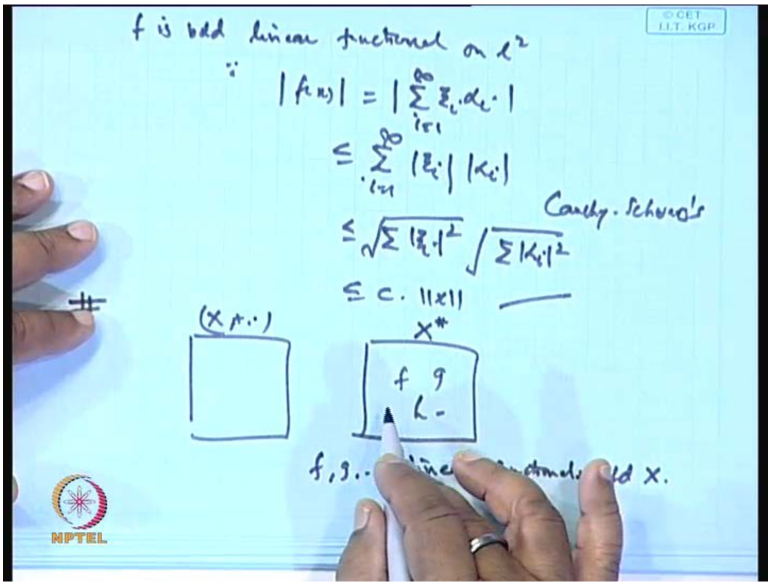

The bounded linearity follows, because f is bounded linear functional on l 2; because what is the mod f x? Mod f x is mod of sigma x i i alpha i, i is 1 to infinity, which is less than equal to sigma i is 1 to infinity mod x i i mod alpha i. Now, use the Cauchy Schwarz inequality. So, by Cauchy Schwarz inequality, this can be written as, sigma mod x i i square under root into under root sigma mod alpha i square i is 1 to infinity. But this part is nothing, but the norm of x, and this is a constant C. So, it can be written as C times norm of x. And, this shows that, f is a bounded linear functional. So, we have seen the so many examples of the bounded linear functional as well as the bounded linear operator. Now, with the help of this functional, since our x be a normed space, and one can introduce that on it, the functional, linear functional, bounded linear functional on x.

Now, what will be the corresponding space of the linear functional or bounded linear functional. That, we see from here. Suppose, x is this vector space, let us take. This is a vector space and we are considering another set f, g, h, etcetera which are, f, g, h, these are the linear functional, functional defined on x. So, let it be denoted by this, say, x star, clear. Now, this linear functional, whether it does it form a vector space again or not; if it forms a vector space, then, it will be usefully structured. And, we see that, if we define the suitably the mapping on this bounded linear functional…

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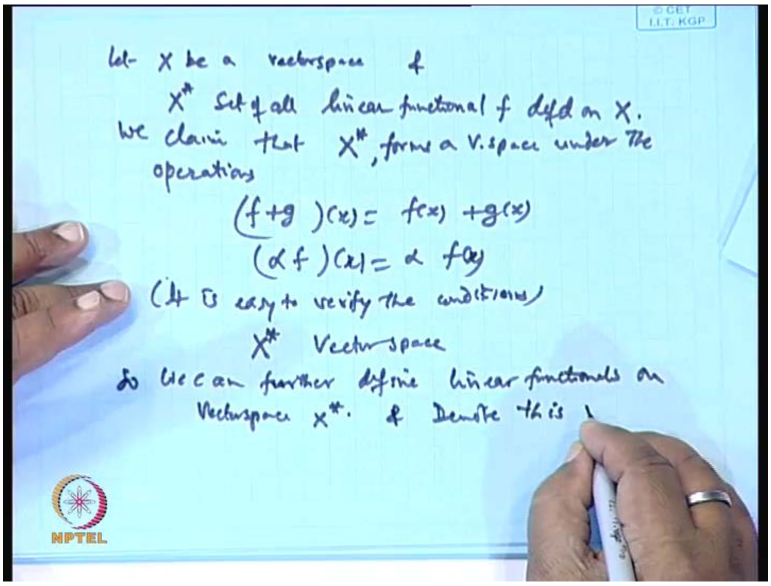

So, let X be a, a vector space; X be a vector space and X star be the set of all linear functional defined on X. We claim that, X star forms a vector space, forms a vector space, under these two operation, under the operations f plus g x is f x plus g x, where alpha f x equal to alpha times f x; that is one x star. If I introduce the addition, scalar multiplication in this fashion, then, all the property of the vector space are satisfied and this will be shown that, x star it is easy to prove. So, it is easy to verify the conditions, conditions. So, this space x star, becomes a vector space; this space becomes a vector space, ok. So, now, we are having this space x and then, this also becomes a vector space; then, one can further think, the linear function defined on x star. So, we can further write, another space x double star, say, mapping is psi, eta, etcetera, which are the functional on x star and if it is again a vector space, then, the question arise, what will be the relation between x star, ok.

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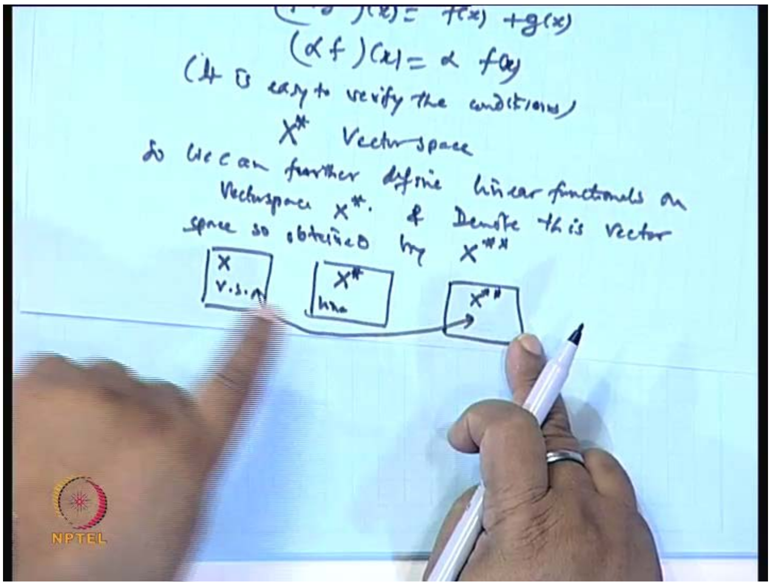

So, we can further define linear functionals on the vector space X star and, and denote this vector space so obtained, by X double star, clear. In a similar way, you can introduce the concept of addition, (( )). Now, my question is, we have started with a vector space X; this is a vector space; then, we have got another vector space, which is a linear functional; then, another vector space X double star. My question is, what is the relation between these two? Can we connect X and X double star? If we are able to connect such a, by such a mapping, by a mapping, then, this will be a useful mapping and one can identify these two spaces; so many properties in these two. So, this we will discuss next time. Thank you.