Ordinary Differential Equations (noc 24 ma 78)
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Week-01
Lecture-05
Lipchitz Continuity

hello students welcome to this lecture on preliminaries for the ode course now uh in this lecture we are going to talk about something called a lipchitcz continuity okay now what is that so essentially what we have is this let's say that if you have a function f:[a,b]-ℝ, right so it is defined on a closed interval [a,b] and the range is in ℝ okay so real valued function

and you are assuming that you are starting with a point x, right, x, which is in open (a,b). So, basically, x is any point.

So, you have a, you have b, and x is an interior point of (a,b), right, okay.

So, we say that f is continuous at the point x, okay, let me just put it x.

If you, if you, let us say, if you give an ε,

you will have a δ depending on ε, right?

Such that f(x)-f(x)< ε for all |x-x| < δ.

So, please remember this δ depends on ε and of course the point x.

And also we know that, I mean, if this dependence, this dependence is independent.

So, basically if this δ is independent of x.

So, if this holds for any x in (a, b), let us say, then x

f is uniformly continuous, right?

Uniformly continuous.

So, basically, it does not really depend on any δ, sorry, x , uniformly continuous in a, b, right?

This is what we know from our real analysis.

So, essentially what I am trying to say is this.

See, for a point x , if you are given an ε, you have to find a δ which depends of course on the ε and the point x such that this happens.

So, basically you will have that f(x)-f(x)can be made less than ε for all |x-x | $< \delta$

This is what you need to do.

to show that this is continuous.

And of course, if this, for any ε greater than , if you can find a δ which is independent of x , then we say it is uniformly continuous in the whole interval a, b. So, basically it is independent of any such point, right.

Now, what are the examples?

Examples we already know that, let us say f(x)=x^n, n$\in \mathbb{N}$

let us say.

So any polynomial, so this is continuous, continuous in $\mathbb{R}$, right?

Continuous in $\mathbb{R}$. It does not really matter what you are going to take in whatever interval it is, it is $\mathbb{R}$. And also you can have f(x)=exp(x)

So that is also continuous, continuous in $\mathbb{R}$.

And similarly, you know that there are other examples which you can construct in the right.

So, I mean, there are lots of examples which you can construct in $\mathbb{R}$. So, that is live continuity.

But the thing is, what is missing from continuity is this.

See, here it is saying that if you are choosing x in a small neighborhood of x , your x also will lie in a small neighborhood of $\mathbb{R}$.

f(x) right that's what it is saying but the thing is it does not give you a control on the growth of f so basically what we want is we need to know how fast f is it's not like exactly how fast f is going because that will give you differentiability we need to know some kind we need to put some kind of control on it and that actually gives us a lot of information on f okay so that is where the concept of leaf lipchticz continuity comes in so let me just define what is leaf lipchticz continuity

Lipchitz continuity, okay?

So, what you have is this.

See, this concept of continuity depends on a point x , right?

So, this is kind of a local concept.

Local concept.

What I mean by local concept is depends on points, right?

Or in a small neighbourhood of that point.

That is what we are looking at, okay?

Of course, if it is uniformly continuous, then it is a kind of non-local.

Non-local or global, you can say non-local.

Non-local means it does not depends at one particular point, but over the whole interval, right?

So, a similar sort of concept is Lipchitz continuity.

What is Lipchitz continuity?

So, if I am defining $f:[a,b]$- $\mathbb{R}$ and we say this is Lipchitz continuous, this is Lipchitz continuous, okay, continuous in [a,b], right.

You see, Lipchitz continuity, we will not define it at one point.

This is a non-local concept.

Please understand, this is non-local, okay.

non-local.

So, what do I mean by this?

I mean that it does not depend at one single point.

So, okay, there exists M, okay, M of course depends on the interval [a,b], okay.

So, M depends on [a,b], okay, but does not depend on any point.

So, M, such that, okay, so there is a real number M such that,

Whatever $f(x)-f(y) \leq M|x-y|$ you choose, does not matter what x,y you choose, right?

So, what I mean by this is, you see, you can of course control the growth of f, how fast f is growing using the information that, see, we already know how fast x is growing, right, the range, sorry, the domain, how fast we are changing the point.

So, we are actually controlling that change in your x, okay.

So, the

uh this thing the domain uh based on that we are changing i mean i am saying that we are changing the uh growth of f we are controlling the growth of it based on the growth of x right that's what you are saying here and this m of course this m you can see that this M is independent of a,b okay so this should hold for all x,y in [a,b] doesn't really matter what x,y you are choosing this is there

Right.

So, that is Lipchitz continuity.

I hope this is okay.

So, what we have is it is Lipchitz continuous over the whole interval a,b. Please understand this is not a local concept.

This is a non-local concept.

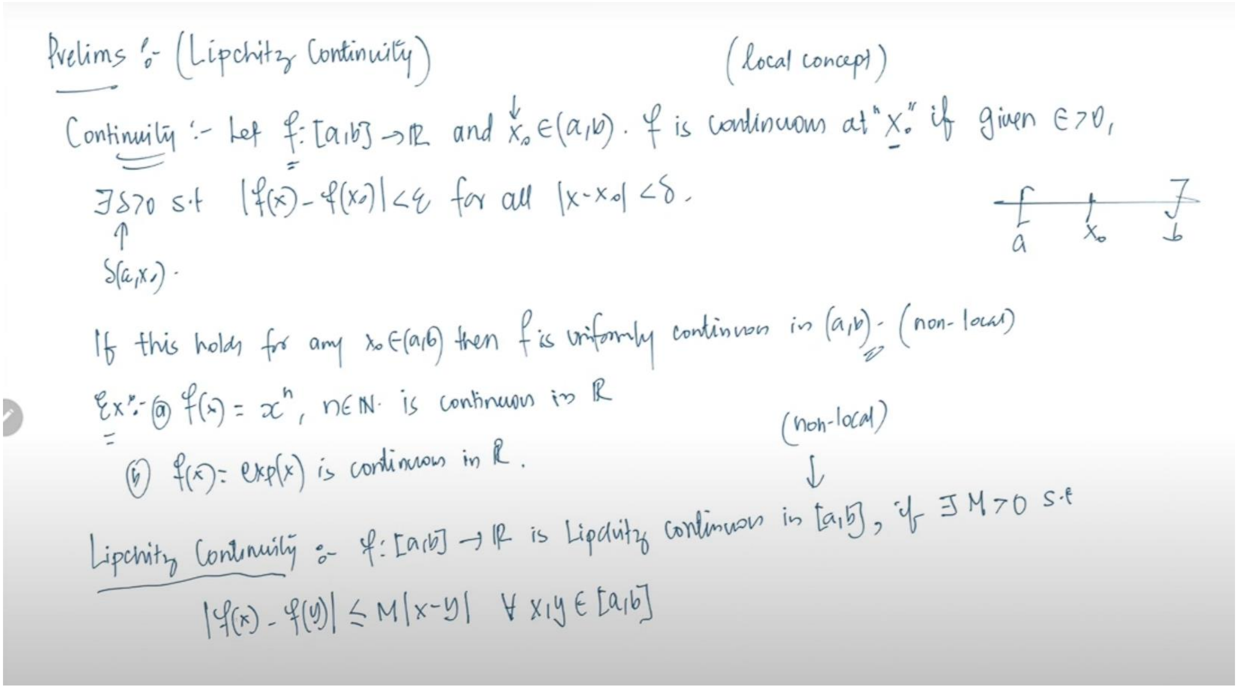
It depends on whatever the interval is defined, right.

If you have a M again, if you have a uniform bound M, right, this is a uniform bound.

And this M should work for all x, y, and ℝ. So basically, this M will help you to actually control f in the whole interval.

Is this clear?

Okay, right.



Now, let's just understand what is so special about Lipchitz continuity.

So with some examples, okay, examples.

okay so f(x)= x in ℝ let's say in ℝ right so let's see if it is continuous or not of course you can see that this is going to be trivially lifted continuous right so |f(x)-f(y)|=|x-y|

Right.

So this is again you can actually write it.

So basically you see I have to find the M such that so I have to write it like this |f(x)-f(y)|=|x-y| $\leq M|x-y|$ right for Lipchitz continuity.

But you can see that if we choose M to be , then this is Lipchitz continuity.

So this M, this M. So here this M is called the Lipchitz constant.

So here this is also this M we will call it as a Lipchitz constant.

constant, okay?

Lipchitz constant.

So, this is a Lipchitz continuous function, okay?

Let us look at some other example.

Let us say f(x)=|x|, okay, in ℝ. Now, we know that this function is continuous, right?

So, f is continuous, right?

This we know, it is continuous in ℝ,

And of course, we also know that f is not differentiable in ℝ, right?

It is not differentiable at the point x, at the point at x=, right?

But the thing is, is it Lipchitz continuous?

Let us just see that.

So, what we have is, see, x can be written as x=x-y+y, right?

This we can do.

This holds for all x, y in ℝ. We can write for any x, y in ℝ , this is, we can write since

Then, you see, |x|=$|x - y + y|$ is always less than mod x minus y plus y, sorry, equals to, yeah, okay.

So, I can take the modulus on both sides.

This is the modulus function, right.

So, you see, then mod x can be written as $|x| \leq |x-y|+|y|$, right.

So, this is triangle inequality.

Of course, triangle inequality holds in $\mathbb{R}$, triangle inequality, equality.

And then you can, of course, write that $|x|-|y| \leq |x-y|$. So, you see, and since x and y, see, it does not really matter if you are writing x and y. If you replace x and y with y and x, then you also have, so by symmetry basically, you also have $|y|-|x| \leq |y-x|$.

And since $|x-y|$ and $|y-x|$, are basically the same thing.

So, these are basically same.

Therefore, you can write that $||y|-|x||$. If you are putting the modular function, $||y|-|x|| \leq |x-y|$. So, what does that give you?

You see, that gives you that $f(y)-f(x) \leq$ *$|y-x|$ or *$|x-y|$. It does not really matter.

So, this holds for all x, y in $\mathbb{R}$.

x, y in $\mathbb{R}$. So, you see this function, this actually says that this is your M, right?

This is your M. So, is a Lipchitz continuous, sorry, $f(x)=|x|$ is Lipchitz continuous.

So, what is the whole interval $\mathbb{R}$, okay?

And by the way, please remember this inequality, it is very, very important.

So, basically it says that $||x|-|y|| \leq |x-y|$, okay?

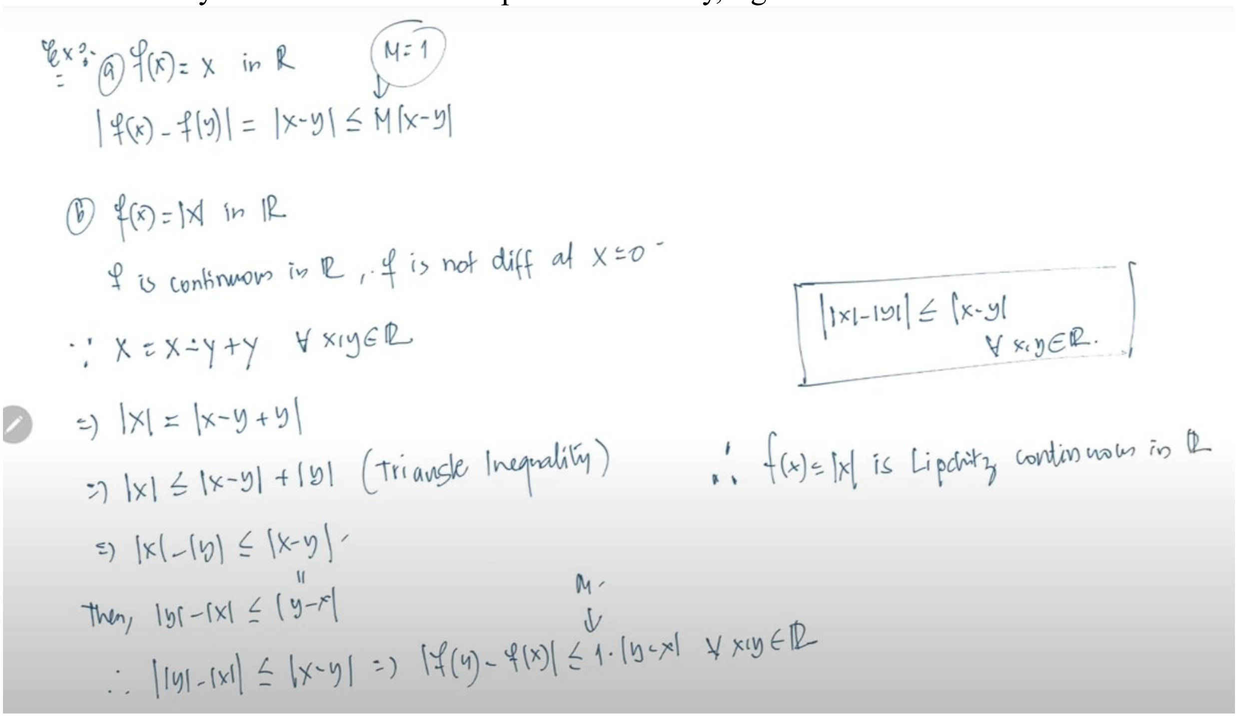
This holds for all x, y in $\mathbb{R}$.

This inequality we are going to use it in many other places but you do realize that you can use this inequality to show that therefore f(x)=|x|, is Lipchitz continuous.

Lipchitz continuous where?

Lipchitz continuous in ℝ.

Now, you have to understand why I am saying in ℝ. Please, whenever you do Lipchitz continuity, make sure that you know where it is Lipchitz continuity, right?



So, for example, let us just put this example, okay?

C, okay?

Let us say f(x)=x .

Let us just see if it is Lipchitz continuity in ℝ or not, okay?

In ℝ or not, okay?

So, you have f(x)-f(y)=| $x - y$ |, right?

So, that will give you |x+y|*|x-y|.

Now, if you, let us say if it is Lipchitz continuous, if you are assuming it is Lipschitz continuous, f(x)-f(y)=| $x \quad - y \quad$ | $= |x + y| * |x - y| \leq$ M*|x-y|, right?

So, that will actually imply that if this is Lipchitz continuous, if, so let me put it this way, if f is Lipchitz, is Lipchitz in ℝ, Lipschitz in ℝ in ℝ,

Then there must exist M such that |x+y|≤M for all x, y in ℝ.

But you do realize that this is of course not possible because I can choose x and y sufficiently big such that this is not true.

This is not true.

Hence, f(x)=x , is not, Lipchitz continuous, Lipchitz continuous, continuous in ℝ, right?

Lipchitz continuous in ℝ. But, but you see,

You see, here the problem is this.

x, y, I am choosing it from ℝ, right?

And ℝ is unbounded.

So, essentially, that is the problem that we cannot bound it with a uniform constant, right?

But if you are choosing f from a bounded set, so basically, let us say f(x)=x in a bounded set, let us say in closed bounded set [a, b]. Let us just choose it like this.

It does not need to have to be closed, but let us just choose it like this, okay?

So, in that case, you see, f(x)-f(y)= $|x - y||x + y| \leq b * |x - y|$

Of course, in a bounded set, so, basically, what is your M here?

M=b, okay?

So, basically, let me write it this way.

See, let me do it properly.

You see, here, $|x+y| \leq |x| + |y| \leq b$

Since, this is triangle inequality, right?

Triangle.

Triangle inequality.

Equality.

Hence, you see, this is less than equal and |x| is always bounded by b, right?

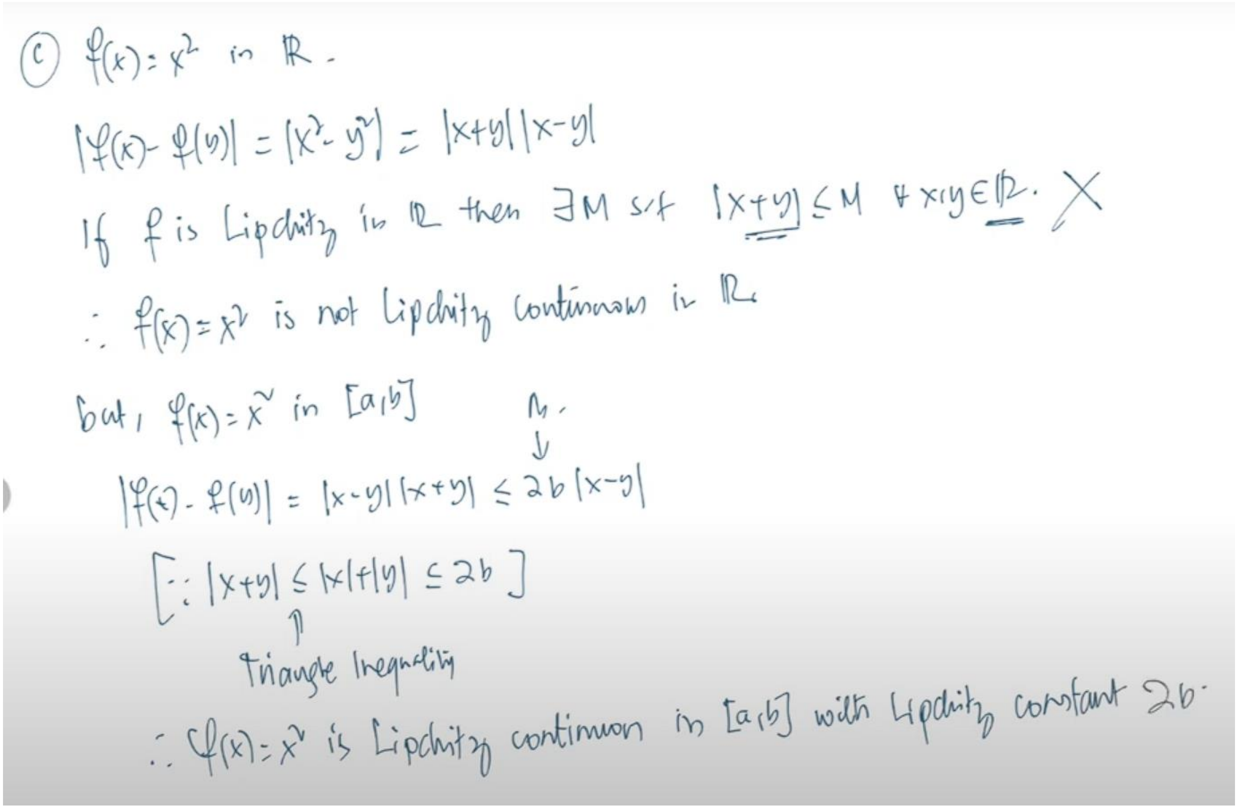
So, this is always bounded by b.

So, therefore, $f(x)=x$

is Lipchitz, is Lipchitz, continuous in, continuous, in closed [a,b],with, what is the Lipchitz constant?

So, with Lipchitz constant, Lipchitz constant, okay, what is the Lipchitz constant?

b.



(c) $f(x) = x^2$ in $\mathbb{R}$.

$|f(x) - f(y)| = |x^2 - y^2| = |x+y||x-y|$

If f is Lipchitz in $\mathbb{R}$ then $\exists M$ s/t $|x+y| \leq M$ $\forall x, y \in \mathbb{R}$. ✗

$\therefore f(x) = x^2$ is not Lipchitz continuous in $\mathbb{R}$.

but, $f(x) = x^2$ in $[a,b]$ M.

$|f(x) - f(y)| = |x-y||x+y| \leq 2b|x-y|$

$[\because |x+y| \leq |x| + |y| \leq 2b]$

Triangle Inequality

$\therefore f(x) = x^2$ is Lipchitz continuous in $[a,b]$ with Lipchitz constant $2b$.

right okay so i hope this is clear to you whether it is Lipchticz continuous or not okay so you have to check one problem now so check f(x)=log x, is lipchticz, so basically uh the question is this can you find a function uh sorry can you find a domain lipchticz in I

okay let's say if I put it does there exist such a I. Clear so basically what i'm trying to say is this does there exist an interval I where f(x) equals now once we have that lipchticzness the concept of lipchticzness clear there is another concept which is called locally lipchticz right locally lipchticz so locally means you see as i told you lipchticz continuity is a

non-local concept or you can say global concept, right?

It depends on the whole, whatever interval you are doing, you have to have the properties satisfied on the whole interval.

Locally Lipchitz is the local version of that.

So, essentially you say that f: [a,b]-ℝ, right?

And you start with x which is in (a,b), right?

Then we say f is locally Lipchitz, locally Lipchitz

locally Lipchitz at x , if there exists a δ such that $f|_{B(x\ ,\delta)}$ Lipchitz continuous, right?

Lipchitz

continuous.

So, essentially what I meant by this is you see at one point if you start with x you need to have $x\ -\varepsilon\ and\ x\ +\varepsilon$ let us say some interval should be there which is of course contained in that in a b. So,

that interval, in that interval, if you restrict f, in that interval, it should be Lipchitz continuous, right?

So, that is there.

So, this is the, I hope the concept is clear, right?

So, again, when it is Lipchitz continuous in whole a, b is for all x, y in a, b, this property, that f(x)-f(y)≤ $M|x-y|$,

okay?

Now, you see f is local lipchticz if for every x , so basically you start with an x , sorry, not for every x , if you start with an x and you say it is local lipchticz at that point,

If you have a neighborhood of such point where f is, when f restricted to that neighborhood is Lipchitz continuous, right?

So, it should not, it does not have to be Lipchitz continuous everywhere, but it has to be in a neighborhood, okay?

So, for example, let us look at an example here.

You see, f(x)= x if you look at in ℝ, right?

This is in ℝ. Then we saw that f is not Lipchitz continuous.

It is not Lipchitz continuous.

right continuous but you can see that uh let's say if you start with x in ℝ okay then uh you can of course choose a δ then there exists a δ okay see any bounded interval so essentially if you start with x and you look at a you can of course find a δ such that $f|_{B(x\ \ ,\delta)} = x\ \ |_{B(x\ \ ,\delta)}$

This function is of course Lipchitz because x is Lipchitz continuous on any closed and bounded interval.

So, this is Lipchitz.

So, what does that gives you?

It gives you that this is a function which is of course not Lipchitz continuous in ℝ, but it is locally Lipchitz.

So, what we have in here is the information is Lipchitz continuity is a much stronger condition.

Locally Lipchitz is a more relaxed condition and it is a bigger space than Lipchitz continuity.

Another remark, small remark I want to add is this.

See,

Let us say that if f is Lipchitz continuous, if f is Lipchitz in whatever Lipchitz, in let us say whatever interval I is, I = [a,b]. It can be ℝ also, I = [a,b]. So, let us say if f is Lipchitz continuous in I, which is [a, b], then f is uniformly continuous, uniformly continuous.

This is very easy to check.

Please do that.

I want you to check this part, continuous.

So, please check.

But, okay, now the question which I want to pose here is this.

Does uniform continuity, does uniform continuity, uniform continuity of f, continuity of f, of f imply Lipchitz continuity?

Lipchitz continuity, okay?

And I will give you a hint.

You think in terms of this sort of function, okay?

g(x)=$\sqrt{|x|}$; $x \in \mathbb{R}$ or you think of it this way or it doesn't have to be ℝ, x is some bounded interval let's say [,] or [-,] whatever it is [-,] or [,] doesn't matter okay so that sort of function think of the square root function and try to answer this question whether uh if f is the uniformly continuous whether this is a this is lipchticz continuous function

And there is again another remark which I want to put here is this.

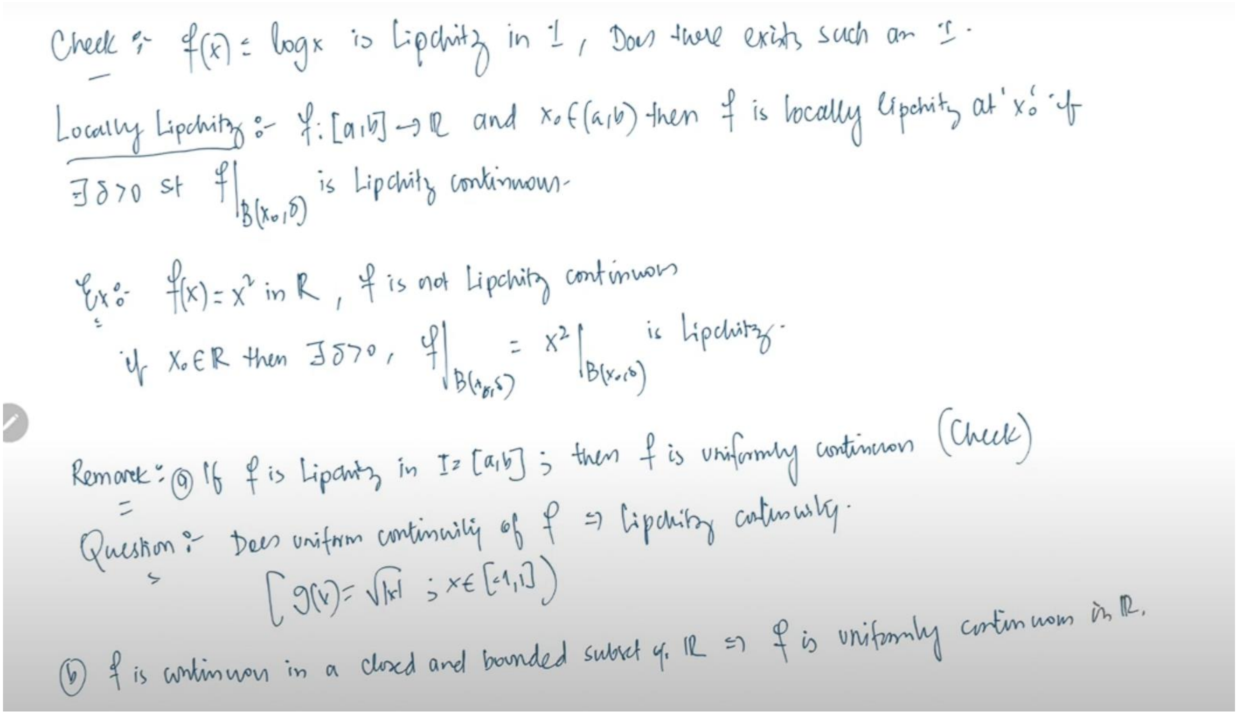
Let us say f is continuous, f is continuous in a closed and bounded interval.

So, f is continuous in a closed and bounded interval, closed and bounded interval, bounded interval or bounded subset, does not matter, bounded subset of ℝ, subset of ℝ.

that will actually imply that f is uniformly continuous, uniformly continuous, okay, uniformly continuous in ℝ, okay.

So, that is there, okay.

So that is more or less the whole idea of Lipchitz continuity.



Now the thing is let me just finish this lecture with a small remark.

Another small remark but this is in a different context.

So C.

You see if let us say I start with a let us say vector space which has some metric property.

So basically you start you can think of it as this.

Let us say that f:x-ℝ , and this is a metric space and x is a metric space.

Okay, now the thing is this, if I ask you that can you, can you, you know, generalize the idea of Lipchitz continuity to this space, can it be done?

Okay, can one generalize, can one generalize the idea?

Okay, generalize the idea, the idea of Lipchitz continuity, the idea of Lipchitz continuity.

lipchticz continuity okay lipchticz continuity actually it can be done and you can of course see that how you can do it so please do it yourself but i'm just giving you an idea once it is a metric space okay so once is the metric space this has a metric right so basically

what you do is this, you see $d(f(x),f(y)) \leq M * d(x,y)$, okay.

And see, this is in $\mathbb{R}$, you see, $d(f(x),f(y))$ is in $\mathbb{R}$, right.

So, basically, $d(f(x)-f(y))$ will actually can be written like this,

$|f(x)-f(y)| \leq M * d(x,y)$. So, you can generalize the concept like this.

Of course, you can replace this x to another matrix space y and then this definition should work.

But the thing is what I want you to do is please check if it does or not.

Please check if the above check if the above idea works.

Idea

Works or not?

Clear?

Okay.