Functional Analysis
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Lecture No. # 10
Banach Spaces and Schauder Basic

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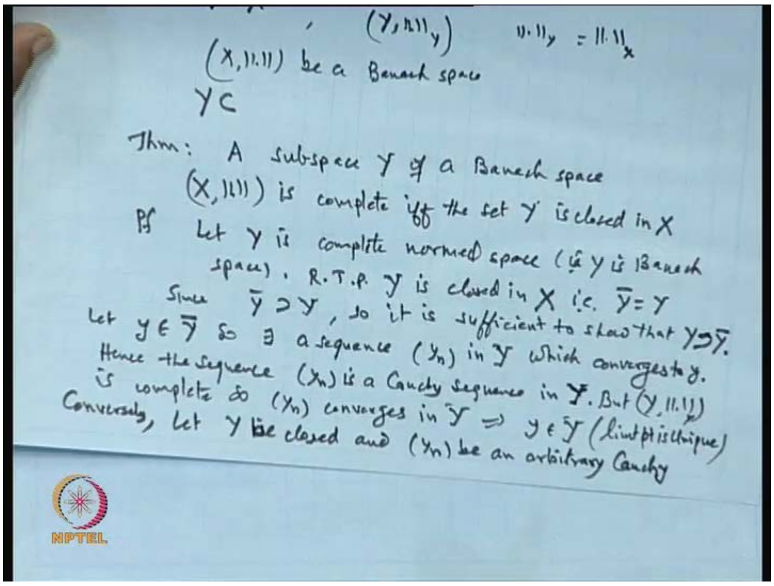

We discussed the normed space and the subspace of the normed space; and, we have seen that, if X be a normed space, be a normed space, then, the subspace of X, we mean a non-empty subset Y of X, Y of X considered as a vector subspace of X and the norm on Y is, basically the induced norm of X on Y. So, whenever the norm of X is defined over the Y, then, the corresponding norm we say, it is the norm on Y induced by the norm on X. So, if a Y, which is a vector subspace of X, it also forms a norm under the norm given by norm of Y, then, we say, Y naught is a normed space, since it is a subspace, vector subspace and the norm is considered to be the induced norm. So, we call it as a norm, subspace of the normed space X norm.

But if X be a norm, instead of the normed space, if X be a Banach space, then, the subspace of X is considered to be a vector subspace of X, together with the norm, same norm, as induced, as defined earlier, it is a induced norm of X on Y. But it is not necessary that, Y should be complete; that is, for having the subspace of a Banach space, Y simply should be a non-subspace of X, not as a complete subspace, ok. So, there are many cases where you have a norm and the subspace Y becomes a normed space, or sub normed space, but it is not, need not be a Banach space. So, in order to get the corresponding subspace is also a normed space, we require the closeness of Y. So, in fact, we have a result, and that theorem says that, a subspace, a subspace capital Y of a Banach space capital X, under this norm, each complete, if and only if, the set Y is closed in X. This we have already proved earlier in one of the results, wherein that metric space and we have seen that, a subspace m of a complete metric space X itself is complete, if and only if the set is closed.

So, the same proof goes; just I will summarize what we have done it. Suppose, we want Y to be a Banach, ok. So, let, let us suppose, Y is a complete normed space, that is a Banach space, complete normed space, ok; that is, Y is given to be a Banach; that is, Y is Banach space. This is known. We wanted to show that, Y is closed in X. Closed means, that is what we want to show that, Y closure must be equal to Y. But always, Y closure always covers Y. So, it is sufficient to show, it is sufficient to show that, Y is, Y covers Y closure. So, in order to show that Y is a proper subset, proper set of Y closure, let us take a point in Y closure. Let us take a point in Y closure. So, suppose, we take a point a y. Let y belongs to Y closure. Now, Y closure is the set of those points of Y, together with its limit point. If y belongs to Y itself, then, there is nothing to prove; it is automatically done; obvious. But if y is not a point of Y, but it is a limit point, so, by definition, there exists a sequence y n in Y, which converges to, to Y, which converges to Y.

Since every convergence sequence is a Cauchy sequence, so, the sequence, hence, the sequence Y n is a Cauchy sequence, is a Cauchy sequence in Y; but Y is given to be complete. But Y is complete, is complete, because it is given to be a Banach space. So, complete means, every Cauchy sequence must be convergent. So, we get the sequence Y n convergent, or converges in capital Y, converges in capital Y. But if a sequence Y n converges, the limit point must be unique. So, it will converge to the same point as the point y, which is that limit point of Y ns. So, this implies that, y belongs to capital Y, because the limit point is unique, because limit point is unique; that is why, we get it convergent to Y, ok. Converse is very easy. Conversely, it is given that Y is closed; Y is closed. So, let Y is closed, or be closed set, closed. And, we wanted to show Y to be a complete. So, let us consider a arbitrary Cauchy sequence. Let Y be closed, and let y n be an arbitrary Cauchy sequence, Cauchy sequence in Y, an arbitrary Cauchy sequence in Y, ok.

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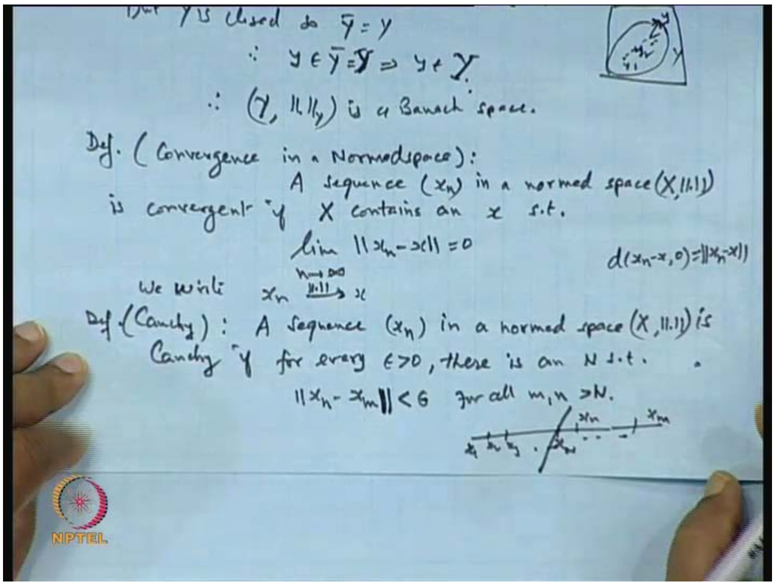

Let us suppose, let x n, this y n converges to Y. Now, this point y, because y n is a point of Y; so, it is a point of X also, point of X also and X is given to be a Banach. So, any Cauchy sequence, it is convergent. So, it must at least a point of X. If I prove that y, which is a point of X, is really a point of Y, then, the Cauchy sequence y n will be convergent in Y and Y becomes complete. So, this y n belongs to Y. Now, y 1, y 2, y n, these are the points of Y; y 1, y 2, these are the points of Y and this sequence converges to Y. So, this Y will be, will act as a limit point of y n, clear. So, y will be the limit point of y n, so, y must be the point of Y closure; but Y is complete ,Y is, sorry, Y is closed. So, Y closure is the same as Y. Therefore, this y, which belongs to Y bar, is basically, is in Y; hence, we get, y belongs to capital Y. So, every Cauchy sequence in y is convergent in Y; therefore, Y, under this induced norm is a Banach space, or complete and that proves it; is it clear now?

So, basically, we, this result which we have gone through, says that, if we want this any subspace to be a complete one, then, it must be closed; that is all. This, now, we define few more concepts like convergence, Cauchy sequence in the normed space. The convergence in a normed space. A sequence x n in a normed space X norm is convergent, or is said to be convergent, if capital X contains an element x, such that, limit of this (( ))

norm of x n minus x, as n tends to infinity, goes to 0; clear? Basically, this norm of x n minus x, this is basically the distance between x n minus x, and 0, because this way, we are defining as norm of x n minus x. So, when we are taking a sequence x n as a convergence sequence, it means, there must be a point x available in X, whose distance from x n keeps on reducing and reduces to 0, when n is sufficiently large; goes to 0.

Similarly, we define the concept, such that, then, we say this (( )), we write this, we write x n converges to x in this norm. This is the notation normally we used. Then, the Cauchy sequence is, we define in a similar way, or fundamental sequence; a sequence x n in a normed space X norm is Cauchy, if for every epsilon greater than 0, there is an positive integer n, such that, the norm of x n minus x m is less than epsilon, for all m, n, greater than capital N. So, a sequence in a normed space is said to be Cauchy, if, after a certain stage, the terms of the sequence x 1, x 2, x 3, and so on, say, after the certain stage, if I truncate the series, the difference between any two terms of the sequence x n and x m under this norm, can be made less than epsilon; and, such a sequence is called a Cauchy sequence, or fundamental sequence.

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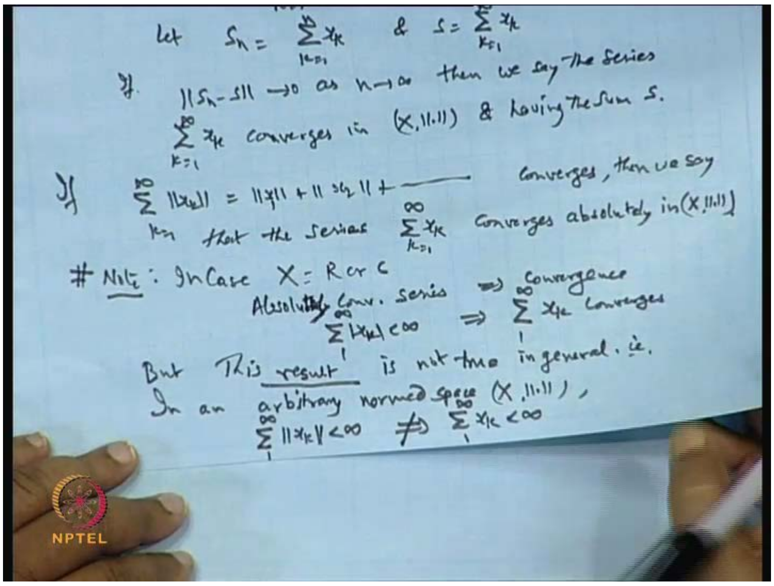

We also define the convergence of an infinite series in the normed space; convergence infinite series in the normed space. Suppose, we have a series sigma x k, k is 1 to infinity and where x k, this is a sequence, belongs to a normed space capital X; X is a normed space; the terms of this sequence are in this. What we mean to say, then, when the sequence, when the series converges in the norm X; the meaning of this is that, if a series, a series sigma x k n is said to converge in the normed space, if the corresponding sequence of the partial sum converges to element S. So, if we take S n as a sum of the first n terms, like this, and S is the entire sum, (( )). So, if the sequence S n minus S goes to 0, as n tends to infinity, then, we say, the series sigma x k, k is 1 to infinity, converges in the normed space X norm and having the sum S, and having the sum S. So, a sigma, an infinite series will be said to be converged to the sum S, if the sequence of the partial sum S n goes to S under the norm, when n tends to infinity.

Now, if the series sigma norm of x k, k is 1 to infinity, that is, norm of x 1 plus norm of x 2 and so on, if this series converges, then, we say, we say that, the series sigma x k, k equal to 1 to infinity, converges absolutely, absolutely in the normed space; converges absolutely in the normed space, or will be said to be absolutely convergent, ok. Now, in case of the real or complex number, in case of, in case X is R or C, that is, when we are dealing with a Banach space of real numbers, or Banach space of, of a complex number, then, a absolutely convergence series, absolutely convergent series, absolutely convergent series implies the convergence; that is, a series sigma mod x k converges, implies, the original series converges; vice versa need not be true; a series may be convergent, but it may not converge absolutely; for example, minus 1 to the power n by n, this series converges, but it does not converge absolutely.

So, in case of the real, complex number, that is, when we replace the X by real or complex point, then, the absolutely convergent series implies the convergence; but this result is not true, is not true, in general; that is, in an arbitrary normed space, in an arbitrary normed space, in an arbitrary normed space X norm, the absolutely convergent series sigma of norm x k convergent, need not imply the convergence of this series. So, here, we have a real generalization, from real to arbitrary vector space; that we do not have that, all the properties, which are enjoyed by the real or complex, series of the real complex number is true or valid, in a general normed space, or in a arbitrary normed space.

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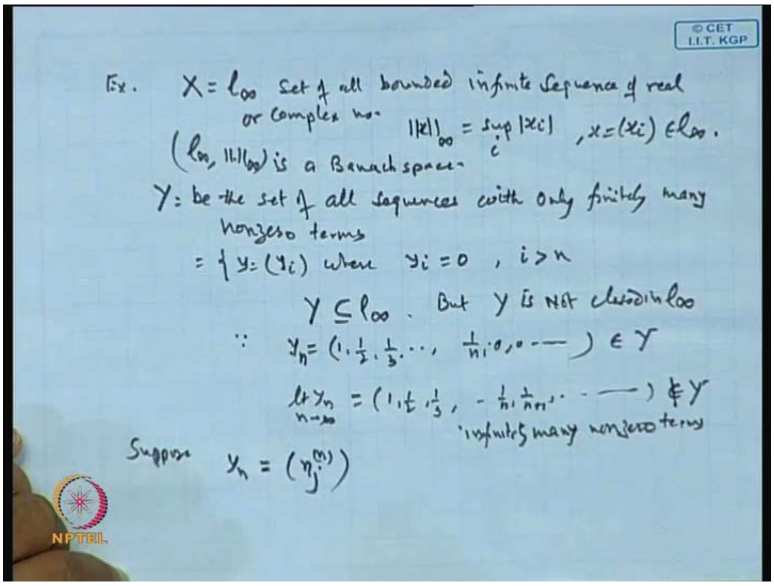

For example, see the example here. Suppose, I take X to be l infinity, set of all bounded, set of all bounded infinite sequence of real or complex number; set of all bounded infinite sequences of real or complex number; and, the norm of this, any element under this, is defined as the supremum of mod x i, where x is equal to x i belongs to l infinity, under sup norm; and, we have seen that, this is a complete norm space, under this x l infinity, under this sup norm, is a Banach space; this, we have seen, ok. Now, if we take Y to be the set of…Let us take Y be the set of, set of all sequences, all sequences with only, with only finitely, many non-zero terms, with only finitely many non-zero terms; that is, the Y is the set of all sequences of this type y i, where y i is 0, after a certain instance, say, i is greater than n; after certain instances, suppose, it is greater than n; all the, only few terms are non-zero, and rest are 0s. So, it is consist of only finitely many non-zero terms.

Now, this Y is obviously a subset of l infinity; because, what is the supremum of Y i; because they are finite, so, its supremum will be bounded, maximum value, among those finite terms. So, it is element of l infinity. But it is not closed; but y is not closed in l, l infinity; why, because, if we choose the sequence y n as 1, half, 1 by 3, 1 by n and 0, 0, 0. Suppose, this is the y n elements; this is in capital Y. So, y 1 is this 1, 0, 0; y 2 is 1, 1 by 2, 0, 0; y 3 is 1, 1 by 2, 1 by 3, 0, 0; every time you are taking only finite number of non-zero terms. So, it is the point in Y. But what is the limit of y n?

This goes to 1, 1 by 2, 1 by 3, 1 by n, n plus 1 and so on, a large number of infinitely many non-zero terms, many non-zero terms. So, it may not be a point in Y; in fact, it will not be a point of Y, because Y contains only those sequences, where finite number terms are non-zero and rest are 0. So, this is not be a point of Y; therefore, it is not a closed set. So, once it is not closed, Y will not be complete, ok. Now, let us take another same. Suppose, I take, suppose, I choose a sequence Y n as the sequence eta j n, where eta j n is equal to 1 by n square, if j is equal to n, and equal to 0, if j is not equal to n, ok. Suppose, I take this; it means, what we are taking is that, we are choosing basically, the y 1 as the sequence 1, 0, 0, 0; y 2, we are taking a sequence 0, 1 by 2 square, 0, 0, 0, is it not; and y 3; similarly, y n, we are taking a sequence 0, 0, 0, 1 by n square, 0, 0, 0. So, basically, we are taking this sequences, which are obviously in Y, because Y is the set of all finite, or sequence which are having only finite number of non-zero terms. So, only 1 term is non-zero; rest are 0s. So, they are all belonging to Y, and in fact, it will be a subset of l infinity. Supremum will be the same as 1, 1 by 2 square, 1 by n square, norm of…

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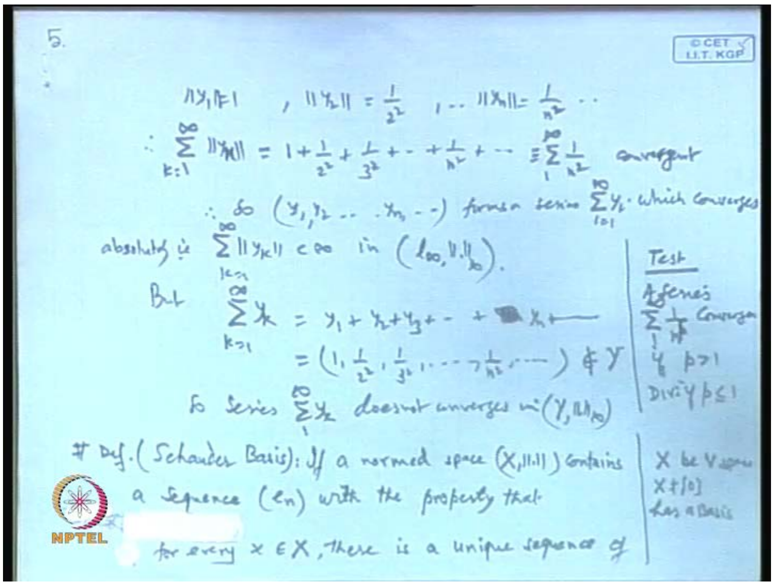

So, what we get, norm of y 1 is 1, because supremum of this 1; norm of y 2 is 1 by 2 square; norm of y n equal to 1 by n square and so on, because there is only one term. So, supremum will attain only at that point, which is non-zero; rest are 0. So, what you have, is this series. So, sigma of norm of y n, n is 1 to infinity, if I look this series, then, what you get, y k, let us be y k, k is 1 to n. So, we get norm of y 1, plus norm of y 2, plus norm of y 3, plus norm of y n and so on; this series is of the form sigma 1 by n square, n is 1 to infinity, where p is greater than 1. So, it is a convergent series, is a convergent; well known result. So, this series converges; it means, the sequence y 1, y 2, y n, so, the sequence y 1, y 2, y n and so on, this sequence, which is in l 1, each term is in l 1, and the corresponding series, forms a series, series, which converges, which converges absolutely; that is, norm of y k, k is 1 to infinity, is finite, converges in the norm l infinity, this; is it correct? So, this forms a...So, we have a sequence y 1, y 2, y n, all these points belonging to l infinity, and this forms a series, which is convergent absolutely. But the series itself, y k, k is equal to 1 to infinity, that is equal to y 1, plus y 2, plus y 3, y n and so on.

This is basically, 1, 1 by 2 square, 3 square, and so on; because y 1 is 1, 0, 0; y 2 is 0, half square. So, when you combine, you are getting like 1, 1 by 2 square, 1 by 3 square, and so on, 1 by n square and so on, where large number of terms are non-zero. So, this will not be a point in Y. This summation is convergent. This is Y; this series test, the test is like this. An infinite series, a series sigma 1 upon n to the power t, n is 1 to infinity converges, if p is greater than 1; diverges if p is less than or equal to 1. This is a standard result. In fact, its proof can be given, clear? You can see in any real analysis book; you can get the proof. A series sigma 1 upon n to the power of p, this series converges when p is greater than 1, and diverges, if p is less than or equal to 1. So, I am using that result, because p is 2 here; sigma 1 by n square, p is 2. So, it is a convergent series. So, this series converges, but here, it is not a point in Y. So, it means, this series does not converge; y k, k is 1 to infinity does not converge in, it does not converge in Y, in Y, under the norm of l infinity, ok. So, though the series converge absolutely, but it is not convergent. So, what we conclude here is that, in an arbitrary norm space, absolutely convergence of the series need not imply, always the convergence of the series; and that is we have seen here. So, this is very interesting results.

Then, we have another concept, which we call it as a Schauder basis. You know, in case of the vector space, if X be a vector space, and it is a non-zero vector is different from 0, that is X is not equal to 0 vector, then, every vector space has a basis, which we call it as a Hamel base. Basis means, a sequence, a set of points, which forms a linearly independent set and spans the whole, is class X; the linear combinations of this any element of X can be expressed as a linear combination of the basis element, then, that collection, we call it as a basis. Now, here in case of the normed space, it is not only the two structure, addition and scalar multiplication is given, but we also use the norm. So, whenever we define a basis, if we take up this as a definition that, a class as set of points which forms a linearly independent set and spans the entire class X is a basis, then, nowhere the norm is involved.

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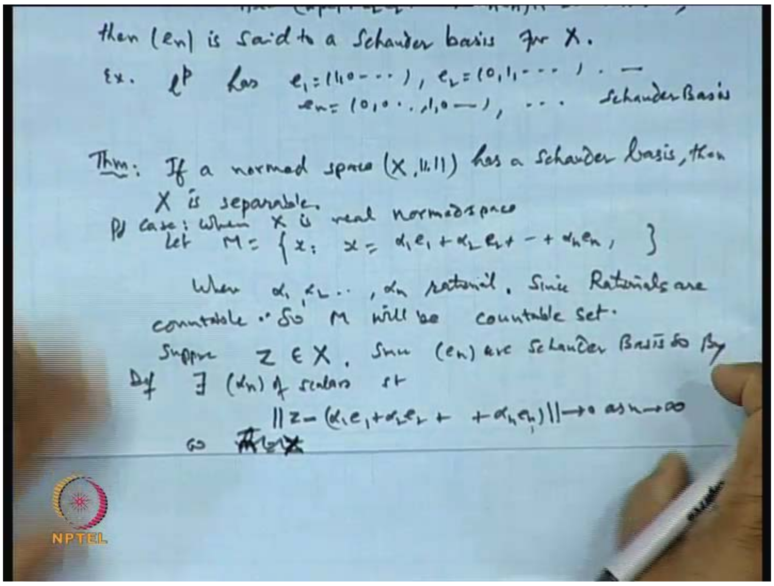

So, in order to have it, the norm concept also in the definition of this, we use the another concept, which called the Schauder basis. So, basically, the Schauder basis is generalized form, you can say, in the defined for a normed space. So, we say, if a normed space X, if a normed space X contains a sequence, contains a sequence e n, with the property, with the property that, for every, for every x belonging to capital X, for every x belonging to capital X, there is, there is a unique sequence of scalars, unique sequence of scalars, alpha n, such that, the norm of x minus alpha 1 e 1 plus alpha 2 e 2 plus alpha n e n, this norm tends to 0, as n tends to infinity; then, we say, then, the sequence e n is said to be Schauder basis, said to be a Schauder base, or Schauder basis for X, ok. So, now, here, the involvement of all the operations are there; norm is involved; addition is involved; the scalar multiplication is involved.

So, basically, it gives all the relation and the definition of the basis. It is called the Schauder basis. For example, the l p space has a Schauder base, has e 1, 1, 0, 0, 0, e 2, 0, 1, 0, 0, and so on; e n's are 0, 0, 0, 1, 0, 0. These are Schauder basis, Schauder basis; at this e 1, e 2, e n is a Schauder basis, ok. Now, there is a very interesting result. The result is, if a normed space X, X has a Schauder basis, has a Schauder basis, then, if normed space X has Schauder basis, then, X is separable, separable. What is the separable? I just told you earlier, a X, if a normed space X is said to be a separable, if it has a countable subset which is dense in itself, dense in it. So, if a space has a subset, which is countable and dense, then, such a space we call it as a separable space; l p space is a separable space, but l infinity is not a separable space; that, we have seen; it does not have the subset, which is both countable as well as dense, clear. So, what this result says that, if a normed space has a Schauder basis, then, it must be a separable. It means, this also gives a hint that, all the normed space need not have a Schauder basis; if it is a separable, then, it cannot have a Schauder basis.

If it has a Schauder basis, then, it must be a separable space; but all the separable space, whether it has a Schauder basis or not, that is the question which is answered in 1973 by Enflo and he showed that by giving a counter example that, every separable space need not to have a Schauder basis, ok. So, other way around it is not true; that is, if a space is separable, it has to, it has a Schauder basis; this is not true, because Enflo has given a counter example in 1973 and that he has proved that, a separable Banach space need not have a Schauder base. But one way it is correct; if a normed space has a Schauder basis, it has to be a separable normed space; proof is like this. What is the definition of Schauder basis? Basically, a Schauder basis means that, a sequence e n with the property that, there exists a sequence of a scalars alpha n, such that, this goes to 0. Now, we want X to be a separable. So, we want, it must have a subset, which is both countable and dense. So, let us take M with the set of those points x, where x is the collection of these points, like this; any element is the collection of these and n is arbitrary; n is…Collection means, linear combination of this Schauder basis.

Now, if we do, where alpha 1, alpha 2, alpha n, I am taking to be rational point, rational point. Now, here, we should take different case, if X is, let us take the case one, when X is real Banach space, real normed space. It means the scalars are reals; field of a scalar is a real number. So, alpha 1, alpha 2, alpha n are real. Now, what we are doing is, we are collecting the set point x, which can be expressed as a linear combination of e 1, e 2, e n, where the coefficients are rationals. Consider, all the such…Now, since the rational numbers are countable, are countable, so, M will be countable set, countable set, is it not; because all possible combination you can choose, but all possible combination can be obtained by changing the alpha 1, alpha 2, alpha n only; and this alpha 1, alpha 2, alpha n, can assume any rational value. Therefore, it can be a all the rational numbers when you use, it becomes a countable set. So, M is countable,e clear. Now, if we take any point x. Suppose, I choose a point say Z, belongs to capital X, clear. then, by definition of this, since e n's are Schauder basis, so, by definition, there exist a sequence alpha n of scalars such that, norm of Z minus alpha 1 e 1, alpha 2 e 2, alpha n e n, this must go to 0, as n tends to infinity, clear; must go to 0, is it clear. So, we are getting this.

Now, if Z is a point in X, we can always find a sequence alpha n of scalar such that, this is true. Now, what is alp ha n? Alpha n may be real, may be rational. If they are rational, then, this combination will be a point of M; if it is real, then, we can find out a corresponding sequence of rational points, which approximate these alpha 1, alpha 2, alpha n, because every real number can be approximated by means of a rational. So, it means, corresponding to alpha 1, alpha 2, alpha n, we can always obtain the rational points here as scalar, so that, the combination of this will, will approximate this combination and that will approximate Z. So, any arbitrary element of X can be expressed as a linear combination of the elements of M. So, it becomes dense. So, this implies…So, M closure will be dense. M closure will be Z, sorry, M closure will be X; I will write here next.

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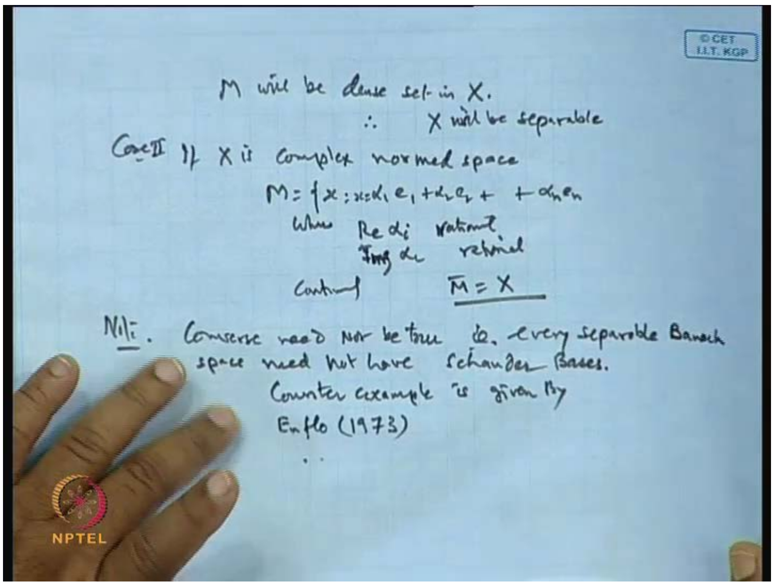

So, M will be dense set in X; therefore, X will be separable. Now, if suppose, X is, if X is a complex normed space, then, what you do is that, you choose the M, which is alpha 1 e 1, alpha 2 e 2, alpha n e n, where real part of alpha i is rational; imaginary part of alpha i is also rational, ok; so that, we can get only that, any real number can be approximated by means of these; real part of this is rational; imaginary part is rational. So, if any complex number, there a plus i b, then, if a is, and b are rational, fine; otherwise, it can be approximated by means of the rational points. So, again, that same, continue this way, we get again M closure will be X. So, it will be dense set, ok. Converse need not be true; that is, that is, every separable Banach space, separable Banach space need not have Schauder basis; and the counter example is given, given by Enflo, 1973; he have shown this example, clear; ok. So, we are not going in detail, because it is a big exercise, example, but it is available and if you, this solution is given in this Acta Math 130, Acta Math 130; this is reference number. So, if you want, you can go through it.

Now, this, just like a metric space we have seen, if a metric space is not complete, one can go for the completion of the metric space, by using the isometric concept. So, similarly, here also, we can, every normed space can be completed under the concept of, of, with respect to the isometry, ok. So, the result is like this. This simply gives the, the completion. Let X be a normed space; then, there is, there is a Banach space X head and an isometry, A from X onto a subspace W of X head, which is dense in X head. This space X head is unique, except for isometry; except for isometry, except for isometry.

So, the meaning of this is that, if, suppose we have this space X, which is not a complete metric space, that, one can always obtain a completion of it, X head in such a way that, it has a dense subspace W and an isometry A, one-one onto mapping, which preserves the distances between X and W such that, W closure is X head, ok. So, corresponding to X, we can always find a complete metric space X head, in such a way that, it has a closed subset, subspace W, which is dense in X and has a one to one relation isometry between X and X head; and this space will be here; except for isometry; that is, if we have another, say, suppose, another complete metric space corresponding to X, then, it should have the property that, it will have a W, it has a W, say delta, W delta, another set, which has an isometry between X and W, and the closure of W delta is X double head; but, what is the relation between X and X head; they should also have a isometry; a isometric; these two spaces will be isometric, ok.

So, the space will be unique; if there is another space, then, they should be isometric to each other, clear. So, it is…Now, this almost completes the concepts in this normed space, particularly, this general normed space. We will go next time the, what is the finite normed space, and what is the simpler properties in, in the normed space, which are finite in nature; finite normed space, what are the properties enjoyed by them. Now, the slight thing we have, just like we have defined a semi metric, in a similar way, here, we also define the semi norm. The first property is usual; norm is a real value, nonnegative number. And, the second property only, we say that, this property p of x equal to 0, implies x equal to 0, this need not holds good, need not be true, in case of the semi normed space; but, rest of the property remains true, ok. Other property will be true. An example is that, if we take the norm of X as the limit of the sequence X n, as n tends to infinity, then, it is a semi normed space, because X n equal to 1 by n, limit will come out to be 0, but the sequence is different from 0. So, if we take x n to be 1 by n, then, norm of X comes out to be 0, but X is, x n is not equal to 0, ok.

So, this way, we can, it correspondingly, we can define the concept of semi norms and similarly, the continuity of the norm. The norm is a continuous function, with respect to the both the operation, addition and scalar multiplication and continuity of the norms. So, these two concepts are parallel and in fact, this result will help you in getting the continuity; norm of X minus norm of Y, modulus of this. So, if we take this, any sequence X n converges to Y, the corresponding norm of X can be converged to norm of Y. So, it is a continuous function. Similarly, for the scalar multiplication, one can show.

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No, no. Norm is a continuous function. Norm is a continuous function. If we take this one, here, what is, X n converges to x in the norm; then, norm of X n will converge to norm of X. So, it means norm is a continuous function. Otherwise, next time, I will give you in detail, what is that. Thank you.