Functional Analysis

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Module No. # 01

Lecture No. # 12

Compactness of Matric/ Normed Spaces

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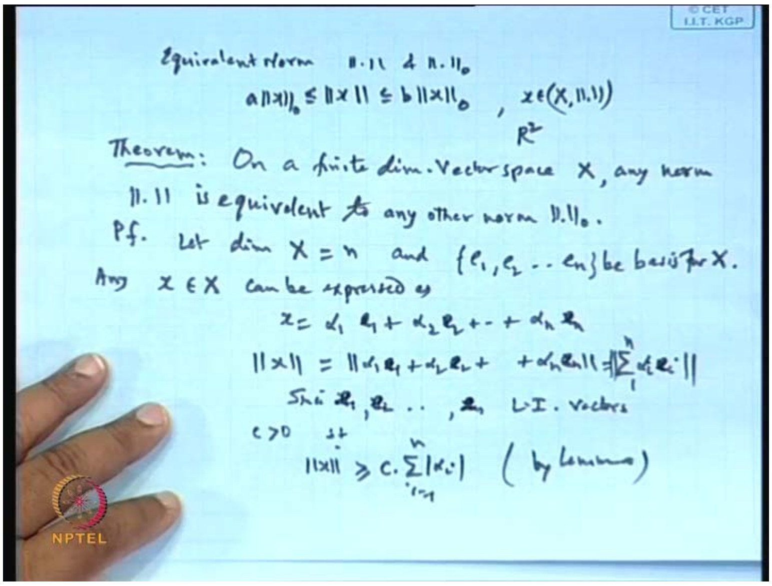

In the last lecture, we have introduced the concept of equivalent norm, which is defined a, we say the two norms are said to be equivalent, the norm this and norm 0 are said to be equivalent, if there exists the number, a number a and b, such that, this condition holds good, for all x belonging to the normed space X. And, we have also discussed about an example, where all the norms defined over R 2 in a very different way, they are equivalent norm, basically. In fact, we have, in general, the result which says that, for a finite dimensional case, the, on a finite dimensional vector space, any norm is equivalent to any other norm. So, on a finite dimensional vector space X, any norm, that is, norm of this is, equivalent to any other norm, that is, norm of 0 defined on it. So, this is general results. Since R 2 is a finite dimensional case, so, that is why, we are getting with the, in example, in the previous example, the norm, which we have defined in the form of norm x 1, norm x 2, norm x infinity, they are coming to be equivalent norms.

So, let us see the proof of this. Suppose, the dimension of the vector space X is n, and let e 1, e 2, e n be the basis elements for this, basis for the vector space X. So, any element of this, so, x, if any arbitrary element, it can be expressed as, can be expressed in terms of the basis element, expressed as a linear combination of the basis element, alpha x 1, alpha 1 x 1, alpha 2 x 2 plus alpha n x n. Now, if we take the norm of this, with respect to the first one, then, what we get is norm of alpha 1 x 1 plus alpha 2 x 2 plus alpha n x n; or, this is equivalent to sigma norm of this; sigma alpha i x i, i is 1 to n under the norm. Now, since x 1, x 2, x n, these are the linearly independent vectors, because they are the elements of, I am writing e 1, e 2; so, let it be e 1, e 2 here; e 1, e 2, e n. So, here is e n and this is also e 1, e 2, e n and since e 1, e 2, e n, these are the linearly independent vectors, because they are the basis elements for x.

So, according to that one lemma, which we have proved earlier that, in case of a linearly independent vector, one cannot get, or cannot expect, a vector involving large number of scalars by the minimum length. So, we can always get a constant C greater than 0, such that, this norm x greater than equal to C times sigma mod alpha i, i is 1 to n, by lemma, which we have shown earlier. Clear? So, from here, we can get this part, norm sigma of mod alpha i, i is 1 to n is less than equal to 1 by C norm of x; let it be 1.

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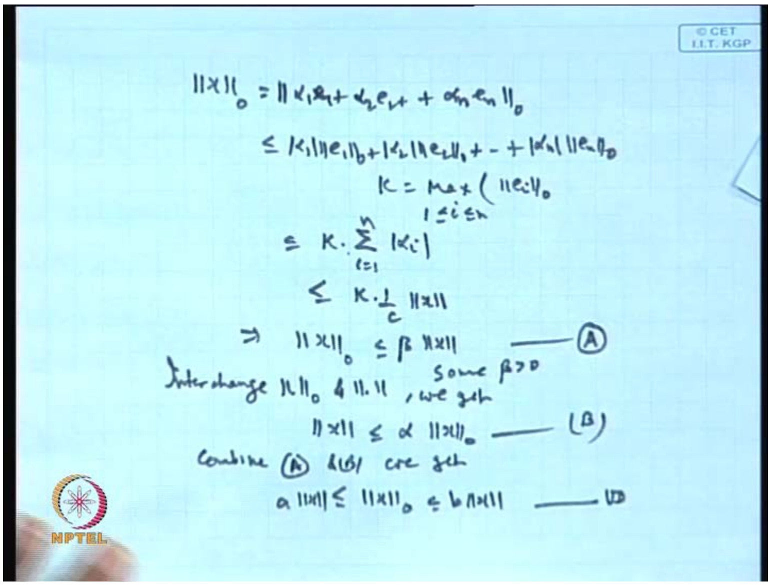

Then, again, we start with this norm x under 0, the norm x, the value of x, under the norm 0. So, this will be equal to norm alpha 1 x 1, alpha 1 e 1, alpha 2 e 2 and then, alpha n e n with 0. Apply the triangle inequality. So, we are getting is, less than equal to mod alpha 1 norm e 1 0 mod alpha 2 norm e 2 0 and so on, mod alpha n norm of e n 0. If I take k to be the maximum of say, norm e i 0, i is 1 to n, then, this whole thing can be written as less than equal to k times sigma i is 1 to n mod alpha i; but we already have it, from this equation 1, sigma of alpha i is less than equal to 1 by C into this. So, we can further write as, this is less than equal to k into 1 by C norm of x, norm of x. And, that gives you the norm of x 0 is less than equal to, say, beta times norm of x, say this. Where beta is some constant greater than 0, some constant beta. Now, if I interchange the position of norm 0 and norm, then, we get, in a similar way, norm of x is less than equal to alpha times norm of x 0.

So, combine A and B, combine A and B, we get a constant, a and b can be obtained such that, norm x lies between, norm x 0 is lying between a times norm x less than equal to b times norm x. So, this is the result which we required, ok. So, any two norms one can, on a finite dimensional space are equivalent norms. The advantage of this result is that, in case of the finite dimensional space, we need not to bother about the kinds of the norm, because they give the same topology.

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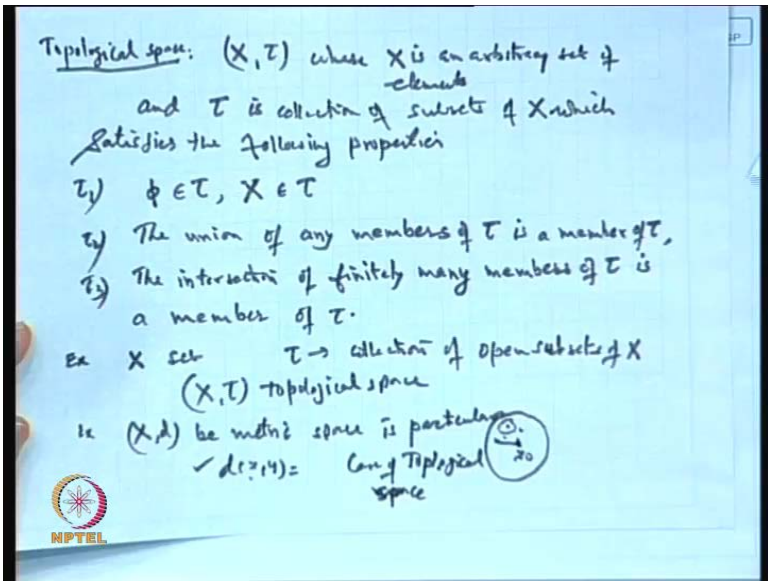

Topology generated by any norm, whether it is norm 0, or norm x will be the same. What is the topology, in fact? The topology means, like a topological space, this is a pair X tau, where X is an arbitrary set of elements, X is an arbitrary set of elements and, and tau is the collection of or the subsets of, and tau is the collection of subsets of X, which satisfies the following property, which satisfies the following property. The first property that we get tau 1, the empty set phi and X both should be the element of tau, must be an element of tau. Second property is that, the union of any member, any member, members of tau is a member, is a member of tau, is a member of tau; any, union of any member of tau is a member of tau.

And third property, the intersection of, intersection of finitely many members of tau is a member of tau. So, if these three properties are satisfied, then, we say the pair X tau is a topological space. In fact, if I go, a general result. Suppose, X be any arbitrary set and tau is the collection of open subsets, open subsets of X, then, obviously, empty set is an open set; X is also an open set; because these are the two non-trivial cases, trivial case; and then, the union of any collection of the open set is again a open. The finite intersection of open set is open, which is well known results. So, this set X with tau will be a topological space, clear. So, collection of all the open subsets of X will form this. In fact, this is the, from here, we have generated these definitions and because, since this is true for all case, so, we can take this definition as an axioms or for the topological space or definition.

Now, in case of the metric space, if X, d be a metric space, then, our distance notion d x y, that also gives you the topology. Because, you take any ball, around the point this; find out the two point, or say x, 0; you can always find the distance between x to x 0 by using this formula. So, an open ball means, if the collection of all such point with distance, we can draw the ball, which is totally contained inside it, such thing, we can always (( )). So, this X, d, with a metric space is a particular case of topological space, topological space. In fact, every metric space is topological space. This satisfy the…What we are interested in, this equivalent norm, when you take the equivalent concept of the equivalent norms, then, this equivalent norms give the same topology; that is, the corresponding metric space, which we get it, you are getting the collection of open set will be the same. So, that is the concept. We will not going detail about this topological space, because it is a another part of the…There it is called a topology, for it. Now, we come for, now, our concept of the compact set. What is the compact set?

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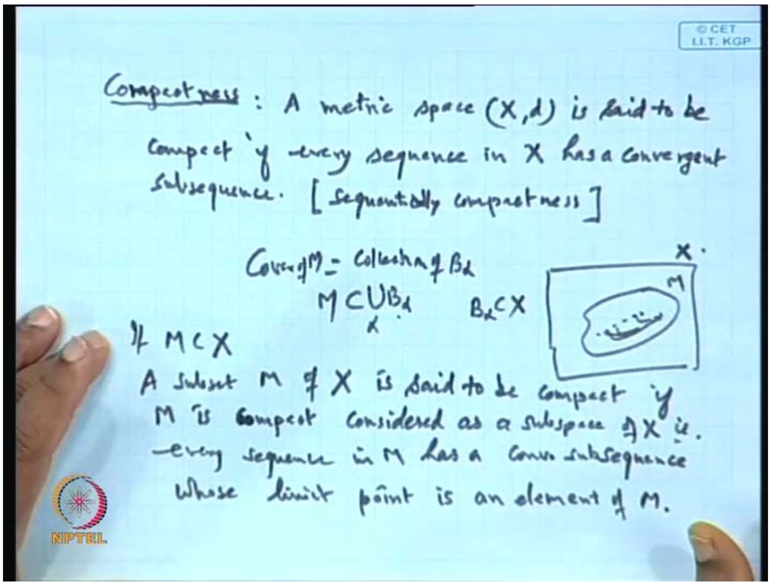

A compactness, compact set or compactness in a metric ((space)), compactness in a normed space. We define a metric space, a metric space X is said to be compact, to be compact, if every sequence in X, in X has a convergent subsequence, convergent subsequence, convergent. This, we also called as a sequentially compact set, definition of the sequentially, sequentially compactness. In fact, there are three ways of defining the compactness in a general topological space, that requires the concept of the open cover. Every open cover has a finite subcover; then, the set will be a compact set. Then, every countable cover has a finite subcover; then, it is also a compact set and third is the sequentially compactness.

The cover, we mean, suppose X be a set and this is our set M. There exists the subsets of X, say B alpha, which is a subset of X, such that, countable union of B alpha covers M. Then, we say, this collection of B alpha is the cover of M. And, when this collection, only finite number of B alpha's are required, then, we say, it only finite number of elements are, are sufficient to cover M then, we say this collection or this cover has a finite subcover. So, we define a set M, in a general topological space as a compact set, if every cover has a finite subcover. B open cover has a finite subcover. Then, if they are countable in number, then, every countable cover has a sub, finite subcover, then also we say, it is a compact set. And third one is, if every sequence in M has a convergent subsequence, which converges to a point in M, then, we say it is a compact set.

So, in case of a metric space, all these three definitions coincide. Therefore, we can pick up any definition of a compact set, either in the form of cover or may be in the form of sequence. This is the most convenient way of defining the compactness in terms of the sequencing. So, that is why, we are choosing a definition of a compact set in a metric space X, a set is compact or space is compact, if every sequence has a convergent subsequence. Now, if M is a subset of M, then, M, how to define the compactness of M? This we say, the, a subset of M, a subset M of X is said to be compact, is said to be compact, if M is compact, if M is compact, considered as a subspace of X, as a subspace of X. It means, the meaning of this is that, every sequence in M must have a subsequence, which converges to a point of M; that is, the meaning is, every sequence in M, every sequence in M has a convergent subsequence, subsequence whose limit point, limit point is, limit point is an element of M, ok.

So, if a sequence whose limit point may not belongs to M, then M will not be considered to be a compact subset of M. So, condition for the compactness for a subset is that, all these points of the sequence must be the point of M and when you take the limiting value, the limit point has to be the point of M; that is the definition. Now, we have a very simple concept in the, of a compact set in a finite dimensional case. In fact, we will say, prove that, in case of the finite dimensional, a compact set means, it is closed and bounded; that is every closed and bounded set is compact and vice versa. But, in general, when the space is not finite dimensional, then, this concept, this closed and bounded set, need not turn out to be a compact set. So, that, we will see.

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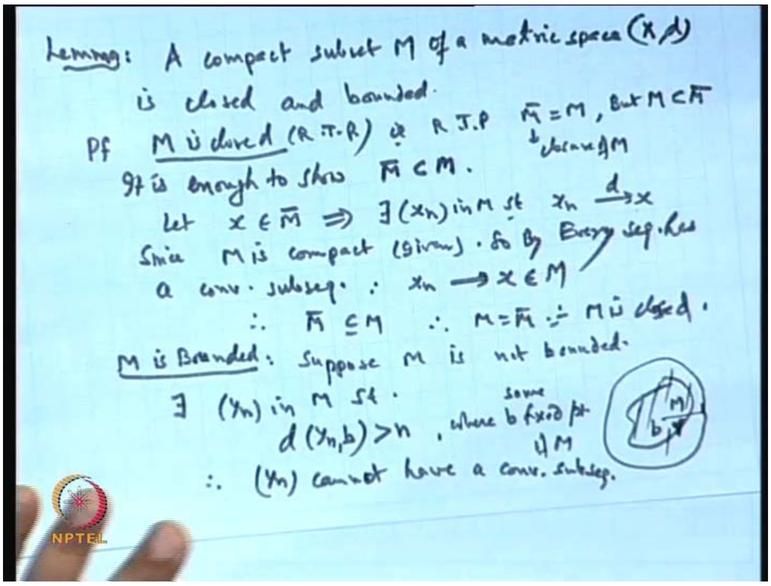

Before going, let us see the one lemma, that shows, the one way, in a general metric space, a compact subset M of a metric space X, d is closed and bounded, is closed and bounded. So, this is the result, true for any arbitrary metric space, whether it is a finite dimensional or whether it is a infinite dimensional. A compact subset of a metric space will always be closed and bounded. The proof is like this. We want M to be closed, M is closed. This is required to prove. So, that is what to prove is, M bar is equal to M; but M is always be a subset of M bar. So, it is enough to prove, enough to show M bar is contained in M. So, finally, if I prove that, M bar is contained, subset of M, then, M will be closed. So, let us take an element x belongs to M bar. What is M bar? M bar is the closure of, this is the closure of M; that is, a set which contains all the points of M, together with its limit point.

So, if x belongs to M bar, either x will be a point of M; then, nothing to prove, because it will be a subset of M. But if x is not a point of M, then, it has to be limit point of M. So, this implies that, there must be a sequence x n in M such that, x n must go to x in the, of course, metric d, whatever the metric d is there. So, there is a sequence which converges to x, but what is given is M is compact. So, every, every sequence must have a convergent subsequence. So, this sequence will have convergent subsequence. But, since M is compact is given, so, by definition, every sequence has a convergent subsequence. So, x n will also have a convergent subsequence, whose limit point will not different from x, because the limit point will remain the same.

So, that way, if x n is a convergent, all of it subsequence will help converge to the same points. So, it has a convergence. So, therefore, this sequence x n has a convergent subsequence, which converges to x and it must be point of M, because M is compact. Therefore, M bar is a subset of M. So, M is equal to M bar. Hence, M is closed. So, if M is a compact set, then, it has to be a closed set. To show the M is bounded, this we will prove by a contradiction. Suppose, it is not true. Suppose, M is unbounded. Suppose, M is not bounded. It means, the meaning is this, that, a set is said to be bounded, if, this is set M, when we say it is bounded, if we are able to draw a ball around a some point b with a positive radius r, such that, all the points of the set M are totally within this ball M, is it not.

So, if we assume M is unbounded, it means, there must be a some sequence of the point in M available, whose distance from b, keeps on increasing as n tends to infinity; means it can, we cannot find any suitable radius r, so that, a ball can be drawn around the point b with a suitable r, such that, all the points are inside the ball, ok. So, if M is not bounded, it means, there exists a sequence y n in M, such that, the distance of y n from some fixed point b is greater than n; where b is a fixed point, n is some fixed point of M. So, as n increases, y distance keeps on increasing. Therefore, this y n, the sequence y n, cannot have a convergent subsequence, because if it is, has a convergent subsequence, its distance from b cannot be an arbitrary large; it will always converge to a certain point and distance will remain less than a finite number.

So, this shows, this sequence y n cannot have a convergent subsequence, which will contradict our, our assumption that, M is compact, because M is given to be compact. So, which contradicts the fact that, M is compact. Because, by definition, a set is said to be compact, if every sequence must have a convergent subsequence. Therefore, our assumption is wrong. The assumption is that, M is unbounded is not correct. Therefore, M is bounded. So, M is bounded, clear. So, this lemma suggest that, if a set is given in a metric space and if it is given to be compact, then, it has to be closed and bounded. But it does not say anything about the converse side. Otherwise, if M is given to be closed and bounded in a arbitrary metric space, whether this set M will be compact or not. Now, in general, this is not true.

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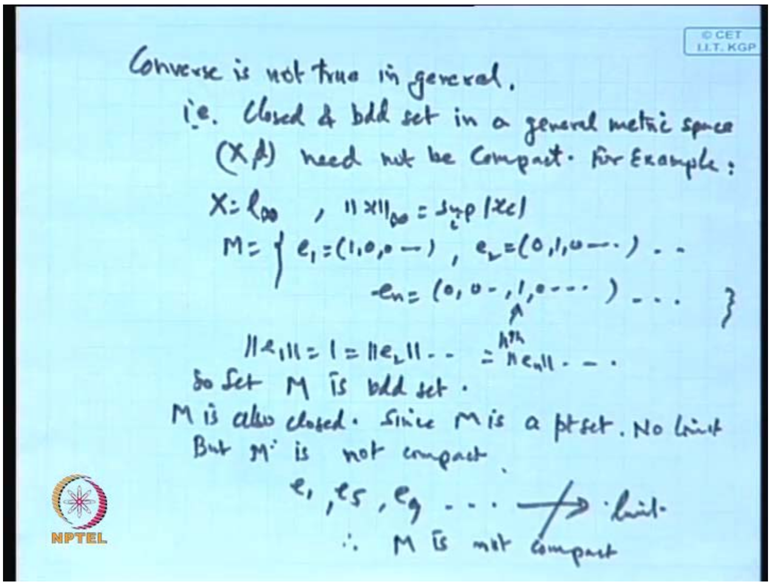

So, converse is not true in general. That is, a set, a closed and bounded set, and bounded set, in a general metric space, in a general metric space, in fact, a infinite dimensional metric space X, d, need not be compact. For example, suppose, I take the set M or X to be l infinity, set of all bounded sequences and M be the set of this type 1, 0, 0, e 2, 0, 1, 0, 0 and so on, e n, 0, 0, 1, 0, 0, like this, it is nth place and like this. Suppose, M is this set. Now, the norm of this is defined in terms of the supremum over i. So, each one is having the norm 1 and so on; each has the norm. It means, the set M, so, so, set M is bounded set, clear. The M is also closed. Why, closed means, all of its limit point must belong to it; but basically, the set M is a point set. Since M is a point set, point set means, each one is a point only. So, they do not have any limit point. So, no limit point is there. No limit point; it means, we can assume, all the limit points lies inside it. So, M is closed. But, M is not compact. Why, because, if it is compact, then, by definition, every sequence must have a convergent subsequence.

Suppose, I take a sequence like e 1, e 5, e 9 and so on. What is the limit point of this or whether this will have any subsequence limit point lies? No, because e 1 is 1, 0, 0, e 5 is 0, 0, 0, 1 and 0 and like this. So, there is no relation between this, and cannot…It means, it cannot go to any limit point. So, a sequence cannot have a, any, we cannot get a sequence, which has a convergent subsequence. Therefore, M, this example shows that, M is not compact. So, though it is closed and bounded, but is not compact. What is the dimension of l infinity? It is infinity. So, here, we have chosen an example, where the space is l infinity and we have got a thing, where M is closed and bounded, but not compact. So, in fact, we will have a result. That result says that, in case of the finite dimensional space, the close and bounded sets are compact.

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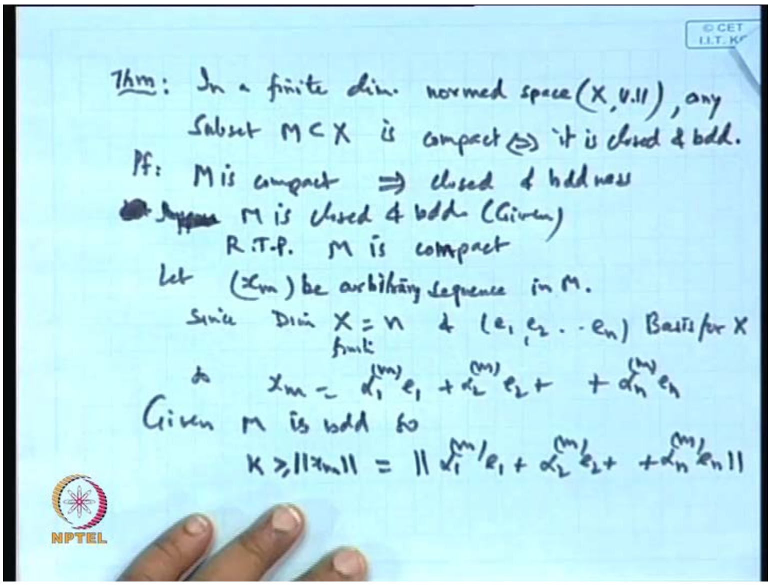

So, that is very good results, that for a finite dimensional case, the closed and bounded sets are compacts. So, in a finite dimensional case, in a finite dimensional normed space X norm, any subset M, any subset M of X is compact, if and only if, it is closed and bounded, and bounded, ok. So, we have already seen the one way that, if M is compact, then, it implies closed and bondedness. Every compact set is a closed and bounded, whether the space is finite dimensional or infinite dimensional. The other way round was a problem, because in case of the infinite dimensional, we have seen, the closed and boundedness space need not imply the compactness. But here, in case of the finite dimensional, we will prove that, even the closed and bounded set will also lead to the set, to be a compact set. So, suppose, M is closed and bounded. This is given, given; let it not suppose.

Let M be, is closed and bounded or given that, M is closed and bounded, clear. We wanted to prove M is compact, M is compact. It means, if I prove that, every sequence in M has a convergent subsequence, then, it will be compact, ok. So, let x m be a sequence, be an arbitrary sequence, arbitrary sequence in M. Now, since the dimension of X is n, which is finite, and let e 1, e 2, e n, these are the basis element for X; so, x m can be expressed in terms of these basis element as alpha 1 m e 1 plus alpha 2 m e 2 plus alpha n m e n. If there exists a scalar alpha 1 m, alpha 2 m, alpha n m, so, that x can, x m can be expressed in terms of this one, clear. Now, given that, M is bounded. So, norm of x m must be bounded by certain number k. k is greater than or equal to norm of x m, because norm of x m is less than equal to k, the length will be finite; but what is the norm? This is equal to alpha 1 m e 1, alpha 2 m e 2, alpha n m e n. Now, again, e 1, e 2, e n are linearly independent vectors, vectors. So, by that lemma, again, that, one cannot expect a vector involving large number of a scalar and the minimum length. It means, there will be a constant C, such that, this thing will hold. Here, C is greater than 0.

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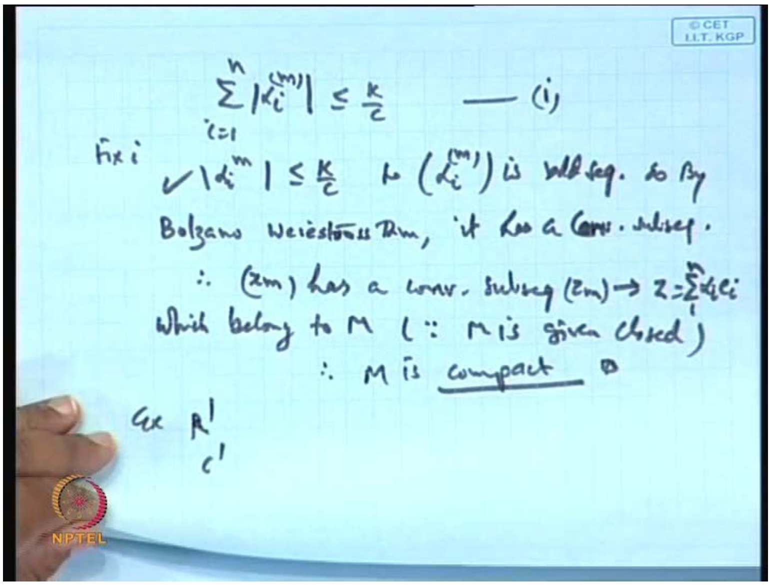

So, from here, we get that, sigma of alpha i mod alpha i m, i is 1 to n, is less than equal to k by C; let it be 1. Let it be 1. Now, if I fixed i, fix i, then, this alpha i m, mod of this, is bounded sequence. So, this sequence alpha i m is a bounded sequence. So, if it is a bounded sequence, then, by Bolzano-Weierstrass property, so, by Bolzano-Weierstrass theorem, every bounded sequence has a convergent subsequence or it has a convergent subsequence. So, by Bolzano-Weierstrass property, it has a convergent subsequence, is it not, which converges to a certain point, say exactly, alpha i m is a convergent subsequence. Therefore, our sequence x m has a convergent subsequence, say z m, which goes to the point z, which is equal to sigma alpha i e i, 1 to n, n elements in this class belongs to M, ok.

Why belongs to M? Which belongs to M because, M is given to be closed. Because, we have assumed M is closed and bounded. So, boundedness, from the boundedness, we have got this part that, since because the boundedness, for each fix i, this sequence of scalars is a bounded sequence. Once it is bounded sequence, apply Bolzano-Werierstrass property, theorem, you will get a subsequence, which converges to alpha i. It means, alpha 1 m will converges to alpha 1; alpha 2 m will converge to alpha 2 and so on. So, the point x m has a convergence subsequence, which converges to alpha 1 e 1, alpha 2 e 2, alpha n e n. Now, this point a must be point of M, because, M is given to be closed. So, any sequence whose limit point, all the limits point must be the point of M. Therefore, this belongs to M. So, what we conclude is that, if we start with an arbitrary sequence in M, then, it has a convergent subsequence, whose limit point belongs to M. Therefore, M is compact. So, this completes the…Clear? So, in a finite dimensional case, the compactness is nothing, but the close and bounded and that is why, in case of the R 1, in case of the C 1, we simply say, closed and bounded sets are compact; like this way, clear; we get this, ok.

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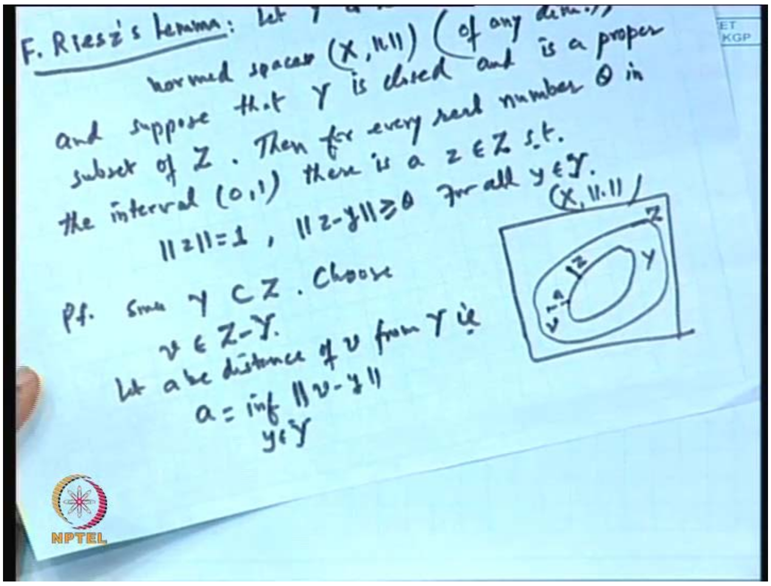

Now, there is another results, which will give you a further deeper study, which will help you in further deeper study of the normed space, for, in case of finite dimensional. The result is known as F. Riesz's lemma, Riesz's lemma. What is this lemma? The lemma says, let Y and Z, Y and Z be subspaces of a normed space, of a normed spaces, normed space X, subspace of a normed space X, of any dimension, of any dimension and suppose, and suppose, that Y is closed, Y is closed and is a proper subset, proper subset of Z; then, this lemma says, then, for every real number theta, every real number theta in the interval 0, 1, in the interval 0, 1, there exist a z or there is a z, belonging to capital Z, such that, the norm of z or length of this vector is 1 and its distance from the element Y is our pre-sign number theta, for all y belonging to capital Y. So, it is very interesting here.

What this shows is that, suppose, here we have a normed space, X norm and let this be a Y and this is our, say Z. Both are subspaces of X, but Y is given to be a closed and proper subset, subspace of, subset of Z; closed as well as proper. Then, what this shows is that, if I take any number theta between 0 and 1, say 1 by 4, or 1 by 2, then, corresponding to that number, one can identify a point z in capital Z, whose length is 1, and whose distance from any elements Y…Now, distance of this, from Y is greater than equal to theta, where Y belongs to this, because Z minus Y is greater than or equal to theta, ok. So, this will be a lemma and which will help you in completing, in proving many results, in case of the finite dimensional. The proof of this lemma goes like this. Since Y is a proper subset of Z, so, let us choose an element v belonging to Z minus Y. So, v will be in a point here.

Since Y is closed, so, find out the distance from v to this Y. So, let a be the distance of v from Y; that is, by formula, the a becomes, infimum of norm v minus y over y belongs to capital Y, ok. Now, obviously, a is greater than 0, because Y is closed. So, we can always find a point, we can get the point, whatever the v, it lies in Y only and v is lying outside of the v. So, the a must be greater than 0. So, clearly, a is greater than 0, as Y is closed, ok.

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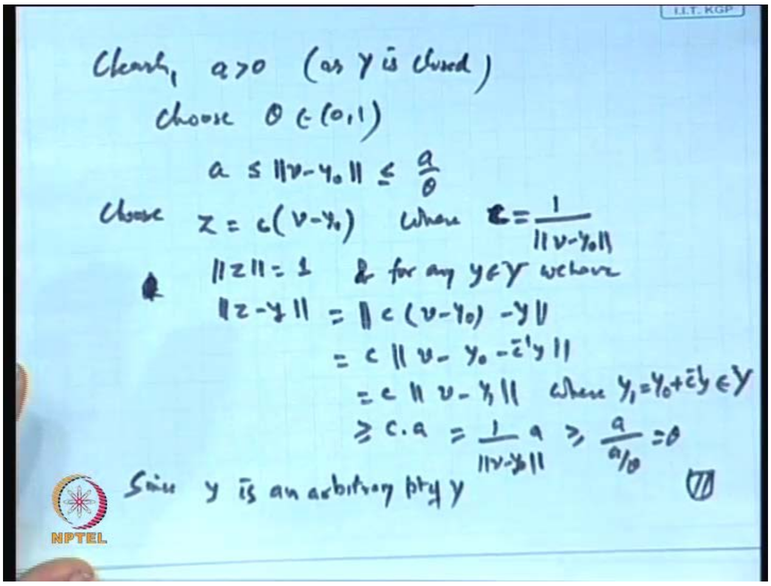

Now, let us pick up the theta. Choose theta, belonging to 0, 1. Now, this infimum definition says that, since the infimum is attained, so, we can find a point y naught in this, where the infimum is attained and that y 0, v, v minus y 0, is exactly a. So, if we picked at any point other than y naught, then, what we get it, we can find out the number between a and a theta by, a and a by theta, a and a by theta, theta lying between 0, 1. So, a by theta is greater than a, and this infimum is attained. So, we can find out the, there is a…Find y 0, such that, infimum that, this is greater than equal to norm of v minus y 0 is less than equal to this; because of the infimum, that is why.

Let us choose z as C into v minus y naught, where c is, oh sorry, C, not z, C is 1 by norm of v minus y 0. Suppose, I take z this. By choosing this way, we can immediately say, mod z is 1. And, what is its distance, from any arbitrary point y, that comes out to be, for any, any y belongs to capital Y, we have, we have norm of this thing y, which is equal to what, norm of C v minus y naught minus y; and, this will be equal to, C, if we take it outside, v minus y naught minus C inverse y, which is equal to C v minus y 1, because, where y 1 stands for y 0 plus C inverse y, which is an element of Y, because Y is a subspace. So, this will be the point in Y, ok.

Now, this C is, is nothing, but this one. So, we saw that, y 1 belongs to (( )). Now, v minus y 1 is greater than a. This is greater than equal to C into a. Why, because, y 1 is an arbitrary point and a is the minimum infimum; so, this. Substitute this, C is 1 upon, into a and this part is greater than equal to what, a by, a by theta, because of this; clear? So, from here, we get, this is equal to theta. So, since y is an arbitrary point, is an arbitrary point of capital Y, so, we get, for any point z minus y is greater than equal to theta and that is the required (( )). This lemma, Reisz's lemma, gives you a immediate one application of this that, in case of the finite dimensional, if a normed space has a property that, a unit ball is compact, then, it must be a finite also.

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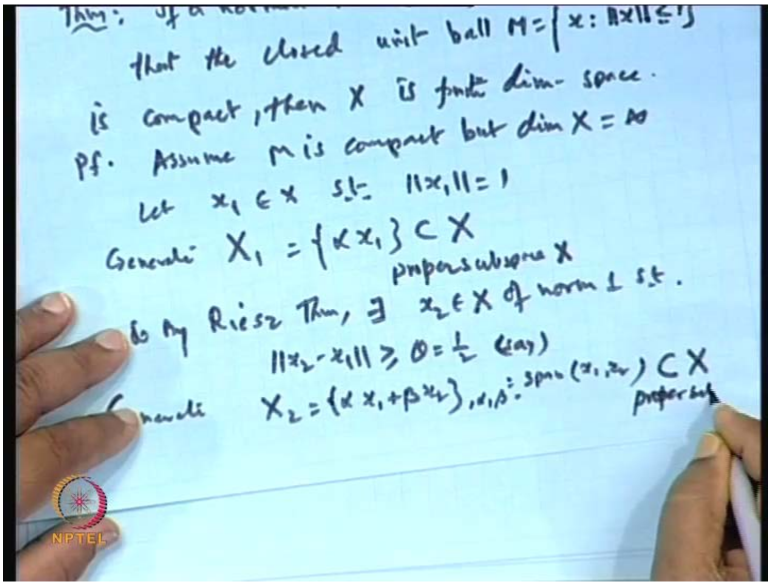

So, if a normed space X has a property, this result, application of this lemma is, if a normed space X norm has a property, has a property that, the closed unit ball, unit ball M, norm of x is less than equal to 1, is compact, then, X is finite dimensional space. So, this is very interesting and important result because, so far, we have seen a…If x be a finite dimensional space, then, a closed and compact bounded set is a compact set; because, unit ball, closed unit ball is a bounded and closed. So, it must be compact. But if I take the other way round, if a normed space has a property that, a closed unit ball is compact, then, it must be a finite dimensional. So, let us see the proof of this. Again, we prove by contradiction. Suppose, M is compact, given M is, assume that M is compact, but X, dimension of X is not a finite; is infinite. So, once its dimension is infinite, then, we can find out the so many points inside it, x 1, x 2, x n, so on.

So, let us picked up a point x 1 of norm 1. So, let x 1 belongs to X, such that, norm of x 1 is 1. Now, with the help of this x 1, generate a space X 1. Generate X 1 as a linear combination of this, which is a subspace of X of dimension 1. Now, this is a proper subset of X. This is a proper subspace of X. So, by this previous lemma, by Riesz's lemma, we can find a point x 2, which is in X, having the norm 1 and whose distance from x 1 can be made greater than equal to theta. So, we can…So, by Riesz's represent, Riesz's theorem or lemma, we can, there exist an element x 2 belonging to X of norm 1, such that, the distance from x 2 minus x 1 is greater than equal to theta, say half, say half. Now, we have got the two points x 1 and x 2. So, generate, with the help of these two point, another space X 2, which is the linear combination of alpha x 1 plus beta x 2; linear combination of this, generates this one; all alpha, beta are constants. So, it is a span of x 1 comma x 2. Now, this is a subset of X again and again, a proper subspace of X. So, again by Riesz's lemma, we can further find…

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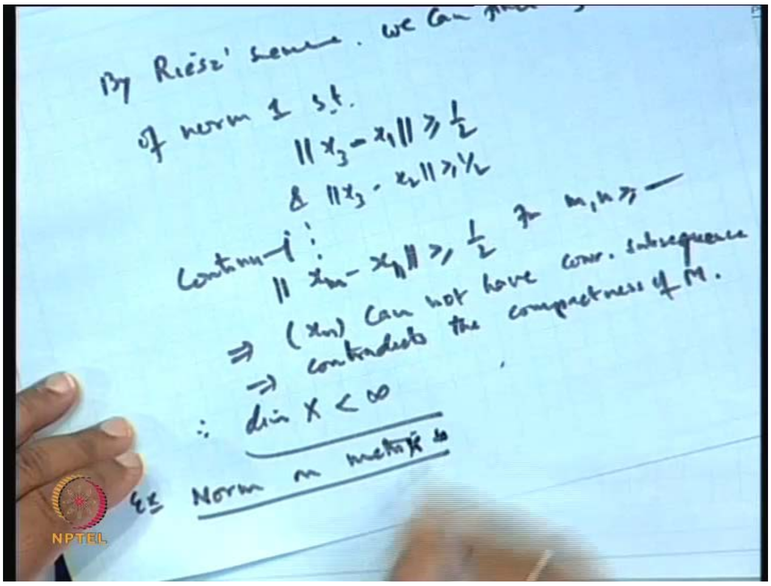

So, by Riesz's lemma or Riesz's theorem, we can find x 3 in X of norm 1 and of norm 1, such that, its distance from the elements of x 2 will be greater than equal to the value theta, say half, greater than equal to half. So, if we continue this way, we get a sequence norm of x 1, x m minus, continue this way, we get, this is greater than equal to half, for n and m is large; for any n and m greater than equal to certain points. What this shows? This shows that, this sequence x n, the sequence x n cannot be, cannot have a convergent subsequence, have convergent subsequence.

Why, because if a sequence is convergent, it must be a Cauchy sequence, because it is a sequence of real or complex number. So, if it is a convergence sequence, it has to, every Cauchy sequence it, it is a Cauchy sequence; convergent means it is Cauchy. So, distance between x m minus x n must be less than epsilon. But here, the distance between x m and x n is greater than equal to half, arbitrary, a, say, greater than equal to say epsilon, which is greater than, which is equal to half.

Therefore, this sequence cannot have a convergent subsequence. Hence, if it cannot be convergent subsequence, hence, the set M is not compact. So, this contradicts the compactness of M. This contradicts the compactness of M. Hence, our assumption, that is, the X has a dimension of infinity is wrong; that is, dimension of X is (( )). So, the dimension of X must be finite; that is what exactly we prove. So, this compactness…Now, there are certain problems, which we will discuss here. Now, R n and C n, these are the compacts or not? Let us see few problems here. As we have discussed last time, there were some examples for the norm on a metric space, norm on the matrix, norm on matrix.

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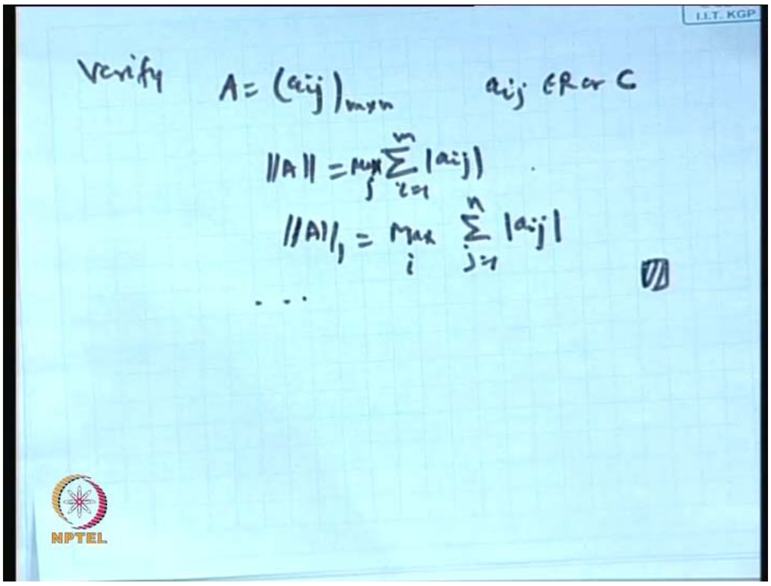

Let us see just one or two here. Suppose, A be a matrix a i j of order m cross n, with fixed m and the a i j may be either real or complex. Then, we can introduce the norm in so many…The norm of this A can b written as sigma, like this a i j, a i j and then, this way, a i j, mod a i j, summation is suppose, i is 1 to n,1 to m, and then, maximum value you can choose over j. You verify whether this is a norm. Verify whether this is norm. Another way is, I am just saying, norm 1 is the maximum of i sigma j, 1 to n, a i j, whether it is norm; and then, you can also define in other way. So, first, you verify these are the norms and whether they are equivalent norm or not, that will give from the…Thank you, thank you.