Ordinary Differential Equations (noc 24 ma 78)

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Week-01
Lecture-06

Cauchy-Schwatz and Gronwall's Inequality

students and in this video we are going to talk about some inequalities okay so the first inequality which we are going to do here is called a Cauchy Schwarz inequality Cauchy Schwarz

Schwarz inequality.

So, probably some of you have heard about this because it is a very, very famous inequality.

But in case you did not and since we are going to use it a lot of time in the course, so we are going to talk about this inequality.

So, what is this?

So, essentially, let us say you have an inner product space, right?

So, (V,<.), is an inner product space, okay, inner product space, let us say, inner product space, okay?

Now, in this inner product space, let us say that you have two vectors u and v, okay?

So, for all u and v, sorry, u and v,

For all v in V. So, you start with any two vectors u and v in v, this inequality should hold.

So, $| < u,v| \quad \leq < u,u * < v,v$

right so so this is what Cauchy-Schwartz was inequality says and this is very very simple and of course uh the thing is um uh how to prove this thing okay so this is a Cauchy-Schwartz inequality and the point is this how do you prove such a inequality right uh so before i do that one simple remark i want to put down here remark

See, every inner product gives rise to a Euclidean l norm, right.

So, we know that every inner product gives rise to an, sorry, Euclidean, Euclidean should be P,

Euclidean, that is l norm.

So, and the thing is where one defines the norm as this, $||u||:=\sqrt{< u,u}$

So, and you always know that u itself is always non-negative real number.

So, all of this makes sense.

So, in terms of norm, so in terms of norm, in terms of norm, of norm, so basically if you are working in a norm linear space that is, the Cauchy Schwarz inequality takes this form.

So, $<u,v \leq ||u||||v||$.

Clear?

So, this is the Cauchy-Schwartz inequality, right?

Now, let us look at what is the proof of such an inequality.

So, the first thing, proof.

See, first of all, u and v are any two vectors, right?

So, this is basically a vector space.

So, u and v are any two vectors, right?

So, let us say, let u and v and v

So, basically I start with any u,v in V, it does not really matter what, but any two arbitrary u and v. So, let u and v be such that u=λv. What does that mean?

It means that u and v are linearly dependent.

So, let us say they are linearly dependent.

So, please understand what we are trying to do here.

For any two vectors u and v, we want to prove that the norm, the inner product of u, v, square of that is always dominated by the inner product of u with itself and the inner product of v with itself, right?

So, and for any two vectors u and v, they either will be linearly dependent or they are linearly independent.

So, the first case is this, that let us say they are linearly dependent.

So, in this case, in this case,

In this case, you see that norm of <u,v=< λv,v, right?

And that is nothing but $\lambda < v, v$ with inner product of itself, right?

So, in this case, you can see that what happens is,

If I am taking the mod and square of that, so the mod and square of that, it becomes $| < u, v|$ =$| < \lambda v, v|$ =$|\lambda < v, v|$,right?

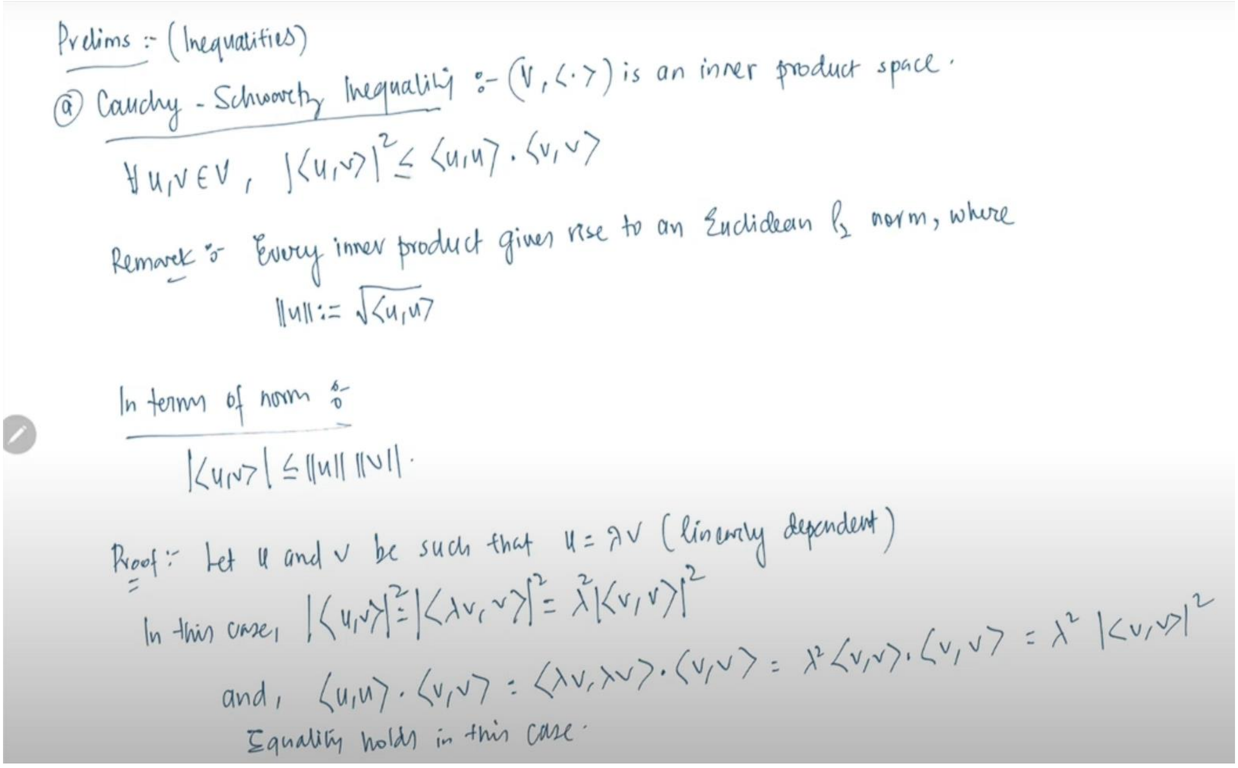
So, this is nothing but equality.

So, basically you can actually write it like this and okay.

So, let us do the other part and <u,u<v,v=$< \lambda v, \lambda v$.<v.v=λ $| < v, v|$, let us just calculate that.

So, essentially the equality holds, equality holds in this case, equality holds

in this case.



So, you see this less than equals to, equals to holds in this case.

So, what happens that if they are not linearly dependent.

So, basically they can be linearly independent also.

So, this is case .

If they are linearly independent, then

Look at this factor.

u$\neq \lambda * v$v for any $\lambda \, in$ $\mathbb{R}$, right?

Of course, you see, if one of them is , then nothing doesn't matter.

I mean, the equality holds.

So, basically, we are assuming u and v are non-zero.

So, in this case, look at this inner product <u-$\lambda v * < u - \lambda v$, the inner product of this particular vector.

This inner product is of course always going to be strictly positive, because if equality holds, because if equality holds, so basically the inner product is equals to , holds, then u = λ v.

right so for any so i can say that the inner product of <u- λv and inner product with itself is always positive for any so i can say this holds holds holds for any λ in $\mathbb{R}$ right this holds for any λ of course i'm assuming u and v not so that that's here okay λ right so

You see, now what is happening is this.

If this is true, then let us break it up.

You see, if you break it up, <u,u- $\lambda < u, v + \lambda \quad < v, v$, right?

minus times lambda u inner product with v plus v inner product with v this is what we are going to get which is strictly positive right now if we write down you see what is happening is this so this is nothing but you think of <u,u as a $\lambda < u, v$ is b and $\lambda \quad < v, v$ is c okay so you have this relation $\lambda \quad c - \lambda \quad b + a$.

is strictly positive.

So, this is a quadratic, right?

This is a quadratic.

okay.

And if this is positive, it means that it is never achieving for any λ .

What does that mean?

It means that discriminant is always going to be negative, which means that $b \quad - ac <$, is going to be negative, okay.

What does that mean?

That will imply that what is b, okay, what is b here?

So,

Let me just put it this way that if we write it like this, this particular thing happens.

Let me write it this way.

So, if we denote this as, so, now what happens is, what is b?

b is nothing but $| < u, v|$, the square of that, minus a is <u,u, u inner product with u times c is <v,u.

itself.

So, $| < u, v| \quad - < u, u * < v, u <$, is always has to be negative.

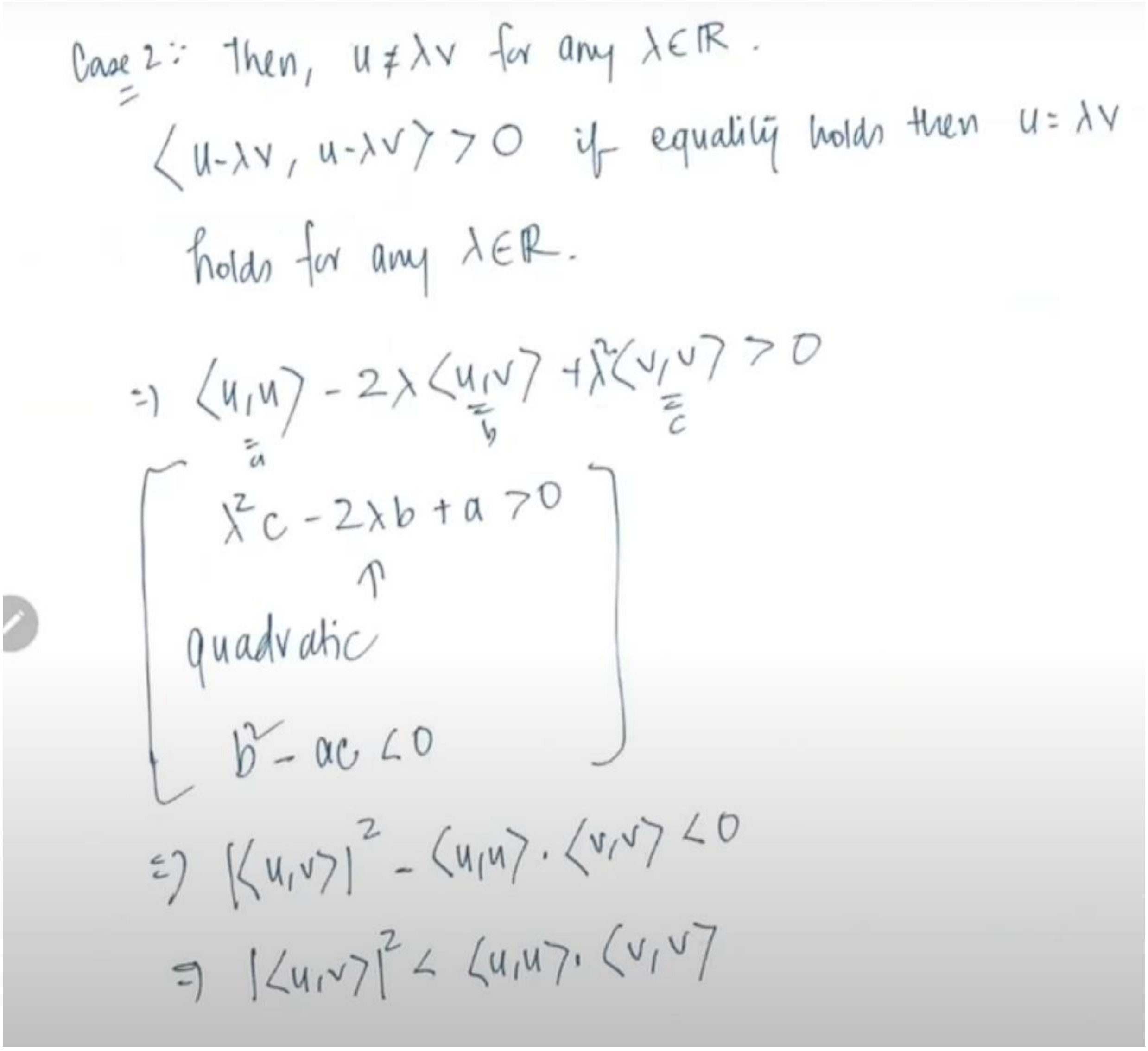
What does that imply?

It implies that $| < u, v| \quad << u, u < v, v$

Sorry, equality does not hold here.

So the inequality holds here and hence so you do realize that when they are dependent then the equality holds and when they are independent then of course the strict inequality holds and hence we have this Cauchy-Schwarz inequality.

So is that clear that the Cauchy-Schwarz inequality holds in an inner product space this sort of thing happens and of course if it is a norm linear space then you can of course write it in terms of norm.



So that is there right.

So, let me put a small remark here.

This is like not a very important remark, but I should make this in any case.

See, if u,v is in $\mathbb{R}^n$, let us say $\mathbb{R}$ or $\mathbb{R}^n$, does not matter$\mathbb{R}$, let us say, let us write it as $\mathbb{R}^n$.

then you can talk about the inner product of u and v, which is nothing but the dot product of u and v. So, these

So, if you take the modulus of this, this means that $|\langle u,v| \leq ||u||||v||$ is always dominated by, so since $\mathbb{R}^n$ is a norm linear space, you can always dominate it by, sorry, norm of v, right?

So, this is always true, sorry, this is always true, okay?

This is always true.

So, I hope this is clear.

So, now, so this is the first inequality which I need to understand, we need to understand.

And the second inequality which we need to do is something called a Gronwall's inequality, Gronwall's inequality.

okay.

This inequality although may seem very, you know, out of the place, but this is also a very, very fundamental inequality which one can actually study in the ODE context, okay.

So, what is that?

So, let us say that let u(x), p(x) and q(x), okay, be non-negative, be

non-negative continuous function, continuous function in the interval $|x-x \ | \leq a$ less than equal, let us say a, yes.

And you also have this relation, $u(x) \leq p(x) + \int_{x}^{x} q(t)u(t)dt$ for $|x - x \ | \leq a$

So, you have to have a modulus here.

So for any $|x-x \ |$ for any x between $x \ - a$ and $x \ + a$, let us, I mean, assume that u(x) satisfies this.

Then, the following identity holds.

Then, the following inequality holds.

also what is this inequality it says that $u(x) \leq p(x) + \left| \int_{x}^{x} p(t)q(t)\exp\left(\int_{t}^{x} |q(\xi)d\xi|\right)dt \right|$ for $|x-x \ | \leq a$, right

Since the first relation is given to u , is given in terms of u is given, u is dominated by a quantity which also depends on u, you see, also depends on u. And the second inequality says that if this sort of, you know,

inequality holds for the first inequality, then the second will also hold and the property of second is this, see u is independent of u on the right hand side.

So, the left hand side u is there, but in the right hand side, there is no u. So, this particular thing holds and this is one of the most fundamental results which we have to learn for this course.

So, this is all hold for all x in between this interval.

So, let us look at the proof of this, proof of that.

okay so we'll do it so we do this we do this for $x \ \leq x \leq x \ + a$, interval okay you can do it for the other one also exactly the same thing doesn't really matter um so you can you can just check the other particles okay doesn't matter okay so um we'll do it for x between x and x plus a right so what we do is we define we define

define $r(x) := \int_{x}^{x} q(t)u(t)dt$, new function

This is the, this is how we write it, define.

So if we do that that will imply $r(x \ ) =$ because integral from x to x whatever quantity it is is going to be .

And of course, we also have that you see q and t are continuous functions.

So, fundamental theorem of calculus says that the integral of those, if you define a new function out of it, then it is also going to be differentiable.

So, we can talk about r'(x)= q(x)*u(x).

Right.

Now, you see from , what can we say?

So, from , one can say that u(x)≤ $p(x) + r(x)$, okay right?

And that, we can also write it as this.

See, if we multiply it by q(x), then we, and q is non, you see, q is non-negative function, non-negative function.

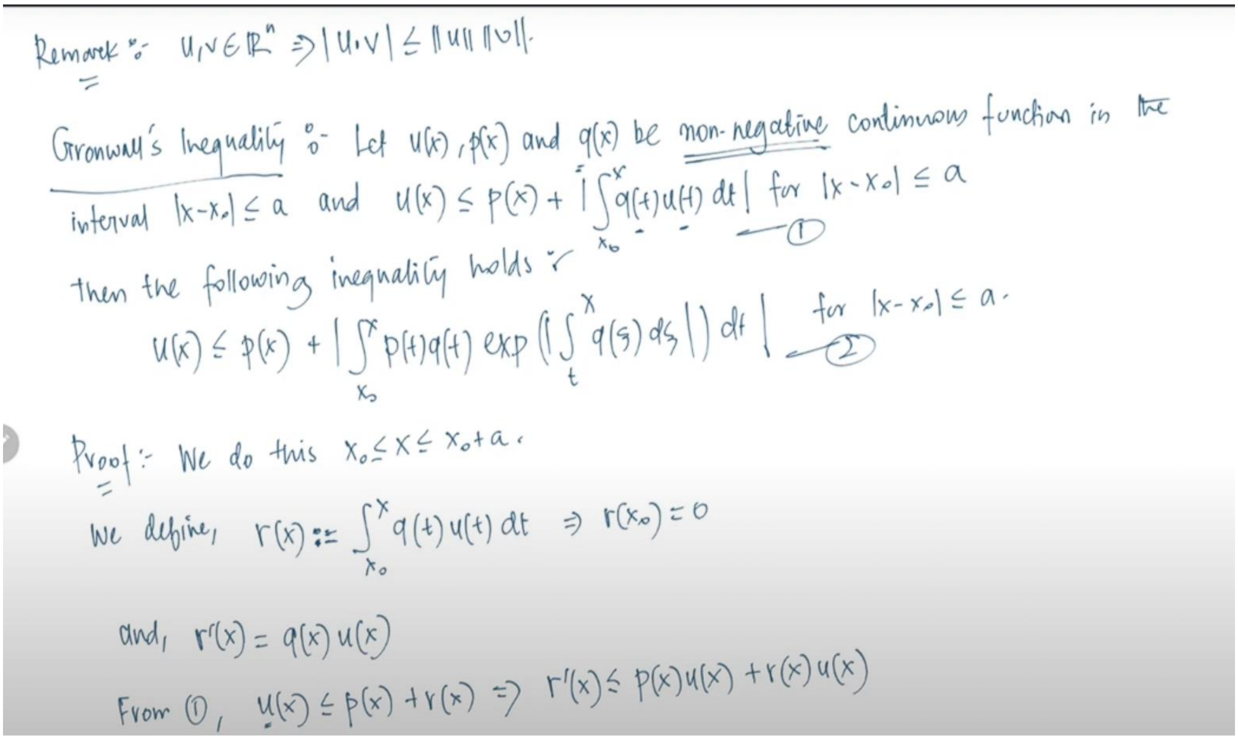
So, all of the functions here are non-negative, okay.

So, if you multiply it by q(x), what happens is, this becomes q(x), u(x) becomes q*r, sorry, q*u, so which is r, so r'.

So, this becomes $r'(x) \leq p(x)u(x) + r(x)u(x)$

I hope this is clear, right?

Now, you see, this is basically an inequality where it looks like this, no?



Remark: $u, v \in \mathbb{R}^n \Rightarrow |u \cdot v| \le \|u\| \|v\|$.

Gronwall's Inequality: Let $u(x)$, $p(x)$ and $q(x)$ be non-negative continuous function in the interval $|x-x_0| \le a$ and $u(x) \le p(x) + \left|\int_{x_0}^{x} q(t)u(t)\,dt\right|$ for $|x-x_0| \le a$ — (1)

then the following inequality holds:

$$u(x) \le p(x) + \left|\int_{x_0}^{x} p(t)q(t) \exp\left(\left|\int_{t}^{x} q(s)\,ds\right|\right) dt\right| \quad \text{for } |x-x_0| \le a. \text{ — (2)}$$

Proof: We do this $x_0 \le x \le x_0 + a$.

We define, $r(x) := \int_{x_0}^{x} q(t)u(t)\,dt \Rightarrow r(x_0) = 0$

and, $r'(x) = q(x)u(x)$

From (1), $u(x) \le p(x) + r(x) \Rightarrow r'(x) \le p(x)u(x) + r(x)u(x)$

$r'(x) - q(x)r(x) \le p(x)q(x)$

This is what it means, right?

Now, you see this is basically nothing but an ODE, right?

And what we can do is if we multiply, so if we multiply the above with this integral, $\exp(-\int_{x}^{x} q(\xi)d\xi) = \varphi(x)$ $(Integrating\ factor)$, so the integrating factor, right?

We are just finding out the integrating factor x to x,

What do we get?

So, $[\varphi(x) + r(x)]' \le p(x)q(x)\varphi(x)$

Q of x and phi of x, right?

So, we define φ like to be this function, the indicating factor, right?

So, once that is there, you see now if you indicate both sides, okay, what do you get?

You get r(x)$\leq \int_x^x p(t)q(t)\exp(\int_t^x q(\xi)d\xi)dt$ and this phi of x, by phi of x, it will be, right?

So, once you have this, you see, so basically this is integrating, integrating above, integrating the above expression, above.

expression.

So, once you have this, you see the inequality.

So, you see the difference between this and this is this.

You see, this is u(x)≤ p(x) times this particular term, right?

So, and we also have this particular term.

So, you have

You see, u(x)≤ $p(x)$+r(x), right?

And r(x), we already got that it is dominated by, r(x) is dominated by this.

So, therefore, you have your result.

So, therefore, one has u(x)$\leq p(x) + \int_x^x p(t)q(t)\exp(\int_t^x q(\xi)d\xi)dt$

Clear?

So, this is how we are going to get it.

Now, you see the original expression, this has a modulus sign here, okay.

So, that you will get once you do the $x \quad - a$ part also, okay.

So, once you do it, you put it together and then we have this Rhon thin equality, okay.

I hope this is clear, okay.

Now, once that is true, what we can also do is essentially put a little corollary, okay, corollary here.

So, you see that if you put in this inequality, if you put p(x)≡, then what does it mean?

It means u(x) is always less than equal plus x to x.

So, basically it means that u(x)≤

But since u(x)

u is non-negative.

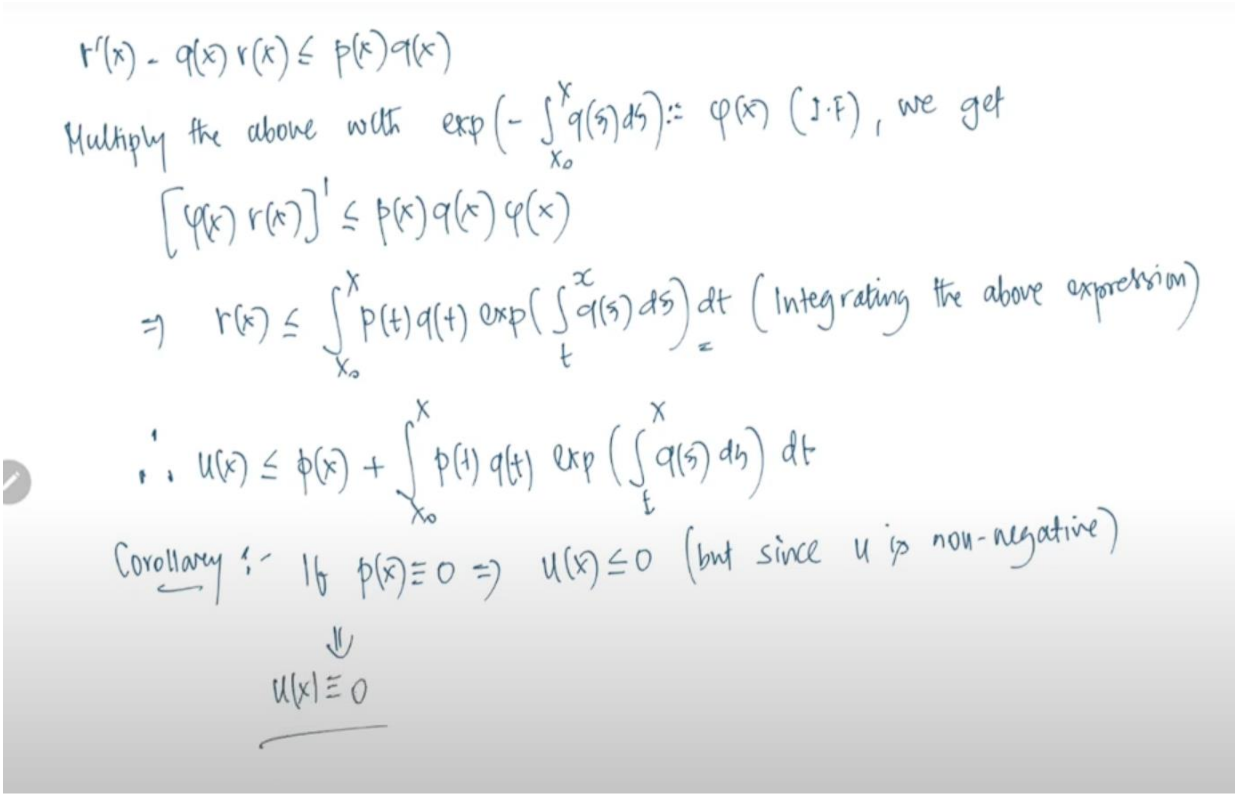
You see, u is non-negative.

This is what our assumption is non-negative.

Non-negative, okay.

That will imply that u(x)≡.

Clear?



Okay.

Now, another corollary which I am writing without proof, okay.

And what I want you to do is please check this corollary.

Check this, okay.

You have to do it yourself.

So, the thing is, let us say the function, the function P(x) is non-decreasing, is non-decreasing.

let us say this interval $[x\ \ , x\ \ + a]$ and it is non-increasing and it is non-increasing.

increasing in the other part.

What is the other part?

$[x \quad - a, x \quad]$.

Then u(x)≤ $p(x) +$ exp(| $\int_{x}^{x} \; q(\xi)d\xi$|)

So, this holds for all x -$x \quad \leq a$

So, I want you guys to check this part.

Please do this.

This is very easy.

You have to just use this Gronwall's inequality properly and you will get the result.

It is not very difficult.

Exactly the same sort of thing.

So, with this we are going to end this particular lecture.

Check :- The function $p(x)$ is non-decreasing in $[x_0, x_0+a]$ and non-increasing in $[x_0-a, x_0]$, then

$$u(x) \leq p(x) \exp\left(\left|\int_{x_0}^{x} q(\xi)d\xi\right|\right) \text{ for all } |x-x_0| \leq a.$$