Ordinary Differential Equations (noc 24 ma 78)

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Week-02

Lecture-08

Differential Inequalities

Welcome students and in this video we are basically trying to look at first order differential equations.

Yes, so as I have already explained we generally are not going to study this in much details because solving this sort of equations we already know there are some techniques integrating factors and when the equation is exact that sort of thing right we know how to solve those things.

So, it is not the primary concern here in this video.

Essentially, what are first order equations?

How does it look like?

So, the general solution, the general form will look like this.

General form of a first order ODE will look like this.

will look like, how does it look like?

y prime equals to f of xy, right?

Now, if f of xy, let us say, it is of the form a xy plus b of x, right?

Then, let us call it .

is inhomogeneous

homogeneous, inhomogeneous is when, when b is not , not , linear equation, right, linear equation and it is very easy to solve.

You just find out the integrating factor, you can solve it, right?

Okay.

Now, it may happen that f of x, y may not be linear in y. So, basically, it can have y square, y cube, that sort of thing, okay?

So, for example, let us just say that if I, let me just put it this way.

Let us say, if I ask you to solve this equation, y prime,

plus a y or forget about a y let us just say this so motivation this is let me put it this way this is this is a motivational motivation of what we are trying to do in this part motivation see let us look at this y prime plus y equals to let us say x square so this is a linear equation right linear equation equation and it is

Relatively easy to solve.

No problem, you just find out indicating factor, it is done.

Now let us say if I change it to y prime plus y square equals to x square.

Now change it to this, this equation.

Now you see, you do realize that this equation is anyway not correct.

very easy.

So, this is comparatively difficult to solve.

And you understand that, I mean, you can make it more complicated by just adding, let us say, this sort of equation e power y and all.

So, what happens is, although linear equations are very easy to solve, nonlinear equations, although they are first order, they can be very, very complicated.

Clear?

So, what do we do in such a case?

So, what I am trying to say is this, in most of the cases,

most of the non-linear cases non-linear cases solving the equation equation is not

easy right or it may be very very difficult not easy means it may be very difficult okay so what we do is in those kind of cases we take we try to find out whatever information we can get for a solution okay and to do that we look at something called a differential differential inequality

Differential inequality.

So, what is this differential inequality?

So, let us say a function y of x is said to be a solution of the differential inequality of the

differential inequality, differential inequality y prime greater than f of x, y. In this interval, let us say x , x plus a. Yes.

If a

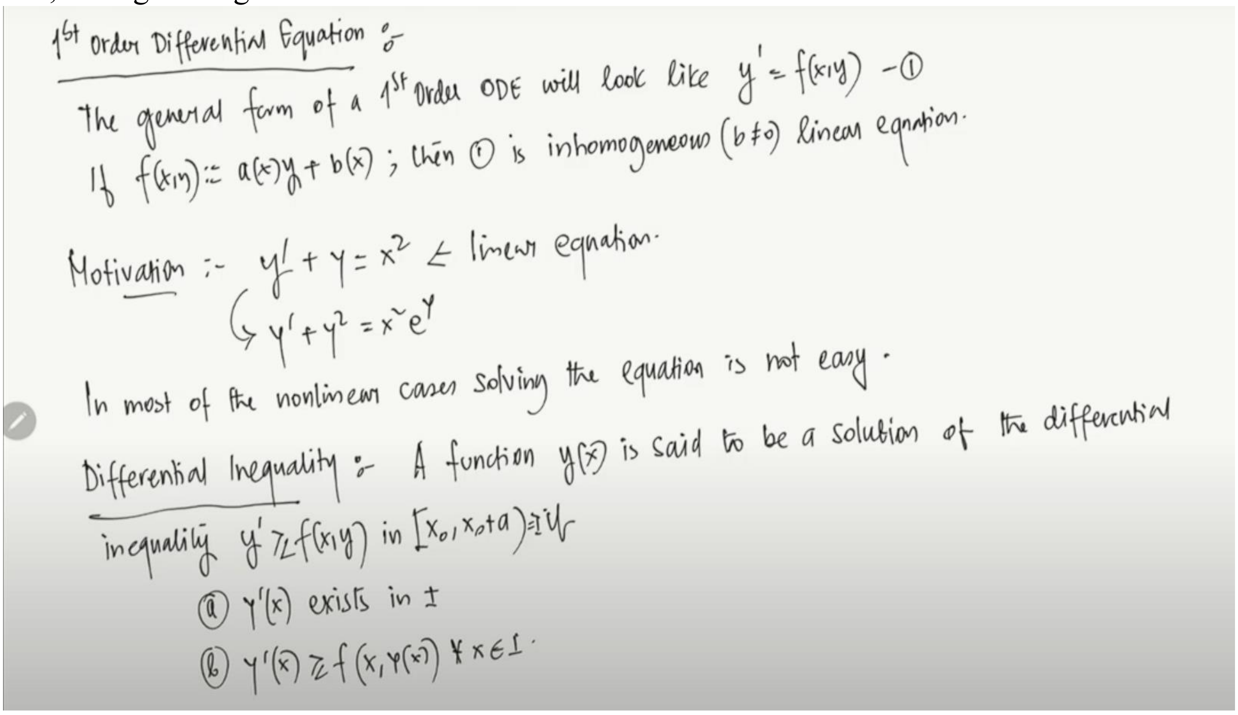
y prime of x exists in this interval.

So, let us call this interval as i in i and b. y prime of x should always be greater than f of x y of x. So, this should be satisfied for all x, right?

For all x in

i. Then we call it a solution.

So, basically if I change it to let us say less than, so y prime is less than f of x, then we will have this, the sign changes.



And if it is greater than equal, let us say, then also we just have the equality here.

So, we can just modify accordingly.

So, what is an example of such a thing?

What is, where do we use something like this?

See, the thing is, for example, for example, yx equals to cot of x, okay?

So, this you can see is a solution, solution of y prime

which is strictly less than minus y square, okay?

And it is in what is the interval here?

and , clear?

So, that is the interval.

Now, the thing is this.

So, you do understand what I am trying to say.

See, the idea is this.

For most nonlinear equation, at least first order, here we are talking about first order, okay?

we may or may not be, it may or may not be possible to find an explicit solution, right?

So, you may not be able to solve that problem, yes?

And then what you do is, let us say you have this equation y prime equals to f of xy, okay?

You find two equations which something like this, y

is less than f of x y and you find a y such that y greater than f of x y .

Yes, and of course, some initial conditions should be there and you want your y to lie between y and y .

You understand?

Something like this.

So, basically you see, if you can find a y such that this happens and if you can find a y such that this happens, you expect your y lie between y and y.

So, basically you get an, you know, quite nice cushion between two functions you can sandwich your original solution.

So, you will get an idea of what your solution is doing essentially.

So, with this we are going to write down our first theorem here.

So, whether this sort of thing can or may work or not, let us see.

So, let f of x, y be continuous in a domain D. Domain D, whatever the domain is, does not matter.

I mean, wherever it is defined, let us just say.

And y of x and y of x.

clear, be the solutions, solutions of the differential inequality, of the differential inequalities.

So, these are called differential inequalities, right?

y prime is less than equal f of xy, okay?

And

y prime is greater than equal f of xy, right, in whatever the interval where y and y are defined, the common interval.

Then, if y of x, okay, is less than y of x,

So essentially what I am doing is I am saying that let us put it this way.

See y is following some rule and y is following some rule.

And so essentially I want to track my y between y and y.

So basically I am saying that let us say that y starts from this is y and x is let us say x is x is .

So, wherever it is starting, y is below y.

So, y may be like this.

Let us just say this is y.

And if y and y satisfies those two equations, then you know that y will maybe a solution like this, y will be in between y and y.

And what is the relation between y, y, y?

See, y satisfies y prime equals to f.

And y and y satisfy these two conditions.

So if the initial conditions, initially they are below y is below y, then that will imply that y of x is below y of x.

Okay.

See this is the motivation.

This is the motivation.

Okay.

We want to go here but for that we are starting with this theorem.

Okay.

We will come to this but we are starting with this theorem.

So basically what we are saying is this if you have two relations such that y prime is less than f and y prime is greater than f and if y is below y all the time.

Okay, initially, initially, then it is always below y is always below y for all time.

Okay, so for all x in i. I hope this is clear.

So, what I am saying is if y and y both satisfies the equation and y starts below y, then y stays below y for all x. Clear?

So, what is the proof?

Let us look at the proof.

This is the proof.

So, let us say that this is not true.

So, let us assume that y of x is not equals to, sorry, is not dominated by y of x for

all x in i what does it mean it means that there is a x in i such that this is not true yeah okay then this set so then the set a let's just call it a and how do we define it is the set of all those x in i such that y of x is greater than equal y of x okay this set is non-empty

This is clear.

Why?

Because you see if y, so I am starting out with y below y.

If y stays below y for all x we are done.

So basically we are saying it does not.

So basically there is a point x.

where y dominates y, at least, okay, at least dominates y, so greater than equal y, right, okay.

So, if that is the case, then this set A is, of course, non-empty, yes, right.

So, this is a non-empty set and, of course, this is bounded below because the set is bounded below from x, okay.

See, x is the initial point.

So, this I, I will write it as x, x plus some A, let us say, open.

clear okay so the it is bounded below by x and it is non-empty so basically what i can say is there is a glb so glb of a exist and let us call it x star yeah glb is the greatest lower bound sorry i made a spelling mistake greatest lower bound

Clear?

Okay.

So, the GLB exists.

Yes.

And is given by x star.

And you also know that x .

So, what is the relation between x and x star?

x is strictly less than x star.

Why?

Because you see at the point x , y of x naught is strictly less than y of x .

Clear?

Okay.

So, x star has to be greater than x .

Right?

Okay.

So, and what happens?

To y and y at the point x.

Since it is GLB, y at the point x star equals to y at the point x star.

It has to be right.

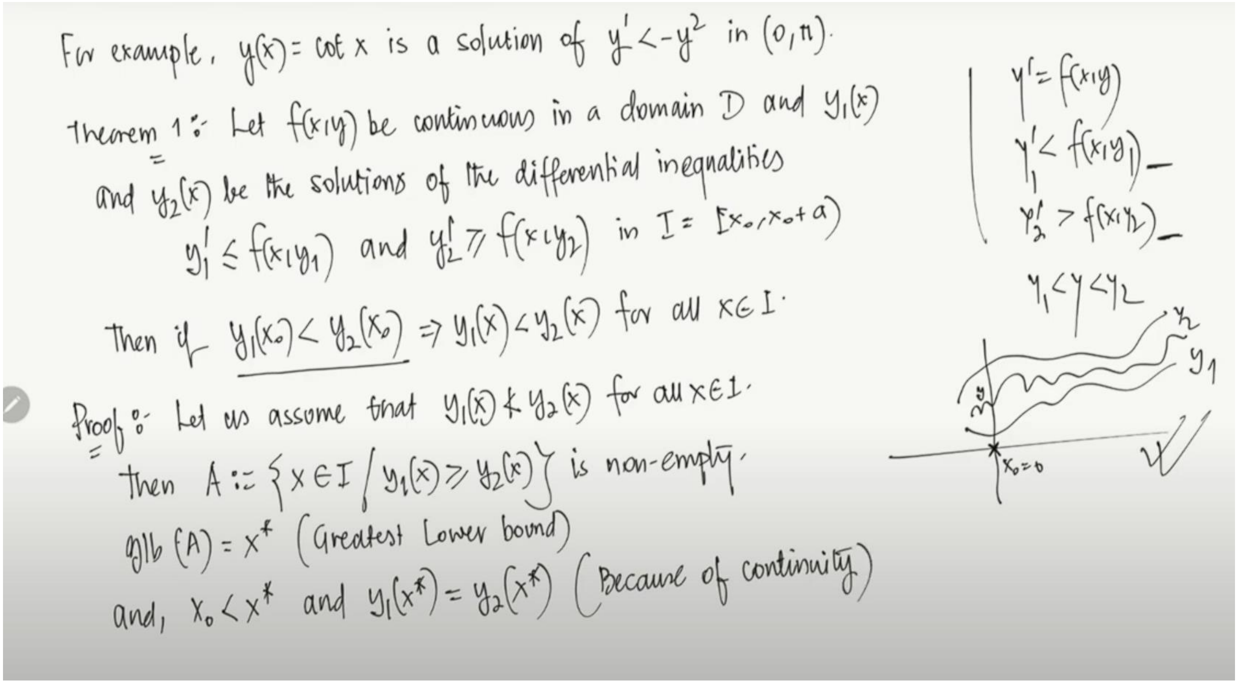
Because y and y are continuous functions.

Because of continuity.

Continuity.

Okay?

Right.



So, if this is the case, what is happening is this, see, for h negative, okay, let us look at y x star plus h is dominated by y x star plus h, right, and hence, and hence, y prime

x star minus so basically if i am approaching y prime from the left of x star okay that will be given by limit h tends to y of x star plus h minus y of x star by h right so this is again dominated by limit h tends to

y of x star plus h minus y of x star by h. Let me put it this way, by h. So, why this greater than equals to sign, let us just understand that.

See, at x star, y and y are basically same.

So,

nothing happens here y see for h negative y of x star plus h is dominated by y of x plus h see for h negative y of x star plus h is dominated by y of x star plus h right so and since y and y are basically same at x star y of x star plus h minus y of x star

is dominated by y of x star plus h minus y of x star.

Right.

And then I will divide it by h. Right.

Divide it by h. But this h is negative.

So basically this h is negative.

So this sign changes.

Okay.

And that is why this is greater than equals to.

Yeah.

Okay.

And y equals to because I am taking the limit as h tends to .

Right.

So that is why this equals to sign is there because of the limit.

This is equals to is because of the limit.

Okay.

So now what is it?

This is nothing but y prime acting at x star minus .

So, you see, what is y prime at x star?

It is nothing but f of x star y of x star.

See, y prime is dominated by f of xy.

So, let me put it this way.

and y prime dominates f of x y so basically what does that means f of x y of sorry x star y of x star gets dominated by y prime of x star minus and that is again dominated by y prime

of x star minus .

And y prime is dominated by f of x y.

So, this is less than equal f of x star y of x star, right?

Okay.

So, if this is the case, you see, you are saying that at the point x star, f of and what is y at the point x star and y at the point x star, they are basically same.

So, this will imply that f of x star y, sorry, there is y.

Yeah, one mistake.

There should be, this is strictly, at least one should be strictly.

So, either this or this.

Let us do this one.

We need this.

Why we need this?

Because to find the contradiction here.

You see, if this is the case, then what happens is y at the point x star is strictly less than f of x star.

You see, y and y are basically same as x star.

So, this is y of x star.

y and y are basically same at x star.

So, basically I am just replacing it.

So, this strict inequality is required and this is a contradiction.

Okay, why strict inequality I am getting because this is our assumption.

You see, I made the mistake here.

At least you do realize that by making that mistake, you do realize that at least one of them has to be negative, sorry, strict.

Either this or this.

Even if this is strict also, then also the same sort of thing will work, right?

Okay, so you need at least one strict inequality.

Okay, so once that is there, then you have a contradiction.

Therefore, the set A is empty.

Empty, right?

And what does that mean?

It means that y of x, so if it starts below, okay, the initial data, whatever the initial data, let us say x star, if it starts at x star, y starts below y at x star, it stays below, right?

For all x in, let us say i, and what is i?

i is nothing but x star, sorry, x .

x and x plus some a. I do not care what a is, some a. So, that is there.

Now, you see what I am going to do is I am going to write down a corollary of this theorem.

And this is the corollary which we want.

You see, I told you, right, we want to sandwich the solution between those two inequalities.

So, how do I do it?

So, let us say that let f of x, y

Is continuous in some domain D. Continuous in.

A domain D. Whatever the domain is.

Whatever the domain is domain D. So let us just find.

Let us just assume that there is a domain D. Where all of this is continuous.

Now further assume.

Further assume.

That.

A.

yx is a solution of y prime equals to f of xy okay in i and what is i x x plus a clear and b so this is the first condition you need y i am starting with y which is a solution of this equation and

y of x and y of x solves solves the inequalities the differential inequalities that is inequalities what is the inequalities f of x y greater than y prime and

y prime greater than f of xy in x, x plus a.

Clear?

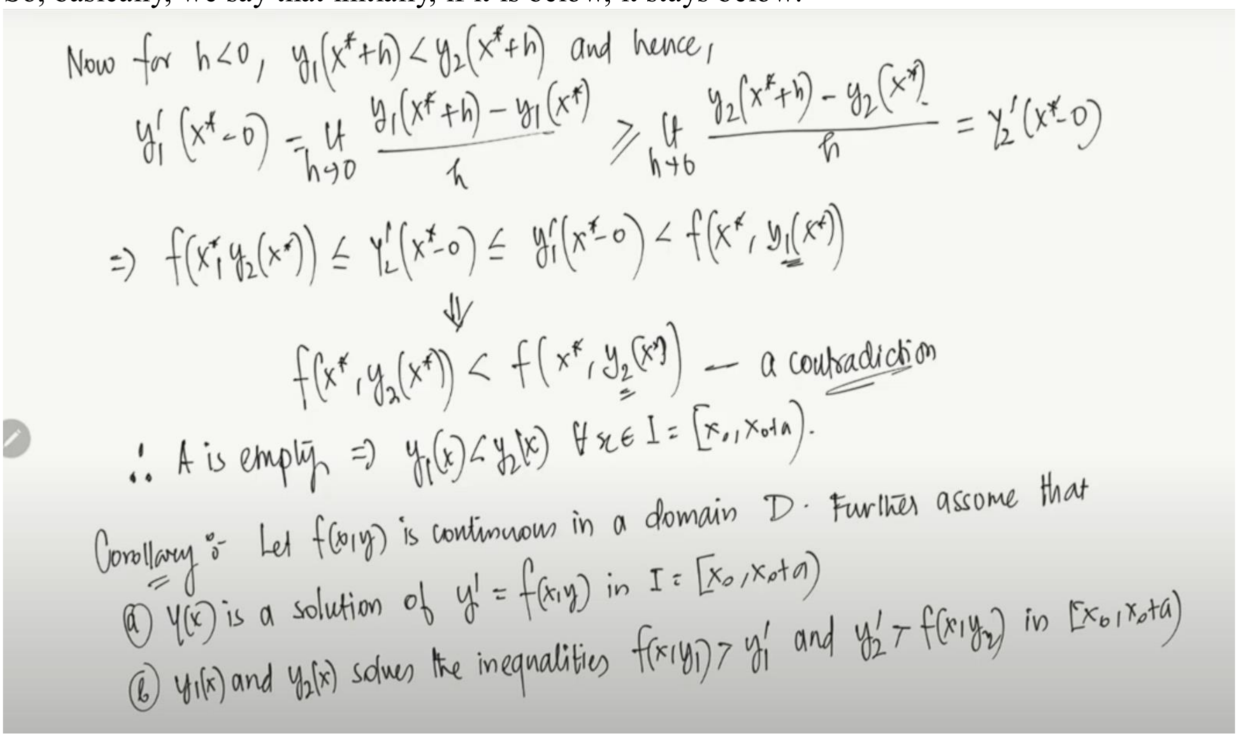
So, this is the second condition.

The third condition, we need another condition.

Why we need another condition?

Because, you know, in the earlier cases, if you remember, we know this data that initially what is happening to the solution.

So, basically, we say that initially, if it is below, it stays below.



So, we need that sort of initial condition.

So, we need to know what does y, y and y does initially.

So, basically, c, y at the point x,

must be dominated by y at the point x less than equals to which is dominated by y at the point x.

Let us just assume this.

Then we can say that as earlier, if it is dominated at the initial data, it will be dominated everywhere.

So basically y of x gets dominated by y of x gets dominated by y of x. Clear?

This holds for all x in

x , x plus A. Why I am writing open X ?

Because at the closed x, we are already given what does Y and Y does.

okay so what is the proof let's look at the proof okay another small remark i have to put it here you see why here this inequality is sorry the equality inequality is strict okay it is not less than equals to but equals to so it's saying that although at initial data they may be equal but they have to you know part ways okay so they have to be strict

once they go out and why is this happening because of this strict inequality you see here both the equality inequality which we wrote are strict okay right so let's look at the proof so we'll just prove the one part let's say we prove this part the other part is exactly the same it doesn't matter okay it's the same thing so let's say that if y which i am defining it as y at the point x

is less than y at the point x.

Yes, if something like this happens, then you see y of x should always be less than y at the point x. This is always true.

Why?

By the previous theorem.

By previous theorem.

Yes.

So, essentially you can see that the previous you know what does it say?

It says that one strict inequality and one less than equals to is fine and if you are starting out with some difference then the difference persist everywhere.

Okay.

Strict difference.

Here also you see the both are strict inequalities.

Right.

So, that is satisfied and now the thing is this.

Sorry, one is equals to and one is strict, right?

And you see at initial data, so they are essentially what they are doing is this is strict initial data.

This is what our assumption is.

And then the solution stays strictly unequal, right?

Okay.

So, this is always there.

Now, let us say that they are same.

Then what happens?

Okay.

So, let us assume, assume.

y is equals to y at the point x .

Let us just assume that.

So, then define, what I will do is this thing, define z of x, which is y of x minus y of x. Here, if we do that, therefore, z prime of x is what?

It is y prime of x minus y of x.

That is what we are going to get and this is greater than equal f of x y at the point x minus f of x y at the point x .

This is what we are going to get evaluated at the point x sorry I just.

Okay, so this is true because you see we assume that y prime is strictly greater than f of x y.

So this is true for all x. So for x also it is true.

Okay, and see this is nothing but because I have assumed that y at the point x is y at the point x.

Okay, so this is true .

So I have z at the point x is strictly greater than .

That is what I am getting.

So that will imply that zx.

is increasing, is increasing in the right of x , in the right of x , right, in the right of x .

So, basically x is the starting point, right, so it is increasing.

in a small neighborhood, let us say, small neighborhood.

So, what is the neighborhood?

So, that is in this neighborhood, let us say at x , x plus some delta, let us just assume that for some delta greater than , for some delta positive, small enough, right.

So, essentially what I am trying to say is this is z prime x is positive, strictly positive,

So, what does that mean?

It means that and we start from x , right?

So, it means that it has to increase at least for some time it has to increase, right?

For some delta.

So, that is good.

So, therefore, if z is increasing, it means that y at the point x plus delta here is strictly less than y at the point x plus delta.

It is increasing, right, at that interval.

So, this is true, right.

Of course, I mean, you can write it by delta by .

So, let me without loss of generality, let us write it as closed interval x plus delta.

Of course, you can do it.

Otherwise, you take delta by .

It does not matter.

So, we can do this.

Okay.

Now, you see.

you have your new initial condition, which is x plus delta.

So, you look at the functions, these functions at x plus delta, they are, if you think of it, it starts from there.

So, y, y starts from x naught plus delta.

So, for all x after x plus delta, y of x must always be less than y of x for all x greater than equal x plus delta.

This will imply this, this is from the previous theorem.

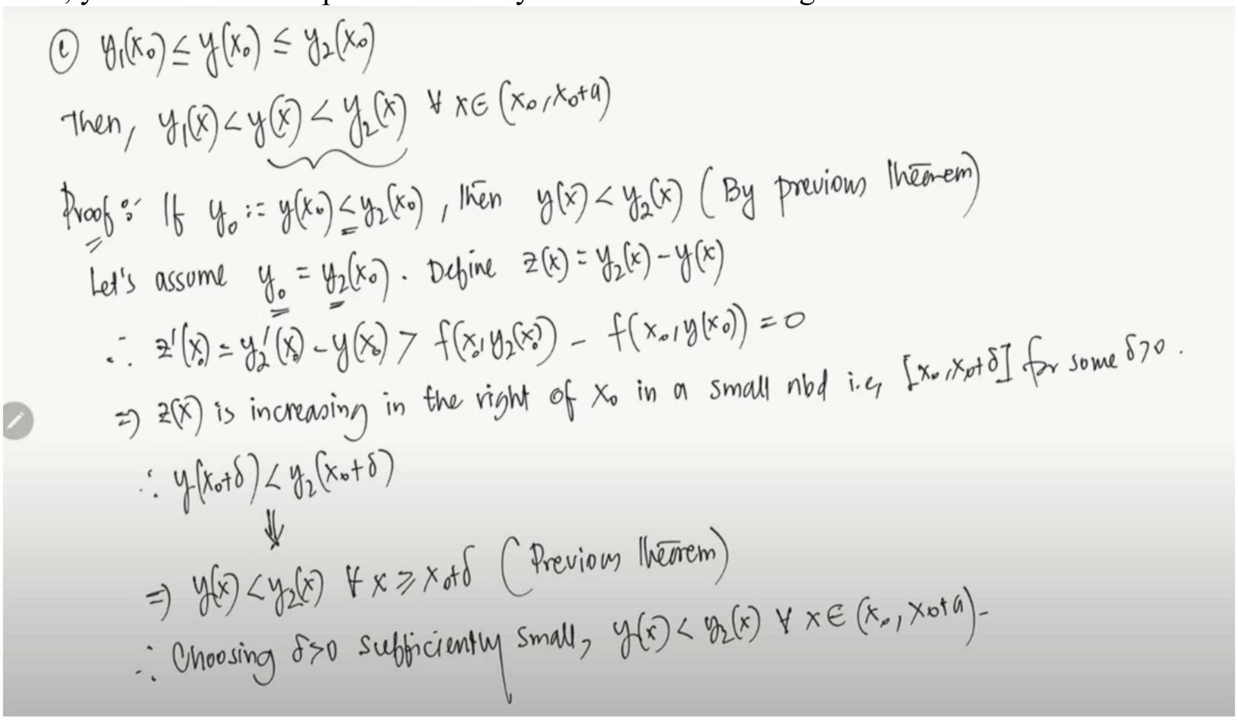
previous theorem, right?

So, you see, what is happening is this.

Therefore, choosing delta greater than , sufficiently small, sufficiently small, one has y of x strictly y of x for all x between x and x plus a.

So, this is true.

Now, you do it the other part also exactly the same sort of thing works.



So, you just do this part and then we are done.

Now, let us look at one example.

So, let me give you an example of how to use this fact example.

So, let us say that consider the initial value problem.

What is the initial value problem?

Y prime equals to Y square plus X square.

Y at the point is and the interval I in the interval I which I am choosing it to be close , open.

yeah and in this interval i want to find out that if i can ah you know without solving this of course i can solve this but the thing is without solving this ah equation can you find a lower bound and upper bound so essentially in this part we are basically looking for a lower bound and upper bound of a solution and you can actually get it using this theorem okay right so how to get something like this see for that you have to produce a y and y which does this right

y prime has to be dominated by f of xy and y prime must dominate f of xy.

And what is f of xy?

f of xy is given here.

You see f of xy here is nothing but x square plus y square.

Okay, you have to find y which gets dominated by f and y which dominates f, right.

So, then you have this equation along with the initial data, of course.

Okay, so how do you find such a thing?

So, take, let us assume, okay, maybe let y of x, okay, is plus xs cube by , okay, and

y at the point , in this case is .

So, which is similar, you see y at the point and y at the point is .

You see, both seem equal to, if the equality holds, it works.

So, you see y of x, if I am choosing it to be plus xs cube by , what happens to y prime of x?

Let us just calculate this.

It is nothing but xs square, which dominates xs square plus

plus xs cube by square, which gets dominated by this, right?

And what is this?

This is nothing but xs square plus y square, right?

So, you see, I have a y such that y prime x is strictly less than xs square plus y square x, which is nothing but f of x y of x, right?

Now, similarly,

Similarly, if you choose y of x to be tan of x plus pi by , if we choose this.

And you see, what is y at the point ?

Of course, you can understand that x equals to is tan pi by , which is nothing but .

Therefore, y prime of x

is equals to what you get it is the derivative it is say square of x plus y by right which is nothing but plus tan square of x square y by clear right now this is dominated by plus sorry

this is dominated by see tan square of x plus pi by is y square right and so this is y square of x plus x square okay see x is always between and okay so dominates x square i can write it like this therefore therefore you have that plus x s cube by here is dominated by y x

Why?

Because of this theorem.

You see, in this case, yx dominates y and dominated by y.

And it will be dominated by tan of x plus by .

Okay?

And this...

holds for all x between and not till but pi by because you see I have this tan right.

So, up till tan it will hold.

So, essentially it is x lies between and pi by .

So, in this interval you can actually get that there is this upper bound and lower bound.

Hence, one finds an appropriate

Upper and lower bound.

Upper and lower bound.

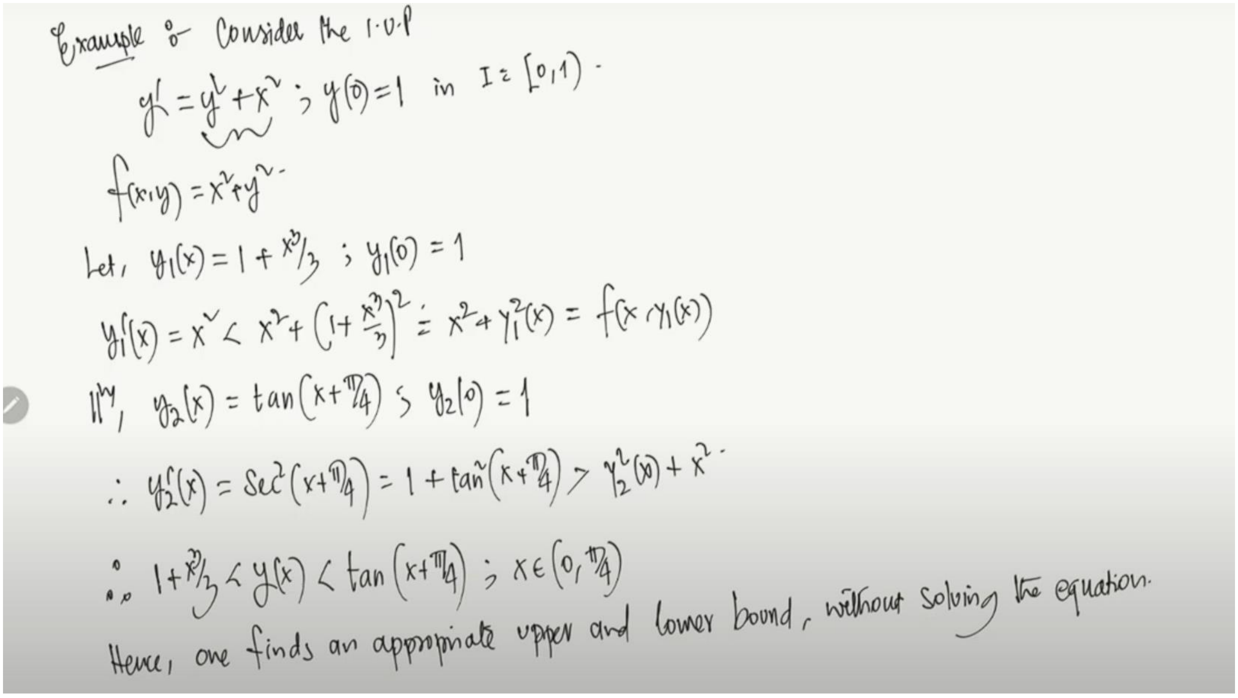
Lower bound.

An important thing is, without solving the equation, right?

Without solving the equation.

The equation.

Clear?



Example : Consider the I.V.P

$$y' = y^2 + x^2 \; ; \; y(0) = 1 \quad \text{in } I = [0,1).$$

$$f(x,y) = x^2 + y^2.$$

Let, $y_1(x) = 1 + x^3/3 \; ; \; y_1(0) = 1$

$$y_1'(x) = x^2 < x^2 + \left(1 + \frac{x^3}{3}\right)^2 = x^2 + y_1^2(x) = f(x, y_1(x))$$

lly, $y_2(x) = \tan(x + \pi/4) \; ; \; y_2(0) = 1$

$$\therefore y_2'(x) = \sec^2(x + \pi/4) = 1 + \tan^2(x + \pi/4) > y_2^2(x) + x^2.$$

$$\therefore 1 + x^3/3 < y(x) < \tan(x + \pi/4) \; ; \; x \in (0, \pi/4)$$

Hence, one finds an appropriate upper and lower bound, without solving the equation.

Is this clear?

Okay.

Right.

So, with this, I am going to end this video.

you