Ordinary Differential Equations (noc 24 ma 78)

Dr Kaushik Bal
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Week-02

Lecture-09

2^{nd} Order Constant Coefficients Linear Equations

Hello everyone and in this video we are going to talk about second order linear equation okay second order linear equation and we are basically starting out with a constant coefficient equation okay equation with constant coefficients with constant coefficients okay and for all

So, for the starting part, we are going to only talk about homogeneous equation and then I am going to show you how to work out the inhomogeneous equation.

So, first of all, we know that let us say a equation.

So, what is the general form?

So, general form will look like this.

General form will look like this.

Form will look like y double prime plus ay prime plus by equals to .

That's a general form.

And A and B are some constants.

And this is a general form of homogeneous.

second order linear equation with constant coefficients clear now i want to we want to know how to solve this problem right and why i am specifically doing this thing i am quite sure you guys already know what to do here but i again this is a revision and i am doing this because i want it to be precise and exactly what is happening because we are going to use this idea in a later part

Okay.

So you see by since in this case we do not know how to solve this problem.

Okay.

Explicitly without you know directly solving directly attacking this problem.

So what we do is this this ODE we want to use something which we know.

So we will convert this ODE to a algebraic equation.

Okay.

Algebraic equation.

So, what we are trying to do here is this, see, let us say that if you, if one assumes, one assumes that there is a function, you know, y of x, let us say, which is e power r of x and that satisfies, let us call it star, satisfies star, okay.

satisfy star.

So, here we have to find r such that this happens.

So, what is the guarantee that something like this will happen?

So, that is the question.

So, basically if we can find some r such that e power rx will satisfy this equation, then we are basically done.

And if something like this happens, then you see y prime of x is nothing but r e power rx and y double prime of x will be

r square e power rx now if you put everything together the equation if if one puts the above values in star values in star okay what do we get we get r square plus a r plus b okay times e power r x this is nothing but right that's what we are going to get and what does that imply that will imply r square plus a r plus b equals to why because e power r x is never going to be

for any x in R, right, for any x in R, okay.

So, that is fine there.

So, this equation, the equation which we derive like this, if we remember, it is called a characteristic equation.

I hope this is the spelling is correct, characteristics, okay.

There is a mistake here.

Let me just put it this way.

Characte

restrict equation okay okay well now if we solve the characteristic equation you do realize that we are going to get two roots right it may be same may be equal or maybe distinct maybe you know complex but we are going to get two roots because this quadratic equation and once we get two roots we can just put it in the function and we can get solutions out of it okay so so first case case one

distinct real roots.

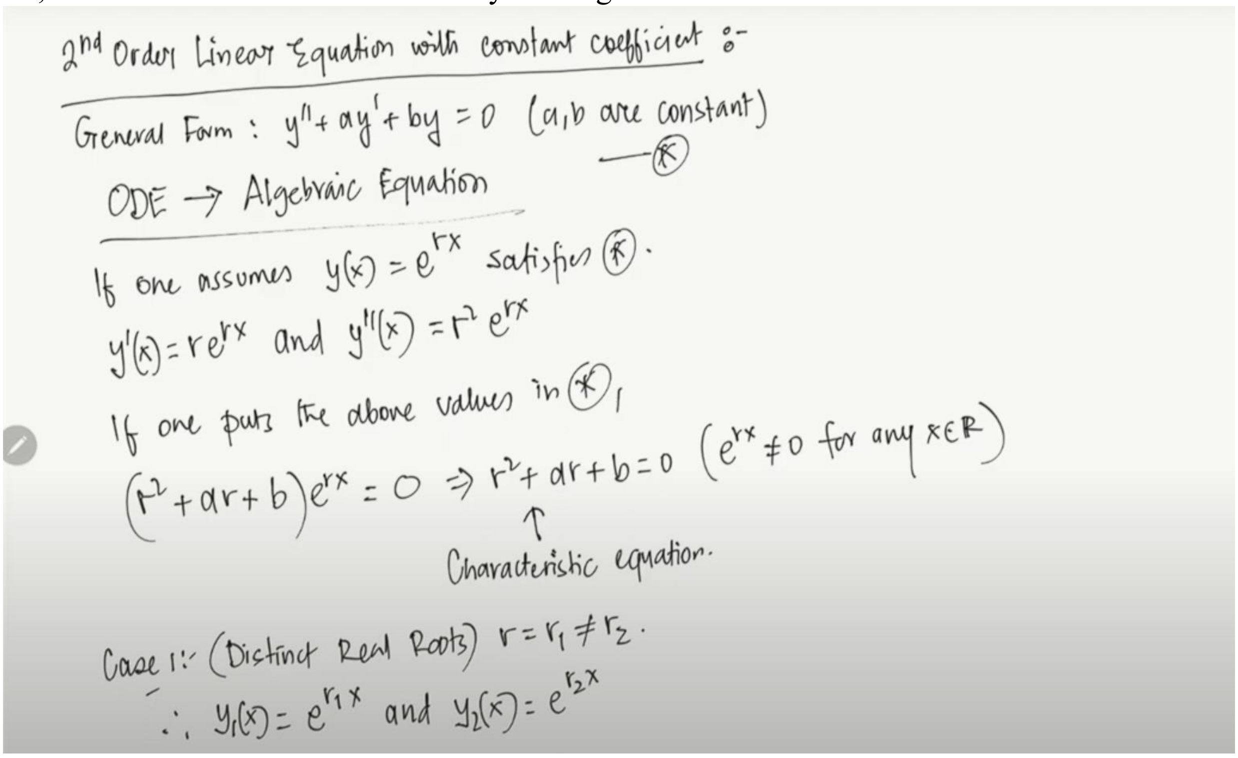
So, what I meant by this is you see there are roots r is nothing but r which is not equals to r.

So, in this case what are the solutions?

Therefore, the solutions which you are getting the first solution which you are getting y of x let us call that you can write it as e power r of x.

and y of x. Another solution you are also going to get, you are going to get it as e power r of x, okay, right.

So, those are the two solutions which you can get out of this.



Now, you see what is happening is this, this is a linear equation, right.

So, check, check that if y and y solves star

solves star, okay.

See, this y and y is of this form, but basically this holds for any two solutions.

So, basically if you have any two solutions, y and y, you do realize that alpha times y, then alpha times y plus beta times y.

So, this is independent of this case, solves star, solves star for any, for any alpha beta

in R. So, this always holds.

And why this holds?

Because this is linear, because of linearity.

So, you need to check this part.

Please check this.

This is one of the characteristics of linear equations.

So, once that is true, you see here what we have is y is a solution, y is a solution.

So, therefore,

y of x. We will call this a general solution.

So we will write it as g. Okay.

This given by y e power r of x. So alpha e power r of x plus beta e power r of x. Okay.

This solves star.

solves star.

So, basically, this is the general solution.

Let us write it like this is the general solution.

What do I mean by general solution?

I mean that all possible solutions of this equation has to look like this.

There is no other way, okay, for any alpha beta in R, okay.

Now, a big question, question, and this is very, very important.

How do you know

How does one know that any solution of star for the, let us say, at least for this case, let us say, real distinct case, distinct solution.

case must look like this must look like y g okay so what is the guarantee here you do understand what i'm trying to say see a general solution is nothing but all possible solutions okay see this equation is a linear equation y equals to is of course a solution you do realize that right you see y equals to so y x identically equals to is always a solution

Always a solution.

Now, basically we are trying to find the other non-trivial solutions.

And what I am trying to say is this, all possible solutions are contained in this family.

So, what is the family?

If you are varying alpha beta, so let us say alpha equals to , you have beta times e power rx.

alpha equals to and beta equals to , you get e power r of x. You do realize all possible combinations are there, linear combinations.

So, basically it says that all possible solutions will look like this, okay.

So, what is the guarantee?

These questions will be answered in another few lectures, okay.

When we are going to talk about systems, I will do it for a system and then that will actually imply in this case also the exact same thing.

Okay, so that is there.

So basically what I am trying to say is this, if you have a, so please remember this, if you have a second order equation, at least constant coefficient equation, if this also holds for variable coefficient, does not matter, but for now, since I am doing constant coefficient, let us just assume that constant coefficient equation, if you have, you always will have two such functions such that the linear combination will actually give you all possible solutions, okay.

right and what do i mean by two functions they are very special functions which we will later call it as a linearly independent function since i did not do it right now we'll do it later in another few classes then it will become much clearer okay let's look at the second case so second case okay case two what is the second case case two is the real

And equal.

And equal.

Okay.

So basically what it means is we are looking for r which is r equals to r.

Okay.

And in this case of course you have one solution y of x given by e power r of x. Okay.

But you can't get another solution y of x because it will be basically same as y of x. Okay.

So in this case what we do is this is a solution.

Is a solution.

solution of star right.

How do you get another solution?

So this is a trick which actually you can use it and this is will be a part of assignment.

So what you can show is this to find another solution what you do is this to find another solution another solution

Let y of x, we will define a new function y of x is nothing but u of x power r of x. r is r or r, whatever you want to call it, basically that.

Now, you see if this is a new function and this u is a smooth function, so is a c infinity function, we will assume that.

So essentially what I am trying to say is this, we are trying to find a C infinity function, a infinitely differentiable function that is, C infinity is infinitely differentiable function such that yx given by ux times e power rx will also satisfy star.

And what is r?

r is r or r, whatever you want to call it.

So if that is the case, then y,

y prime and y double prime you can actually calculate and then that will satisfy the equation.

So, let us just put it together.

If we put it together, it becomes e power rx u double prime plus ru prime plus r square u plus a e power rx u.

u prime plus r u plus b u e power r x equals to .

So, just by substituting the value of y prime and y double prime in the equation, we get this.

It is a very straightforward thing.

Nothing special is happening here.

Okay, so if you see now, the thing is this, you can rewrite this thing and it will look like this, u double prime plus r plus a u prime, see e power rx is basically non-empty, sorry, non-zero, right, so we can just throw it out, plus r square plus ar plus b times u equals to .

Clear?

So, now you see R is a root of the characteristic equation and the characteristic equation is given by R square plus AR plus B. So,

If r is a root, then r square plus a r plus b is equal to .

So, basically this term is gone.

This is .

So, that will imply that your equation looks like this.

u double prime plus r plus a u prime is .

Clear?

This is a first order equation.

See, r is a repeated root.

r is a repeated root.

So, basically that will actually imply that r plus a is equal to .

And if this is the case, then that will imply that u double prime is equal to and I am just looking for u which will satisfy this.

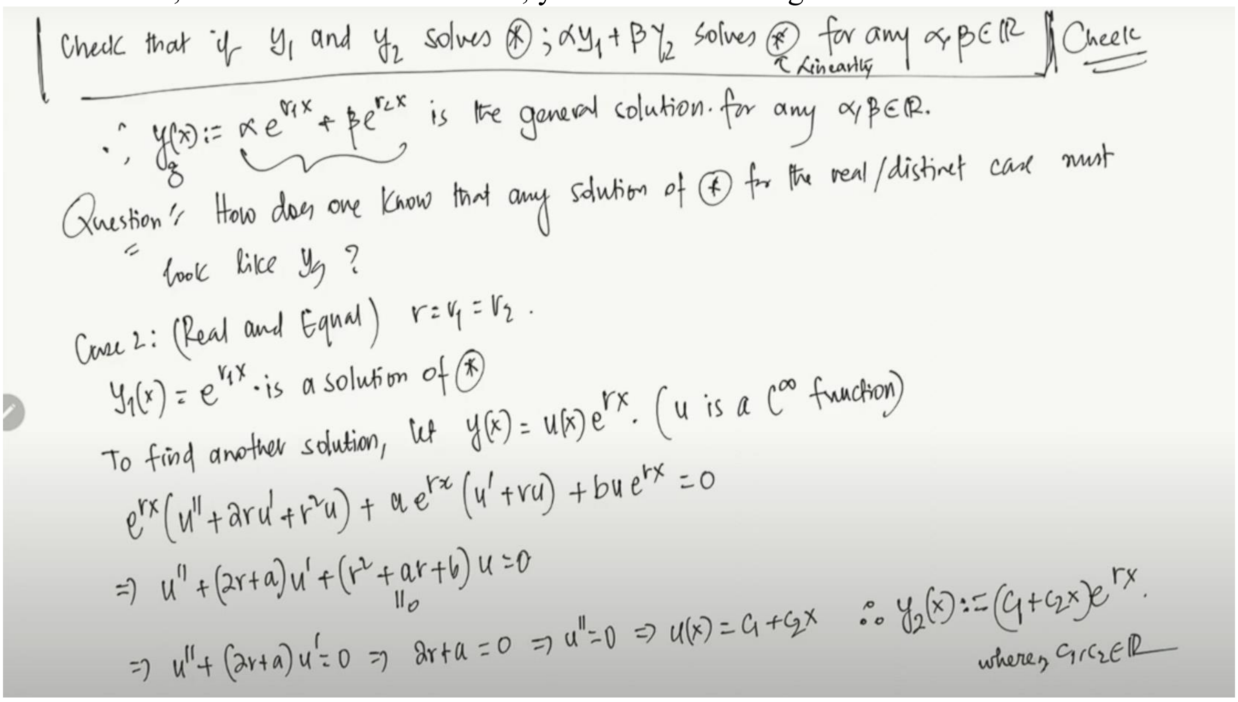
So, ux given by c plus c of x will satisfy such an equation u double prime equals to .

So, therefore, y of x we can define it to be c plus c of x

e power rx.

So, then and what is c and c?

c and c is in r, where c and c is in r. Now, you see what is the general solution in this case?



General solution, so yg is nothing but alpha times y

plus beta times y, as I told you, right.

So, it is alpha times e power rx plus beta times c plus c x times e power rx.

So, essentially this will look like, let us say alpha plus beta times x, beta times x or I mean some constant.

So, basically what I am trying to say is this, it is

alpha plus beta times c let's just call it beta okay times x e power rx so that is the general solution okay that so therefore the general solution so where alpha and beta is in R

So, what is the general solution when both roots are same and real?

It is alpha plus some constant plus another constant times x e power rx.

Now, let us look at the third case, case .

case .

So, what is the case ?

It is the complex conjugate root, complex conjugate root, okay.

So, what I mean by this?

I mean that r, so you will have two roots.

So, if r equals to u

plus iv plus minus iv okay r so basically is plus and let's say is minus if we put it this way so if r is u plus iv and r is u minus iv okay okay so

In this case, what is happening is this.

See, you are going to get two solutions here.

So, you see y of x in this case will look like e power r of x, right?

So, e power ux.

times e power i vx, which is cosine vx plus i sine vx.

And y of x will be the exact same thing with the minus sign here.

So, it is e power ux cosine vx minus i sine vx.

Okay.

Right.

Now we have to understand this that you see this y and y I can write down as a linear combination of this is let us say general solution.

But the problem is this.

Note that here that y or y does not matter or y are both complex.

Are both complex.

Right.

And

The thing is, we are not looking for any complex solution.

We are looking for a real solution.

So, how are we going to get a real solution?

Tell me.

So, what we are going to do is this.

See, we are going to write down a small theorem here.

It is a small theorem.

So, let us say if, let us say y plus iy.

solves or let us call it z plus i z.

So, let us z plus i z solves star.

Then both z and z solves star.

So, z and z are real.

But the thing is, let us say z, which is given by z equals to z plus i z, that solves the equation.

Then, individually z and z, those are all real functions.

They solve stuff.

So, how do you prove it?

Very easy.

So, please check this part.

The solution, I am not going to do it.

Please check.

Substitute z, z prime and z double prime.

In start and then what do you do?

You equate the real and equal parts.

Equate real and equal parts.

Equal parts.

So you get the theorem.

So I am going to use this theorem here.

So what I am going to do is this.

What I have said is this is z which is z plus i z.

If you solve the equation.

So here you see if you see properly it is e power ux.

cosine vx plus i times e power ux sine vx.

And similarly, here also the same sort of thing happens y with a minus sign.

So, here you see, since z equals to z plus iz.

So, here z is this, z is this.

both of them solve the equation.

So, individually z and z solve the equation, right?

And then they are real, right?

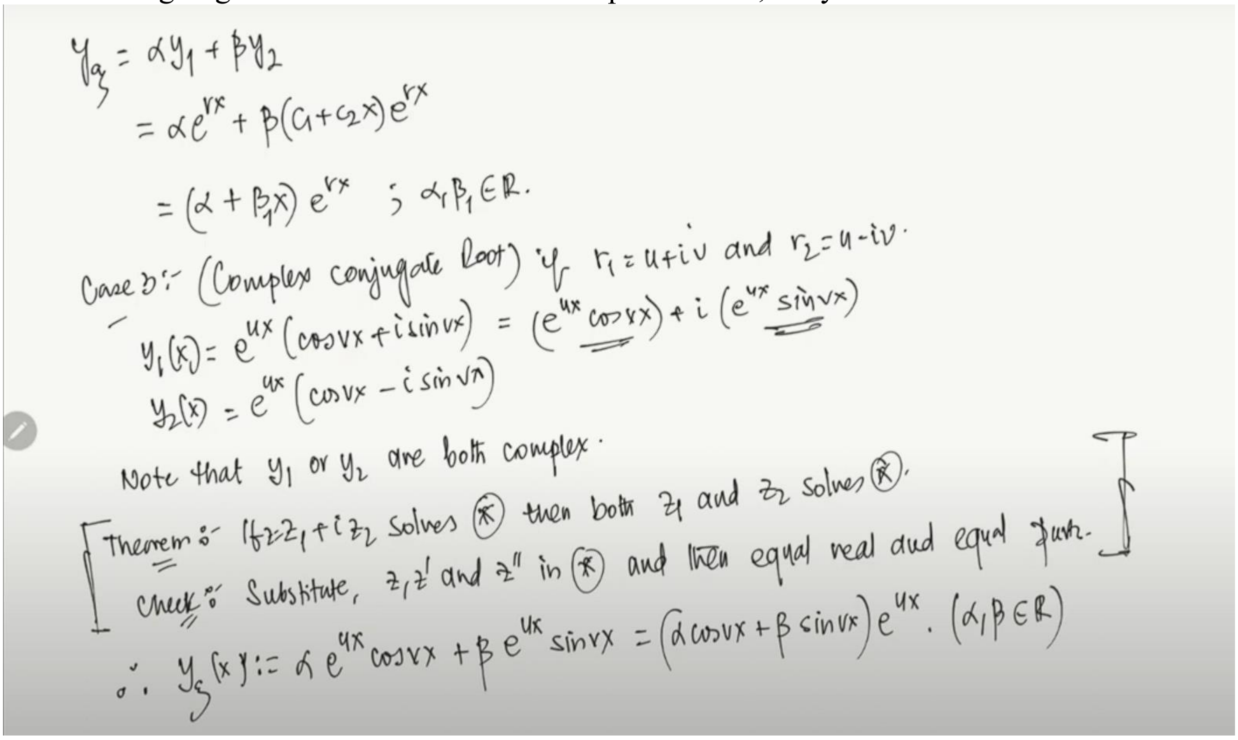
So, therefore, the general solution yg of x can be defined as alpha times y of x, which is e power ux cosine vx plus beta times e power ux

sine vx, clear?

So, if you want to write it properly, you are basically end up with alpha times cosine vx plus beta times sine vx times e power ux, right?

And here alpha beta is in R. I hope this is clear to you.

And we are going to use the same sort of concept later also, okay?



So, I want you to understand what is happening here.

Right.

Now, we move on to a very, very similar situation for a different equation.

So, we move on to something called a Cauchy-Euler equation.

Cauchy-Euler equation, right.

So, what is Cauchy-Euler equation?

Let us just write it down.

You see, essentially, we are looking for a equation which looks like this.

Any equation, any equation of the form

of the form x square y double prime plus a x y prime plus b y equals to , right?

And here I am assuming x is positive, clear?

Then, so basically this sort of equation we call it a Cauchy-Euler equation.

You see one beautiful thing about this equation is the x square, the power of x which is involved with y double prime is , with y prime is and with y is .

So, that is this beautiful structure and this sort of equation is called Cauchy-Euler equation.

Why are we trying to study this equation?

Because we can actually, this is, you see, if you think about it, it is not a constant coefficient equation, but in some sense, it is actually a constant coefficient equation in the, and what sense?

See, this is variable coefficients.

But the thing is, you can use the same sort of technique which we used in the first part to actually derive a solution.

How can we do this?

See, let us assume, assume yx,

to be x power m. Solves star, solves , let us call it .

So, if it does, then you see y prime, y double prime, you can calculate and put it here.

Once you put it, what will happen is this, you are going to get x square m, m minus , x power m minus ,

plus a x m x power m minus plus b x power m this is equal to this is what we are going to get and that will actually imply that m m minus plus a m plus b is equal to this is what we are going to get because x power m x is assumed to be positive it is never going to be right okay now again you see two and again you see this is nothing but a quadratic equation which will actually again have two roots so exact same sort of thing will going to happen what we did here okay it's the same sort of thing so what i want to do is please check this check the theorem stated below okay this is your work here

below and I will state the theorem what happens is this.

See A, so the solutions.

So, basically if M equals to M which is not equals to M.

So, if there are two roots M and M such that this happens that M not equals to M.

If this happens then the general solution as you can see y of x should look like

c times y power, sorry, x power, x power m plus c times x power m.

So, you have to do this part.

Now, if m equals to m, real and equal, then how does it looks like?

yx will look like

c times x power m, this will be the first, of course, you are going to get that, m, plus about the second one, what you are going to do is, you are going to get, see in the earlier case, we got just a x, here we are going to get log x power, x power m, clear, and see, if they are complex, so complex conjugate, so basically, m is u plus iv, plus minus iv,

Then y of x will look like this.

c times x power u cosine u v log x plus c x power u

Okay, so in the both cases x power u is here, sine.

v u plus i v that v log x. Clear?

Okay.

So, you have to check this part.

How do you do it?

Exactly the same sort of thing what we did in the last part.

That same sort of idea will work in this case.

So, you please do this part.

And with this, we are going to more or less close our the study of equations at least second order linear equations.

So, before we finish this thing in the video, let us put down a question here.

Question.

Check if this procedure, okay, not only this, the whole video, whatever we did, works for higher order equation or not.

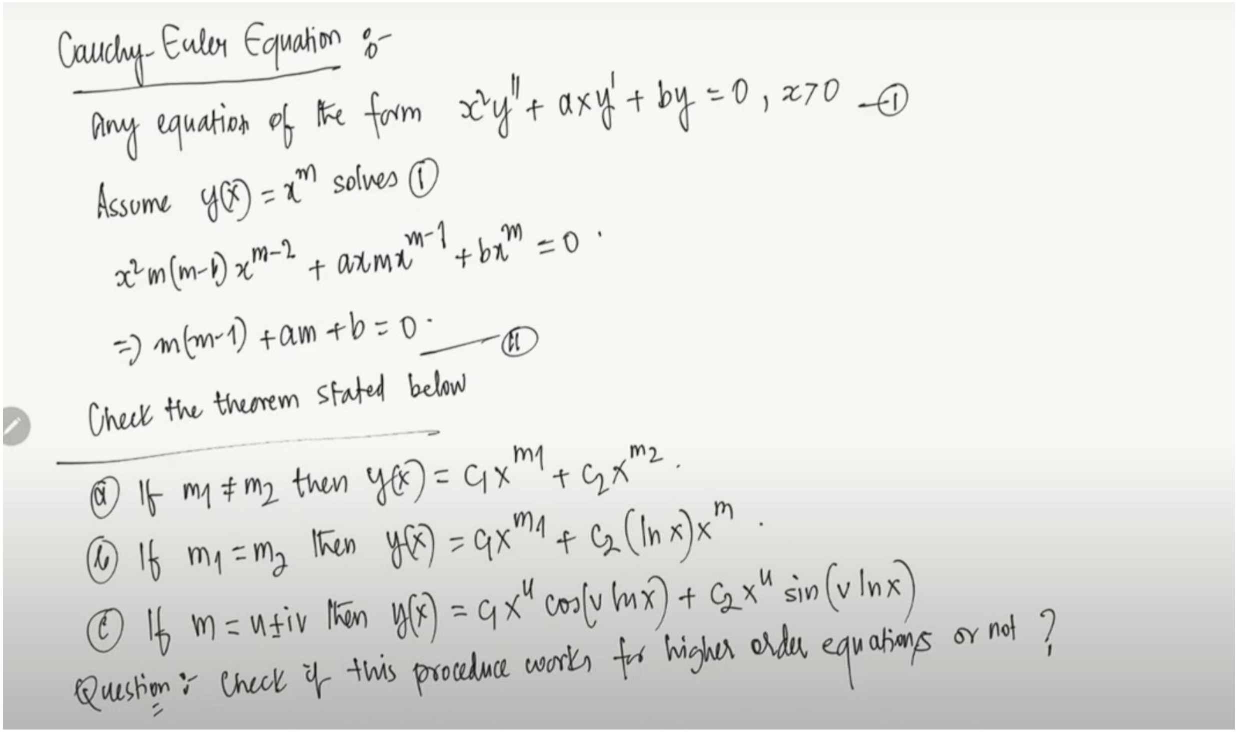
Higher order equations or not.

Okay.

So essentially you see it should work right because you see what you are doing is you are basically you know reducing a ODE to essentially an algebraic equation.

It can be quadratic, it can be third order equation, fourth order equation right but algebraic equation.

If you can solve that equation you actually have all the possible ways of finding the solutions.



Okay.

right.

So, please do that.

So, start with a lesser , I mean, you know, third order equation and then try to do exact same thing.

So, basically, with a constant coefficient or like a Cauchy-Euler equation, you can do that.

So, with this, we are going to end this video.

Thank you.