Functional Analysis

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Module No. # 01

Lecture No. # 01

Metric Spaces with Examples

This is a first course in functional analysis. Functional analysis is an abstract branch of Mathematics, that originated from classical analysis. Its impetus came from linear algebra, ordinary and partial differential equation, approximation theory, linear integral equation, etcetera whose theory had a great effect on the motivation and development of the new ideas. In functional analysis, we deal normally, with the abstract space and we use the abstract approach in connection with the abstract space. The abstract space we mean, are set of unspecified say, point, together with some axioms defined on it. Now, where these axioms are changed, then, corresponding the structure are changed.

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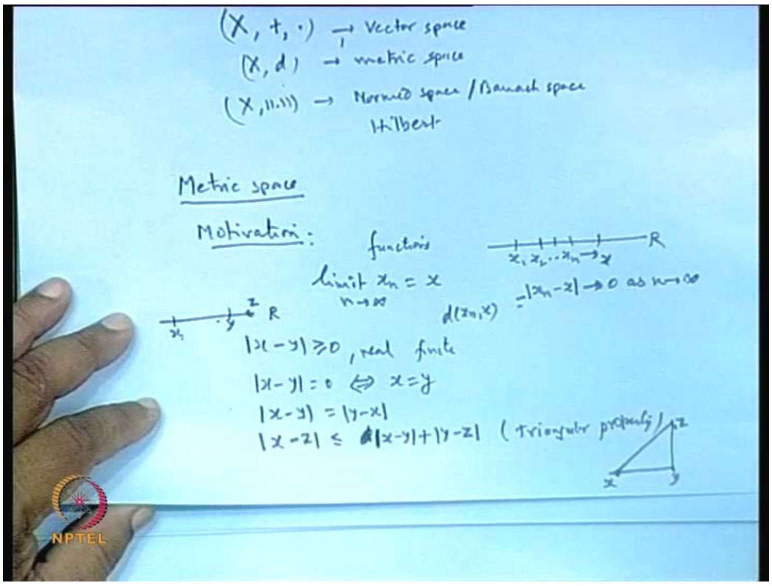

For example, if we take, a set X with the two operation, addition and scalar multiplication and X is over a set, field, say real or complex number, then, this will form an structure and we could know this structure is a vector space or linear space, provided with addition, scalar multiplication, this satisfies some conditions. See, when we take X, set of points X and introduce the notion of the distance d on it, satisfying a certain condition then, this new structure, we call it as a metric space and denote it as X d or X.

When the idea of the length of the vector is analyzed to an arbitrary set X, then, this gives a concept of the normed space, X norm and a complete normed space, we have a, called a Banach space. So, idea of the normed space and Banach space can be introduced. Then, further, the scalar product which is available in the three dimensional plane, (( )), the dot product or scalar triple product, when it is generalized to an arbitrary set X, where arbitrary vector space X, then, we get a new concept, this we call a inner product space, or a complete inner product space, we call it as a Hilbert space.

So, basically, with the help of these various structures defined on a set X or on a vector space X, we get the various types of this structures, like a vector space, metric space, normed space, Banach space and Hilbert space. In this particular course, we will deal, in detail, the concepts of metric space, their rated results, normed space, Banach space, dual spaces, set of all bounded linear functional defined on this and then, inner product space, Hilbert space. There are few results, which we called it as a fundamental theorems of functional analysis; those theorems will also be discussed, which will be used very frequently in the development of the theory of the functional analysis. So, we will start with this today, lecture on metric space.

So, before starting this definition of the metric space, let us see the motivation behind it. In calculus, we study the functions defined on the real line R and during this study of the functions, we also go through the concept of the limits. When we say the limit of the sequence x n is x, when n tends to infinity, the meaning is that, when difference between x n and x is very close, when n is sufficiently large or x n is approaching to x, when n is sufficiently large or when n tends to infinity, what do you mean by this? It means, we are using basically, the idea of the distance function on a real line. Because, if I take this as x and these are the points x 1, x 2, x n, which are tending to x, as n is tending to 0, infinity, it means the difference between x and x n, this difference is coming closer, as n is approaching to infinity.

Now, this difference, absolute difference between the two point is nothing, but the usual notion of the distance over the real line, ok. Now, this particular type of the distance function, which is defined over a real line, satisfy certain conditions. It is always be greater than equal to 0; it cannot be negative; it is a real quantity and finite. So, if we take the two point x, y, let it be on the real line; then, mod x minus y will always be greater than equal to 0, is real and finite number. So, it is a real, finite, nonnegative number. It can be 0, at the most, when x is equal to y and vice versa. Then, whether we measure the distance from x to y or from y to x, the distance remains the same; that is, this is d symmetric in x y. Then, the another property which modulus function enjoy is that, if I choose a point z on the real line, either in between x, y or may be outside of this, then, x minus z will always be less than equal to mod of x minus y plus y minus z.

This property, we also call it as a triangular property, triangular property. Why, because we know, in case of the triangle, if we add the two sides, then, sum of the two side can never be less than the third side; that is, distance from x to y plus y to z will always be greater than equal to distance from x to z. So, that is why, it is called the triangular inequality or triangular property. Now, keeping in mind this, these conditions, which the usual metric function on the real line enjoys, we wanted to extend this idea on a general abstract set X. Why should we fix, restrict our study, only on the real line or only on the complex plane? Let us take x, be an arbitrary set, where the elements may be a set of real numbers, elements may be a complex number or elements may be functions; it may be bounded function; even it may be a infinite sequence, like this. So, when we extend, when we replace this set of real number by an arbitrary set of points, then, the question arise, how to define the notion of the distance between the two point of that set, and this leads to the concept of the metric space.

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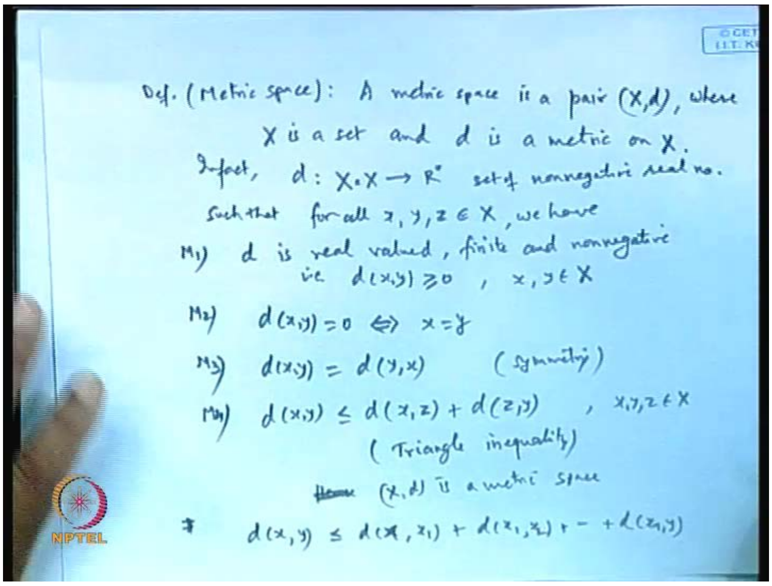

So, we now define the concept or definition of a metric space as follows: a metric space, a metric space is a pair X d, where X is a set and d is a metric, metric on X. Basically, we said d is a… So, in fact, d is a function from X cross X to R 0, set of nonnegative real number, such that, such that, for all x, y, z belonging to X, we have one, number one, the function d is real valued, finite and nonnegative; that is, d of x, that is d x y will always be greater than equal to 0, for all x and y belongs to X, for any x, y belongs to X. So, it will always be greater than or equal to 0. It is a nonnegative quantity, finite and real valued. Second, d of x y is 0, if and only if, x is equal to y; and, third d of x y equal to d of y x, that is, d is a symmetric function, with respect to x and y. So, symmetric or symmetry is there. And, last property, that is, the triangular inequality is d of x y is less than equal to d of x z plus d of z y, where x, y, z are the points in capital X and this property is known as the triangle inequality, inequality, triangle inequality, clear.

So, what we see here is that, we have, taking the idea of the usual notion of the distance function on the real line, and generalized that idea to an arbitrary set X, so that, any two point of this set, if it satisfies these four condition, then, we say the set X, together with the function d, forms a metric space, ok. So, the pair X d is metric; hence, the X d is a metric space, metric or, so, this, hence, X d is a metric space, clear. Now, this triangular inequality, which we have mentioned, can be generalized to n points also. Suppose, we have x 1, x 2, x n, many point and we want to find x y, then, this can be written as x x 1 plus d of x 1 x 2 and so on, say upto, d of x n y. So, this idea can be generalized in general. So, that will be sometimes used for it, ok. Now, let us see, few examples, which based on this definition, to justify whether this metric space or not; example of the metric spaces.

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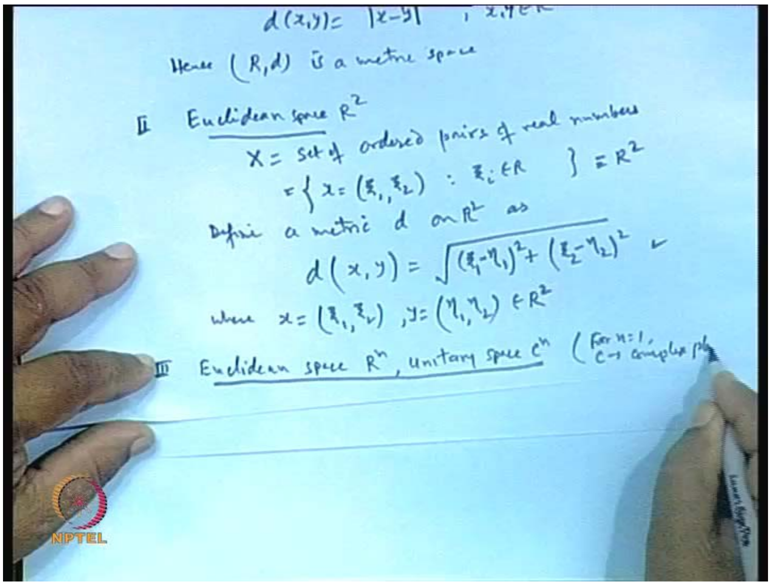

Metric spaces. Very simple example of a metric space, we know, the real line R, with the metric d defined on this. So, let X be the, let X be the set of all real numbers, real numbers and notion of the distance d is defined as mod of x minus y, where are x and y are real; then as we have seen earlier also that, this satisfy all the four conditions of the metric space; hence, this are, under this d, is a metric space. It is easy to prove, because, this already we discussed. If we take the Euclidean space, Euclidean space R 2, the R 2 is the set of, let X be the taken as set of all ordered pairs, set of ordered pairs of real numbers, of real numbers, ordered pairs of real number.

Say for example, x, which is x i 1, x i 2, x is are real, are real; then, this set of pair which is in the R 2 space, if we introduce the notion of the distance R 2, as d of x comma y, is square root of x i 1 minus eta 1 whole square plus x i 2 minus eta 2 whole square, where x is x i 1, x i 2, y is eta 1, eta 2; both are point in R 2. So, this is our space X, which is R 2 space, R 2 space. Then, this notion of the distance, which defined this, forms a metric space. Why, because d, which is defined in this fashion, will be a nonnegative quantity, because of this square and square root; then, it is real, finite and nonnegative. If d x y is 0, then, sum of the two nonnegative quantities is 0, only when individually, the terms are 0. So, x i 1 is eta 1, x i 2 is eta 2, and we get x equal to 1 and vice versa if x equal to y, d of x y will be 0. Then, d of x y is the same as d of y x, which is obviously true, because of the square. If I interchange x i 1 eta 1or x i 2 eta 2, it will not affect the whole thing and we get the d x y is the same as d of y x.

The third, fourth property m four, that is, the triangular inequality requires some more inequality, some more concepts, to prove that, triangle inequality follows. That particular concept or inequality, we call it as the Minkowski's inequality. So, that will be derived later on and with the help of the Minkowski's inequality, it can easily be shown that, it satisfies the triangle inequality, ok. So, time being, we will just assume that, this satisfy the triangle inequality. Hence, all the four conditions of this d functions satisfied. Therefore, this space R 2, under, R 2 under d, forms a metric space, and this, we call it as a Euclidean metric space or Euclidean space, ok. The parallel to this, we have the unitary space or complex (( )). The unitary space, when the points are complex number, then, we get unitary space. So, we generalize this thing and we get Euclidean space R n and unitary space unitary space C n; when n is 1, we get the c, c is a point. So, for n is equal to 1, C is the complex plane, C is the complex plane. So, what is our R n C ns?

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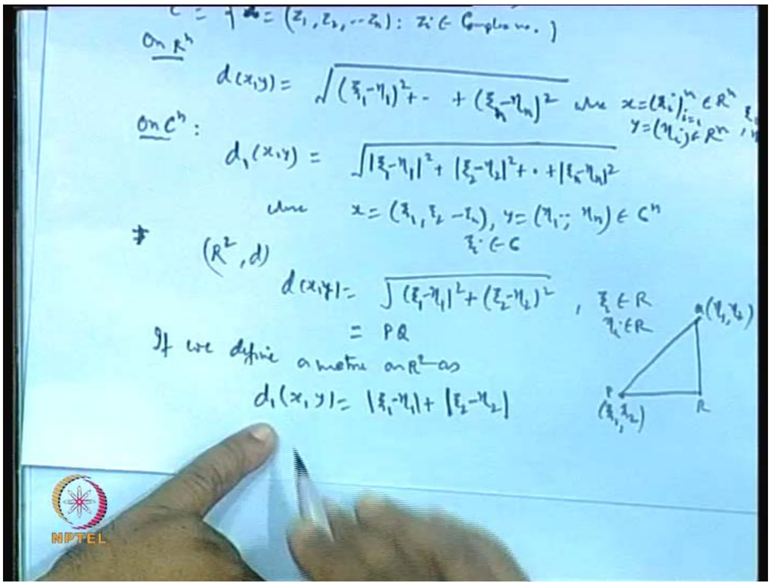

R n is, basically, the set of all n-tuples, x i 1, x i 2, x i n, where x is are real, x is are real, reals and C n is the set of all, say, z 1; let it be, here, say, z, z 1, z 2, z n, where z is all complex numbers, ok. Now, we want to introduce the notion of the distance (( )) as shown, R n, we introduce the notion of the distance d x y as under root of x i 1 minus eta

1 whole square x i 2 minus eta 2 whole square minus x i n minus eta n whole square. And on C n, we introduce the d x y, say, this, as under root. So, I put it the d 1, as this mod of x i 1 minus eta 1 whole square mod of x i 2 minus eta 2 whole square and so on, plus mod of x i n minus eta n whole square, where, in this case, x, x i 1, x i 2, x i n, y, eta 1, eta 2, eta n, this belongs to C n, where each x is are complex number; while in the first case, x, which is x i i, i is 1 to n, y which is eta i, both are in R n, both are in R n, that is they are reals; that is, x is are real; etas are real. Here, they are complex number. So, that is why, the modulus is used, so that, this quantity becomes real and nonnegative one.

Now, as we have seen in R 2, all the conditions are satisfied, except the triangular inequality, which we can be followed, which will be followed really, with the help of the Minkowski's inequality, so that, we could, later on. Now, we have seen the metric on R 2. Let us come back to this R 2, the metric on R 2 d is defined as under root x i 1 minus eta 1 whole square plus x i 2 minus eta 2 whole square, where x is are real, etas are real, ok. Now, this is basically, these are the two points; this is the point x i 1, x i 2; here is the point eta 1, eta 2; and, P is this point, Q is this. So, basically, this is the distance P Q, which is normally considered, when we deal with the problem in the two dimensional plane. But whether this is the only way of defining the notion of the distance, because what is the distance function, what is the metric?

Metric is a function from X cross X to R, which satisfy those four properties; that is, the property which we call it a axiom, metric axiom, axiom of the metric space, satisfies those property. So, if we introduce the d, in some other way, such that, all the four conditions are satisfied, then, that new definition will also (( )) metric on the R 2. For example, here, if we define a metric on R 2 as d 1 x y equal to mode x i 1 minus eta 1 plus mod x i 2 minus eta 2. That is, what we are doing is, we are taking the length P R plus R Q; and, this sum we are saying as d 1 x y, distance between x and y; this satisfy all the four properties. First thing, it is greater than equal to 0; it is 0, only when, this is 0; this is 0; means, x i 1 is eta 1, x i 2 is eta 2; so, x equal to y, and vice versa. It is nonnegative, real, finite and d x y is 0, if and only if, x is 0. It is also symmetric, x, d x y is the same as d of y x, because, this modulus sign, it will not affect, if I interchange x i 1 eta 1 and x i 2 eta 2, (( )).

Then, fourth property, triangle inequality also follows, if I add and subtract zeta 1 here, and zeta 2 here. Suppose, I take a point z, which is another point, zeta 1 and zeta 2. Then, what happen is, when we take d of x y, d 1 of x y, then, this will be equal to x i 1 minus, I am just putting zeta 1 plus zeta 2, zeta 1 plus zeta 1 minus eta 1 and here, I am put, adding and subtracting say zeta 2; and then, apply that angle inequality modulus function. So, what we get is, this is less than equal to this plus this, which is less than equal to mod of x i 1 minus zeta 1 plus x i 2 minus zeta 2 plus zeta 1 minus eta 1 plus zeta 2 minus eta 2; and that is equal to d 1 of x z plus d 1 of z y.

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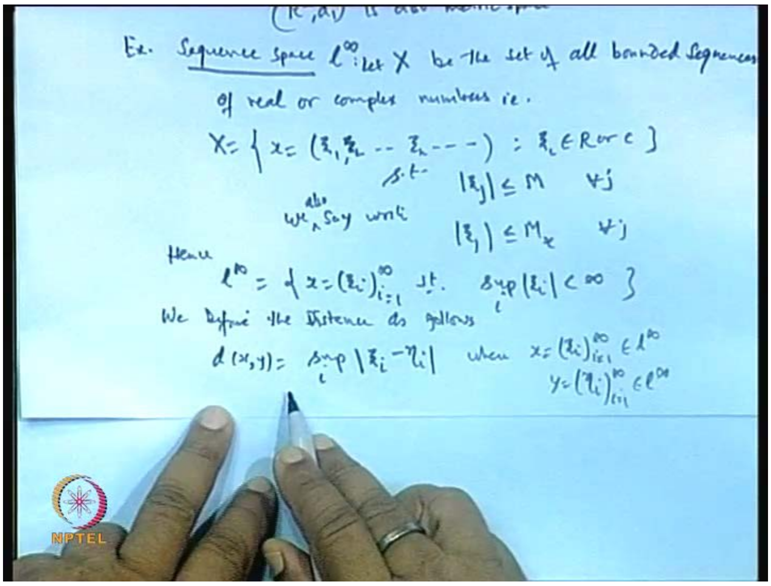

So, triangle inequality is satisfied. Therefore, it also gives a metric on R 2. So, R 2 with this d 1, is also a metric space. So, what we conclude is that, on R 2, we have two types of the metrics; that is, one can define on X, a metric in more than one way. So, only condition is that, function, which d, the way you are defining d, must satisfy those four conditions, clear. So, it is not necessary that, with the set X, we have only one and only one metric, no. X is a set, one can define in so many ways, the metrics on it, ok. So, it is, we get a different type of the metric space, as soon as we change the notion of the distance on it. So, that. Now, till now, we have discussed the set X, where the elements are either real or complex number, or may be ordered pair, ordered n-tuples and like this.

But the function analysis, where (( )), is not only restricted only for the reals or complex. The X may contain the points, where the points may be a functions, may be a sequences and something else also. So, the corresponding example for a metric space and which is very important and useful example, is the sequence space l infinity. Now, here, we take

X, let X be the set of, let X be the set of all bounded, all bounded sequences of, sequences of real or complex numbers, complex numbers, that is, the set X is the set of all infinite sequences, x i 1, x i 2, x i n, so on, where x is are either real or complex; this number, real and complex, such that, such that, such that, the sequence x i is bounded; such that, mod of x i j is less than equal to a constant M, for every j; that is, set of all infinites sequences, which are bounded; that is, each element of x i, x i i is less than equal to M.

Since this bound will depend on X, so, we also say, we also write, this as, mod of x i j is less than equal to M suffix x for every j. So, x is the collection of all those infinite sequence, which are bounded. This collection, we denote this as l infinity. Hence, l infinity is basically, is the set of all bounded sequences x, x i i, i is 1 to infinity, such that, supremum of mod x i i is finite over i. So, set of all bounded sequence, supremum is the… If supremum is M, then, all the terms of the sequence, less than equal to M; and, if we take a number, slightly less than, then, they are the term, which (( )) the condition that, it will be greater than M minus epsilon, those terms. So, this is the definition of l infinity.

Now, on this l infinity, we define the distance, distance as follows. Take the two point x and y, and let us define the distance as supremum x i i minus eta i over i; supremum x i i minus eta i, over i, this, where x is x i i, i is 1 to infinity. So, it is a sequence of this type, x i 1, x i 2; in short, we write x i n, belongs to l infinity; y, which is eta i, i is 1 to infinity; this also belongs to l infinity; and, we are introducing the notion of distance as follows. Now, this clearly satisfy all the conditions of the metric space. First is, it is greater than equal to 0, because of the mod. It is a finite, real quantity, nonnegative. If it is 0, then, supremum of this thing is 0 means, the individual, each term must be 0. So, individual each term is 0 means, x i i must be equal to eta i, for each i. So, x must be equal to y and conversely, if x equal to y, then, x i i must be equal to eta i, for each, each I; therefore, this supremum will be 0 and d will be 0. Then, d of x y is the same as d of y x, because nothing to prove; just we are interchanging and mod of this is there; so, negative sign will be taken care.

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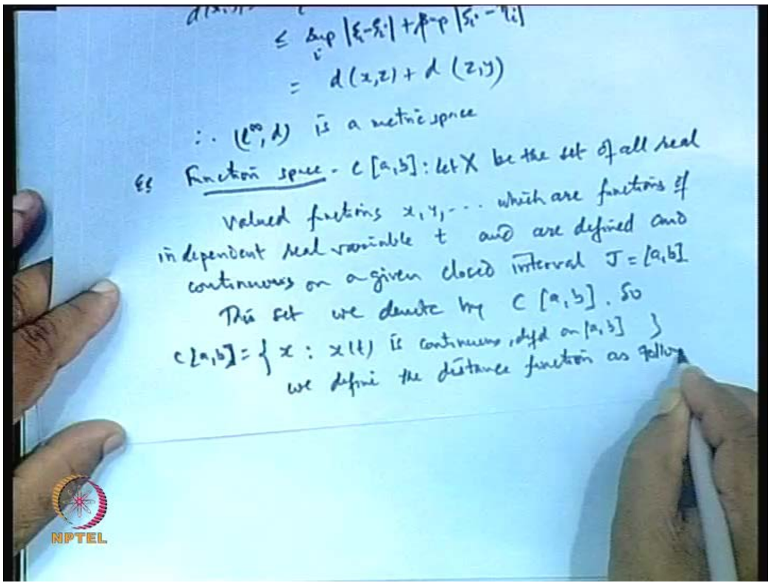

Then, d of x z is less than or d of x y is less than equal to x z, here. So, let us consider, z point which is, suppose, zeta i belongs to l infinity. So, consider this, z equal to zeta i belongs to l infinity and consider d of x y, which is supremum of x i i minus eta i, over i. Now, if I add and subtract, then what happen is, x i i minus zeta i plus zeta i minus eta i and take the supremum over i; this will be further, less than or equal to supremum of i x i i minus zeta i plus supremum of mod of zeta i minus eta i. Now, this is again, d of x z plus d of, d of z y. So, triangle inequality is satisfied by this form. Therefore, l infinity under this metric d, is a metric space, is a metric space and this is a very interesting metric space form.

Now, so far, we have seen the real number, complex number, then, set of sequences, one of the example of set of sequence, we have seen l infinity, now, we will see the function space. Now, here, we take X, function space, say, c a b. What is c a b? Here, we say c a b is taken as a set X, let X be the set of, be the set of, X means c a b, is the set of all, all real valued function or c a b, real valued functions, functions, x comma y and so on, which are functions of independent real variable t, t and are defined, defined and continuous, and continuous, continuous on a given closed interval j, j. This set, we denote by c a b. Just like previous one, this entire set, we denote this by l infinity. Similarly, this set we denote by c a b. So, c a b is the class of all functions x, where the x t is continuous, well defined on the closed interval a b, on the closed interval a b; set of all continuous functions which are defined on a close interval a b.

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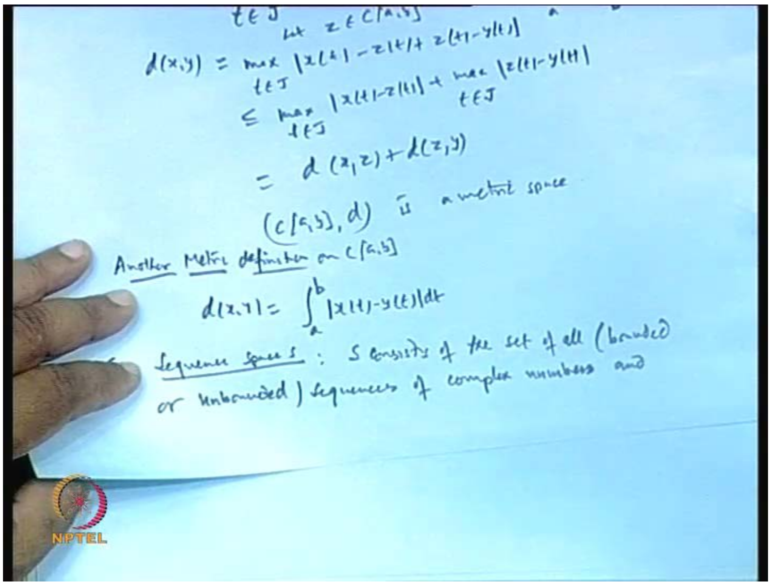

Now, over this c a b, we define the notion of the distance, define the distance function as follows, as follows: d of x y, d of x y as the maximum of mod x t minus y t, when t belongs to the closed interval a b; maximum, where the maximum denotes the maximum value of this difference over all t, over the interval a b. Now, first thing is, whether this definition, this notion of the distance which we have defined, is a well defined function or not; means, whether it is a finite, real value, nonnegative quantity or not. Now, x and y both are continuous function over the closed interval a b; this is the interval a b and functions x and y, both are continuous and say x and y.

So, we know that, if f is a continuous function over the closed interval a b, then, it attains its maximum and minimum value over the interval. So, over this interval a b, the function x as well as the function y will attain its maximum or minimum value. As well as, we know the difference of the two continuous function is a continuous function; so, x minus y will also be continuous function. Therefore, the maximum value will be, will be attained by the function x minus y over the interval a b. Hence, this is well defined thing and it will be a, some finite real quantity. Then, second part is, whether this is 0 implies, x equal to y or not. If I take d of x y to be 0, then, maximum of this thing is 0 for t belongs to J, t is varying over the closed interval a b.

Now, when the difference of this is 0, maximum difference is 0, it means, this is only possible when x and y are basically, identical, like this; otherwise, the difference will not, the maximum difference will not go to 0. If there is slight change in x and y value, the difference can never be 0. So, if x y is equal to 0, means, the maximum difference is 0 means, x and y must be identical, and vice versa, if x equal to y, obviously, the d of x y will be 0. So, this satisfy the second condition also. Third one, it is symmetric in nature, because of the modulus, we would not get, loose anything, if I interchange the x and y position. Then, fourth is triangular inequality. So, to verify the triangular inequality, let us take the another function z in c a b, and then, find out this - minus z t plus z t minus y t and maximum is t over J; now, it is less than equal to maximum of this sum, maximum is taken over J, t, over t, t belongs to J and like this.

So, this is equal to d of x z and this one is equal to d of z y. So, what we see here is that, this metric d, this notion of the distance function d over c a b, satisfy all the condition of the metric space condition, axioms of the metric, metric. Therefore, c a b, under this metric, is a metric space, ok. Now, there is another way of defining the metric space on c a b. So, another metric definition on c a b. Suppose, we define the function d of x y as a to b mod of x t minus y t d t, then, this is well defined thing, where a and b, function x and y both are continuous over a b and since a b is closed, so, obviously, this function x t minus y t, a to b, d t this will represents basically, the area bounded by this function, within the limit x equal to a, x equal to b, and x axis, ok. So, it is well defined thing and d of x y will be there. So, when you change x y, you are getting a different.

Now, this is greater than equal to 0, because this can never be a negative quantity. If it is 0, then, either a equal to b or function must be identically 0. So, a cannot be equal to b. So, therefore, x t must be equal to y t and vice versa; if x equal to y, then, d of x y will be 0. Then, d of x y is d of y x and triangular inequality can also be shown that, it is similarly shown. So, this forms a metric and this is another way of defining metric on c a b. So, c a b, one can define more than one way, the metric here.

Now, another examples are on the general sequence space S. Suppose, we have the space S is the consist of, consist of the set of all vector, the sequence space S consist of, sequence space S consist of the set of all bounded or unbounded, or unbounded sequences of complex numbers, complex numbers, and, the metric d which we define as, and d is defined, is defined as d of x y as sigma i is 1 to infinity, 1 by 2 to the power i mod of x i i minus eta i over 1 plus mod x i i minus eta i; this is x i i, ok.

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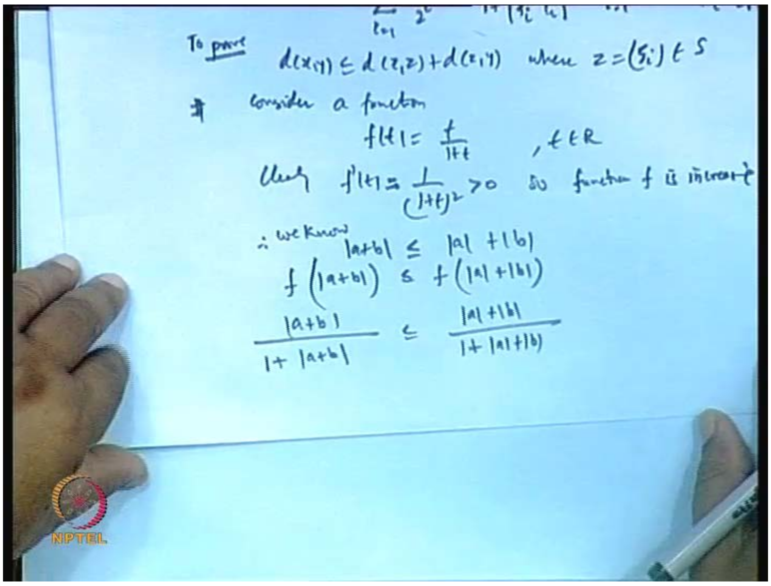

So, this will be mod of x i i minus eta i and this. So, I will repeat again, this is i is equal to 1 to infinity, 1 by 2 i mod x i i minus eta i 1 plus mod x i i minus eta i. Now, the question is that, whether this forms a metric first, well defined and second one is, why will we use the same metric, as we have used in case of the l infinity. The, for l infinity, the sequence was bounded sequence. Therefore, if I take the supremum of mod x i i eta i, supremum of this, it may be unbounded. So, d may not be finite. Therefore, we cannot say, say, d x y is the supremum mod x i i minus eta i, in this case, ok. So, we have to take the definition in such a way, so that, it should be finite, nonnegative, real.

Now, it is a valid definition. Why, because, this entire thing is less than equal to 1; less than 1. So, this entire series is dominated by 1 upon point 2 i, sigma 1 upon point 2 i. Therefore, sigma 1 upon 2 i is a convergent series and we get the sum to be 1, ok. So, this entire series converges and finite sum. Therefore, d x y is a finite, nonnegative, real quantity. Further, d x y is 0. Then, this entire sum is 0; this is as absolute terms. So, individually, each term must be 0. So, x i i must be equal to eta i and vice versa; if x i i is a… So, x equal to y and y equal to n, so on. So, we get, d x y is 0 implies, x equal to y and vice versa. Third is, we can interchange the position of x and y; still, there is no loss of generality; because, we can take the x i and eta i, change, if I change, then, because of the modulus sign, it is not affected.

Then, fourth inequality requires, triangle inequality requires, to prove triangle inequality, d of x y is less than equal to d of x z plus d of z y, where z is zeta i, belongs to, say, S. So, we write first and observe this. Let us consider a function f of t equal to t over 1 plus t. Let us consider this function t defined on R, t is in R. Then, clearly, derivative of this is positive, because derivative is nothing, but 1 over 1 plus t whole square. So, this will be a positive quantity. So, function f is an increasing function, is an increasing. Therefore, the value of this a plus b, because we know, a plus b is less than mod a plus mod b; therefore, f of mod a plus b is less than equal to f of mod a plus mod b. And, this value, if I replace this by t y y mod a plus b, then, we get, mod a plus b over 1 plus mod a plus b is less than equal to mod a plus mod b over 1 plus mod a plus mod b. And, this will be, further less than equal to mod a over 1 plus mod a plus mod b over 1 plus mod b; because I am dropping here b, here a, and this. So, we get this part.

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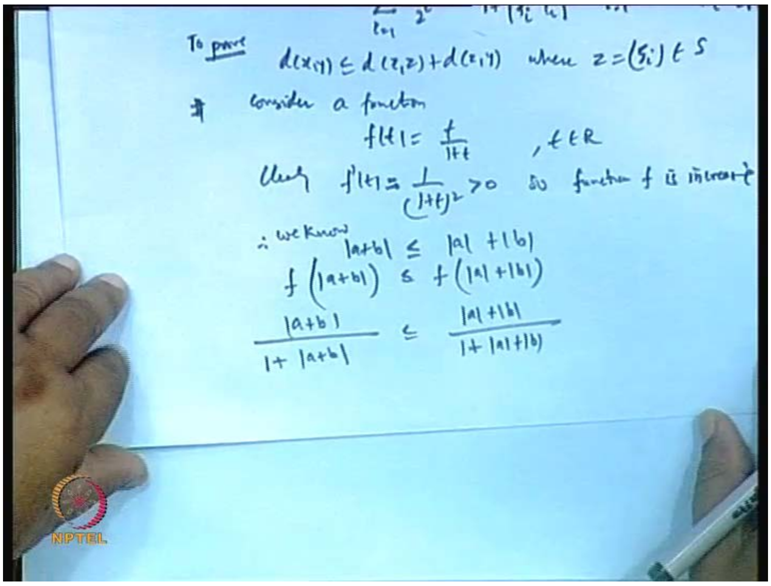

Now, from here, it can be shown, if I take a to be, choose a to be x i i minus zeta i and b, if i choose zeta i minus eta I, then, a plus b, substitute in this A. So, put it in A. What we get is, a plus b, a plus b is mod of x i i minus eta i over 1 plus x i i minus eta i is less than equal to mod x i i minus zeta i over 1 plus x i i minus zeta i plus mod of zeta i minus eta i over 1 plus zeta i minus eta i, is it not; using the A, this inequality, if I put a and b as (( )) this, we get this on. So, multiply by 1 upon 2 i and take the sum. So, we get 1 by 2 i mod x i i minus eta i over 1 plus mod x i i minus eta i is less than equal to 1 by 2 i mod x i i minus zeta i 1 plus x i i minus zeta i plus 1 by 2 i mod of zeta i minus eta i over 1 plus zeta i minus eta i, ok.

Take the summation, i is equal to 1 to infinity, i is equal to 1 to infinity, i is equal to 1 to infinity. Now, because x is x i i, z is zeta i, y is eta i, so, basically, this will give the d of x y and this is equal to d of x z and d of z y. So, this shows, the triangular inequality is is also satisfy by this function. Therefore, S, with this d, is a metric space. This is with this. Now, this is not only the metric space, metric, notion of the metric defined on S. One can go for another also, metric and this another metric is that, in place of 1 by 2 i, we can take mu i, we can take mu i. So, 1 upon 2 i can be replaced by mu i, where sigma mu i is finite and rest will be same. And, the last example is a discrete metric space. Here, we take X to be set of all points and then, notion of the distance x be the set of all points, X is the set of points and define d, define d as d of x y, equal to 0, if x equal to y and 1, if x is not equal to y.

Then, this is a nonnegative, real valued, finite quantity. It is 0, when d x y is 0, when if only x is y. It is symmetric in x y and triangular inequality also follows, because, d x y is less than equal to d of x z plus d of z y, z is any point. If it coincide with x, then, it is 0; but with y, it will different. So, we get the 1. Therefore, all the conditions of the metrics are satisfied by this. Hence, this metric X d is called a, a metric space and we call it as a discrete metric space. And, this space is very much useful, particularly, in establishing the, in giving the counter example.

Thank you very much. Thanks.