Ordinary Differential Equations (noc 24 ma 78)

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Week-04

Lecture 56: Linear two-dimensional phase space dynamics (contd.)

Hello and welcome to another lecture of classical motion of a single particle. We simply continue our discussion on the general aspects of dynamics in a 2D phase space. So, in the previous lecture, we have already showed that if you have a linearized system of equations in 2D phase space, which is given by x dot equal to A x, where A is the coefficient matrix, okay? And x dot is the phase space velocity vector, okay? and x is the phase space state vector ok.

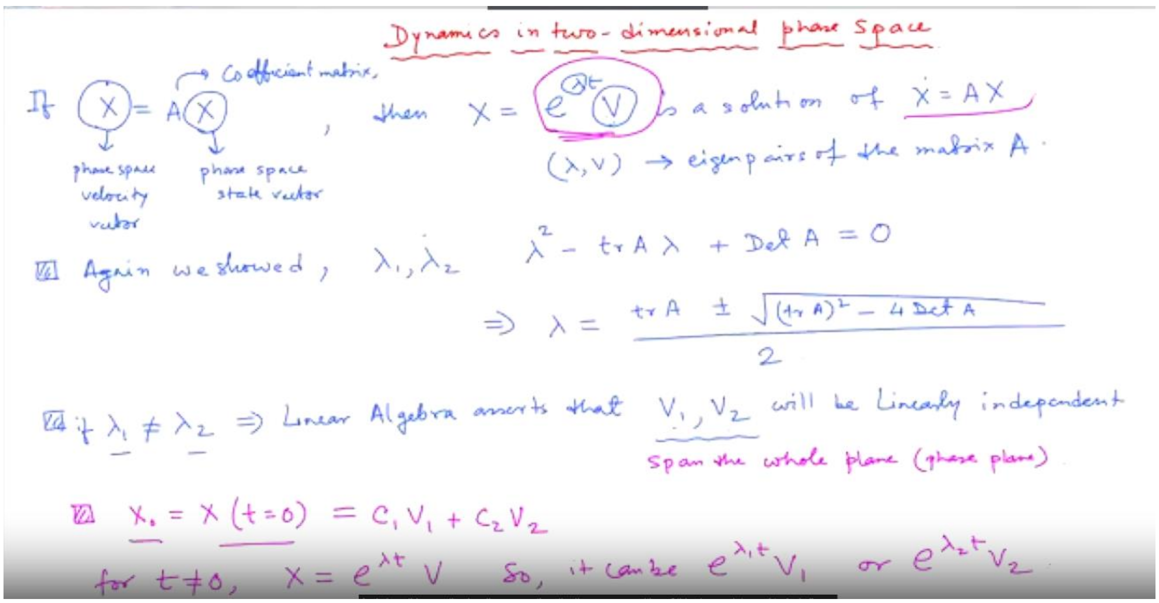
If I can write, so this is composed of ah phase space coordinates, this is composed of phase space velocities ok, then we actually if yeah then we actually can show x is equal to e to the lambda t times v is a solution of X dot equal to A X ok. Now of course, here this lambda and V they are not arbitrary. So, V is a vector lambda is a number and they are nothing, but the Eigen pairs of the matrix A ok. So, again we showed that

lambda 1 and lambda 2, these two eigenvalues, okay, they are simply can be written in for 2D phase space for a linearized system as lambda square minus trace of A times lambda plus determinant of A, okay, is equal to 0. And from that, one can also write lambda is equal to trace A plus minus root over trace A whole square minus 4 determinant of A by 2 ok. Now, if lambda 1 is not equal to lambda 2 that means, the matrix A has two distinct eigenvalues for that case Linear algebra says something very interesting.

Linear algebra asserts that V1 and V2 which are the corresponding eigenvectors of lambda 1 and lambda 2 respectively will be linearly independent ok. And that is why they can actually span the whole plane phase plane ok. You can now just think of the initial vector x 0 which has no dependence on time. So, x 0 is nothing, but x at t is equal to 0 then this is just a constant vector and that can actually be then expressed as a linear combination of v 1 and v 2 with constant coefficients.

So, c 1 v 1 plus c 2 v 2 ok and if we want a solution which is proportional I mean which is having the form of e to the power lambda t v and then we actually know that for t not equal to 0. So, ah this is for t is equal to 0. So, for t not equal to 0 ok we have x is equal to actually e to the power lambda t v. So, it can be e to the power lambda 1 t v 1 or e to the power lambda 2 t V 2.

And since this equation is a linear equation, the linear superposition of this since x dot equal to Ax is linear. So, the solution for this equation will be given by x equal to some c 1 e to the power lambda 1 t v 1 plus c 2 e to the power lambda 2 t v 2 and here i have chosen this c 1 and c 2 exactly equal to this c 1 and c 2 so that if you now write t is equal to 0 you can get back this expression okay for x is equal to 0 Now, this is because and this is the general solution of X. So, this is the general solution.



Now, we have something more interesting here at this point although this is mathematically a little bit deep or profound just for the time being you just take this as an information ok so i can actually also claim this is the only possible solution for x this one why that is because of two reasons first since x is equal to C1 e to the lambda 1 T v1 plus C2 e to the lambda 2 T v2 is the linear combination of the solutions to x dot equal to Ax and also this solution satisfies the initial condition, okay? Then the existence and uniqueness theorem for first order linear ODEs says that X is the only solution.

X means this form, okay? And this is mathematically quite interesting, okay? We now use this thing to understand the phase space dynamics and also the phase space trajectory, the stability, the nature of the fixed points, okay, for different type of generalized 2D phase space flows, okay. So, I just take one by one as examples.

$X_0 = X(t=0) \in c_1 V_1 + c_2 V_2$

for $t \neq 0$, $X = e^{\lambda t} V$ So, it can be $e^{\lambda_1 t} V_1$ or $e^{\lambda_2 t} V_2$

Since, $\dot{X} = AX$ is Linear, the solution for this eq^n will be given by

$$X = \boxed{c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2}$$

general solution

* (i) Since $X = \boxed{c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2}$ is the Linear combination of the solutions to $\dot{X} = AX$

and (ii) $X(0) = X_0$ $\Longrightarrow$ The existence & uniqueness theorem for 1st order linear ODE $\Rightarrow$ X is the only solution

So, I will in this part, I will take several examples. So, the first example is x dot equal to x plus y and y dot equal to 4 x minus 2 y ok. Till now most of the cases you have seen that for x dot it is either equal to x or minus x or a y or minus y, but both of x and y they never appear. ok.

Same similar thing for y dot and of course, from this construction you can see that this is some general 2D phase space flow, but it is not actually corresponding to our usual classical mechanics case. So, once again this is to give you an initiation of the general aspect of non-linear dynamics. So, ah if you have this and the initial condition is given by x 0 y 0 equal to 2 comma minus 3, then we have to find the nature of the fixed points and also the finally, the general solution ok, that is our principle objective ok. So, if I write this as x is equal to x dot equal to A x, then A will be equal to 1 1 4 minus 2 ok.

And so, for this system trace of A is equal to 1 minus 2 that is minus 1 and determinant of A, determinant I sometimes write with a capital D sometimes small d. So, please I mean bear with that. So, determinant is equal to minus 2 minus 4 which will be minus 6 ok. And so, the characteristic equation satisfied by the eigenvalues will be lambda square minus trace A times lambda which is nothing but plus lambda ok and then plus determinant of A which is minus 6 equal to 0 and from that we can write lambda is equal to 2 and minus 3. So, we have the case of distinct root ok.

So, we can proceed without problem because we know that the 2 eigenvectors V 1 V 2 will be linearly independent ok. So, now, let us find the eigenvectors for lambda 1 equal to 2 let us say lambda 1 equal to 2 and lambda 2 equal to minus 3 ok. we have so this a 1 minus 2 1 4 minus 2 minus 2 ok times u 1 and v 1. So, this is small u 1 and small v 1 which is nothing but represented by the vector v 1 capital V 1 and that is equal to 0 ok.

and here you will see that this is nothing but giving ah minus u plus. So, u 1 is equal to v 1 ok and same from the second equation. So, this is the only one equation I can now construct the normalized eigenvector which can be given by by v 1 capital V 1 ok is equal to 1 over root 1 1 ok. I think constructing I mean normalized eigenvectors you all know the trick ok.

If you do not know just go through some books of linear algebra ok.

Examples:

(1) $\left.\begin{aligned}\dot{x} &= x + y\\ \dot{y} &= 4x - 2y\end{aligned}\right]$ Initial Condition $(x_0, y_0) = (2, -3)$.

$\dot{X} = AX$, then $A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \Rightarrow trA = 1 - 2 = -1$, $Det A = -6$

and so, $\lambda^2 + \lambda - 6 = 0 \Rightarrow \lambda = \boxed{2, -3}$ the two eigenvectors V_1, V_2 will be Linearly independent.

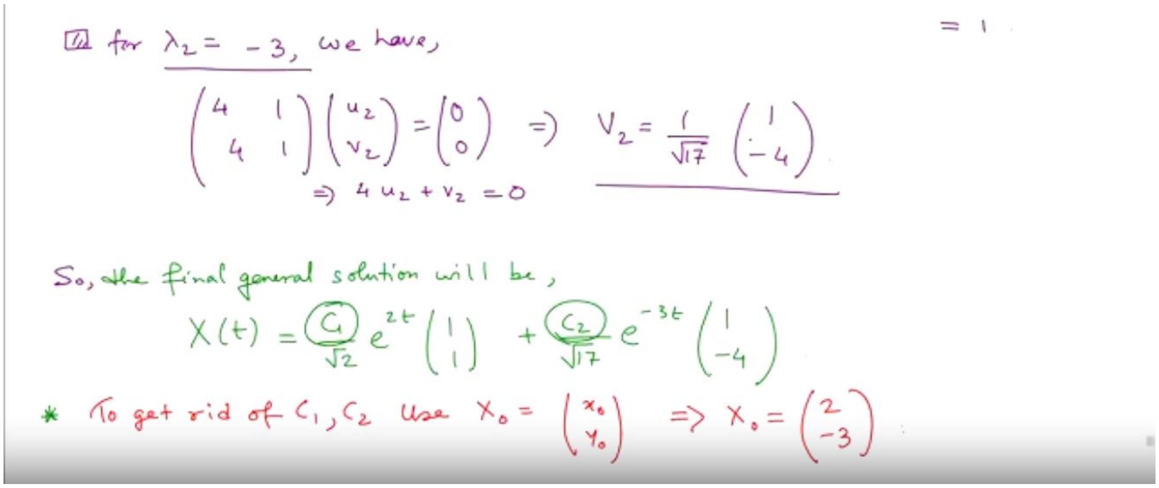
▫ for $\lambda_1 = 2$, we have,

$$\begin{pmatrix} 1-2 & 1 \\ 4 & -2-2 \end{pmatrix} \underbrace{\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}}_{V_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{u_1 = v_1} \Rightarrow \boxed{V_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

So, I mean there can be various possibilities. So, u 1 and v 1 they are just same, but you can write u 1 is equal to 2 v 1 is equal to 2 So, from all these arbitrary choices we just construct V 1 in such a way that the norm or the magnitude of V 1 is 1 ok. If you write instead of ah 1 1 if you just write 1 by root 2 and 1 by root 2 for both U 1 and V 1 ok, then V 1 mod is nothing, but root over half plus half ok is equal to 1 ok.

So, that is the trick. for lambda 2 is equal to minus 3 we have similarly 4 1 4 1 u 2 v 2 is equal to 0 0 and from that you actually again can construct a normalized eigenvector which is given by 1 over root 17 1 minus 4 ok. So, because here the constitutive relation becomes 4 u 2 plus v 2 is equal to 0, ok. So, if you have this two information that lambda 2 v 2 and also lambda 1 v 1, then there is no problem you can write the general solution for x. So, I can write in I do not know like in green.

The final general solution will be X t is equal to C 1 by root 2 e to the 2 t 1 1 plus C 2 by root 17. e to the minus 3 t 1 minus 4 ok. Finally, so this will still give you C 1 and C 2 to get rid of C 1 and C 2 you have to somehow use the initial conditions ok to get rid of C 1 C 2 use the information about x 0 ok that is x 0 y 0 ok and that is nothing but you know that so I mean maybe I should write in a proper way because they are given as columns column vectors so x 0 y 0 so that means x 0 is equal to ok 2 minus 3 ok. And so, of course, you know that this is true for t is equal to 0.



And so, finally, you have two equations. So, C1 over root 2, 1, 1 plus C2 over root 17, 1 minus 4 is equal to 2 minus 3, okay? If you calculate, you can actually just to simplify the scene, you can actually define C1 over root 2 as C1 prime and C2 over root 17 as C2 prime. And finally, you will have two equations. One will be simply C1 prime plus C2 prime equal to 2 and another will be C1 prime minus 4 C2 prime is equal to minus 3.

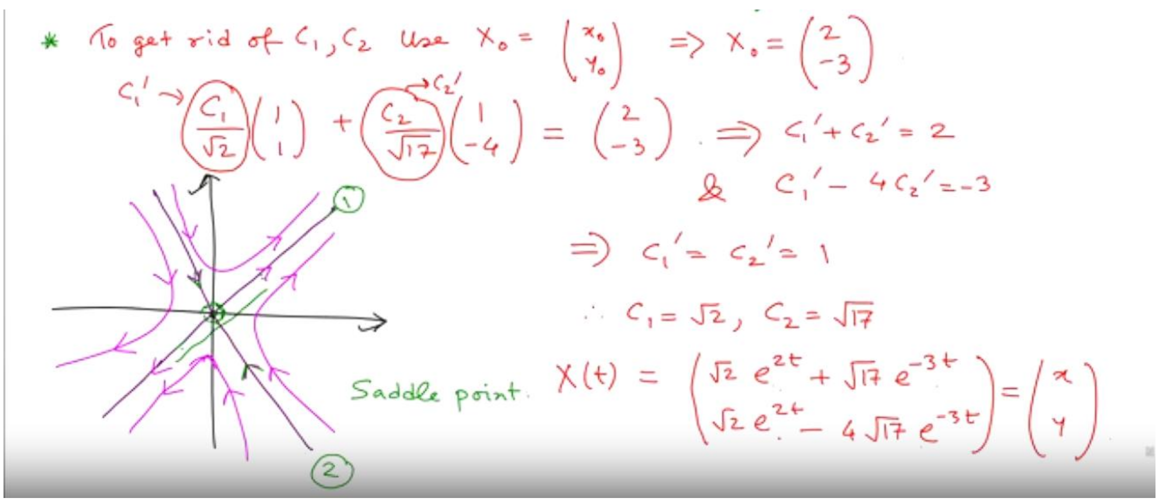
and just doing some algebraic manipulations in one line you can show that C 1 prime is equal to C 2 prime is equal to 1 ok. So, then of course, C 1 will be equal to C 1 will be equal to root 2 and C 2 will be equal to root 17 ok. With that, you can now write the final equation, final solution, sorry, final solution for this dynamics, which is given by X t is equal to root 2 e to the power 2 t plus root 17 e to the power minus 2 t and then root 2 e to the power 2 t minus 4 root 17 e to the power minus 3 t, okay? Sorry, there should be a minus 3 t, not minus 2 t, okay?

So, this is exactly equal to x and y. So, you have this. Now, if you just see a little bit more carefully, you will see that you will have exponentials over here and when they are combined, okay, like this, they will give you hyperbolic solutions. And if you now want to draw this, if you plot that carefully or you use some plotting software, you will see that the hyperbolae are not exactly taking this axis and this axis as their axis, ok.

On the other hand, here the hyperbola they are taking the eigenvectors directions to be their axis. Now, you see let us say I am thinking of this 1 is like 1 eigenvector was like this with 1 1, ok. Another one was 1 minus 4 that can be given by this, ok. And now the hyperbolae will be simply given by this type of curve okay. Here you see I can just want to draw the proper directions one possible direction just to keep the continuity of the phase space intact and you can see if I draw like this then this one should like this, this one also like this and this one like this.

and so on a hyperbola the direction is unique okay and again from continuity this should be like this this like this so here it should come like also so here like this and then it is for this hyperbola it should go like this so this and this okay now you see what is the most interesting thing If you now just concentrate on the eigen directions, you will see along one eigen direction, you will have a repeller, okay? Of course, there is a discrete fixed point. Although I have drawn this a little bit less carefully, I have connected this, but this is a discrete repeller, okay? And you can actually show that

So, with respect to this direction, this direction let us say 1, this fixed point is actually acting like a repeller. But on this other eigen direction 2, they are actually acting like an attractor. And that is exactly the case where we talk about a fixed point whose name is saddle point. And the corresponding fixed point is known as a saddle point. I can go to another example.



which is much more familiar for us and relevant for this class as well, ok, but before that I can also do a simpler one actually. So, let us take this example, ok. So, x dot equal to x and y dot equal to minus y, ok. So, what will be the coefficient matrix A? The coefficient matrix will be simply given by 1 0 0 minus 1, ok.

So, you have two distinct eigenvalues and since this is in diagonal form the eigenvalues are themselves apparent over there lambda is equal to 1 minus 1 ok. So, for lambda 1 is equal to 1 we can actually have 0 0 0 minus 2 times u 1 v 1 is equal to 0 0 from which you can show that v 1 will be equal to 0. So, I can have an eigenvector where u 1 can have any arbitrary value, but v 1 should be 0, ok.

So, I can just take the normalized one. So, normalized v will be equal to 1 0, ok. Similarly, for lambda 2 is equal to minus 1, a is equal to 2, 0, 0, 0, u1, v1 is equal to 0, 0. And from that, we actually have the normalized v2, normalized eigenvector to be 0, 1.

And hence, x is simply given by C1 e to the power t, times 1 0 plus c 2 e to the power minus t 0 1 you now search for the initial conditions if they are provided you can actually find c 1 and c 2 ok. And of course, if you see that this type of thing also gives you if I just write this very simplified equation. So, from this you can easily see that The solutions will be small x will be equal to simply c 1 e to the power t and small y will be c 2 e to the power minus t, ok.

That is exactly we have also found in one of the previous lectures, ok. Now, you have seen from a general perspective, ok. You can actually think of the nature of the fixed point and you can tell me. So, finally, we go to some examples finally, we take an example of classical mechanics.

$$\left.\begin{aligned}\dot{x} &= x\\ \dot{y} &= -y\end{aligned}\right\} \quad A = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

two distinct eigenvalues $\lambda = 1, -1$

$$\therefore \lambda_1 = 1 \Rightarrow \begin{pmatrix}0 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}u_1\\ v_1\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow v_1 = 0$$

$$\therefore \text{Normalized } V = \begin{pmatrix}1\\ 0\end{pmatrix}.$$

Similarly for $\lambda_2 = -1$, $A = \begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}u_1\\ v_1\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$

$$\Rightarrow V_2 = \begin{pmatrix}0\\ 1\end{pmatrix}$$

and hence, $X(t) = c_1 e^{t}\begin{pmatrix}1\\ 0\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}0\\ 1\end{pmatrix}$ $\left\{\begin{aligned} x(t) &= c_1 e^{t}\\ y(t) &= c_2 e^{-t}\end{aligned}\right.$

So, here x dot is equal to y and y dot is equal to x. You can easily understand this is the case of a saddle and where a is equal to 0, 1, 1, 0. This is not the diagonal form, okay. The off diagonal forms are actually filled up. So, here you have lambda 1 plus lambda 2 is equal to 0 and lambda 1 lambda 2 equal to minus 1 which gives you lambda 1 and 2 will be actually plus minus 1, okay. So, they are also the case of distinct eigenvalues and if you calculate

V, you find for lambda is equal to 1, small u1 is equal to V1. And so, capital V1 is nothing but 1 over root 2, 1, 1. And for lambda is equal to minus 1, u2 is equal to minus V2 and that gives you V, capital V2 is equal to 1 over root 2, 1 minus 1, okay? And so, capital Xt will be simply given by C1 over root 2 e to the t, 1, 1 plus C2 by root 2 e to the minus t, 1 minus 1, okay?

And from that, if you just use the initial conditions, you can find the phase-space trajectory and also you can easily see that just I am writing for small x t, it is given by C1 by root 2 e to the t plus C2 by root 2 e to the minus t, already you can smell a hyperbolic solution. But you have to actually do a little bit of coefficient management for that. So, this is the case of the saddle point. Now, I can also write the case where x dot is equal to y and y dot equal to minus x. So, this is the simple harmonic oscillator oscillator ok classical mechanics and we have a is equal to 0 1 minus 1 0 and that is why lambda you can write is plus minus i and you can find.

Finally we take an example of classical mechanics:

$$\left.\begin{aligned}\dot{x} &= y\\ \dot{y} &= x\end{aligned}\right] \Rightarrow A = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \Rightarrow \begin{aligned} &\lambda_1 + \lambda_2 = 0 \\ \& \ &\lambda_1 \lambda_2 = -1 \end{aligned}$$

$$\Rightarrow \underline{\lambda_{1,2} = \pm 1}$$

and if you calculate V, you find,

$$\text{for } \lambda = 1 \Rightarrow u_1 = v_1 \Rightarrow V_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\& \text{ for } \lambda = -1 \Rightarrow u_2 = -v_2 \Rightarrow V_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore \underline{X(t) = \frac{C_1}{\sqrt{2}} e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{C_2}{\sqrt{2}} e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

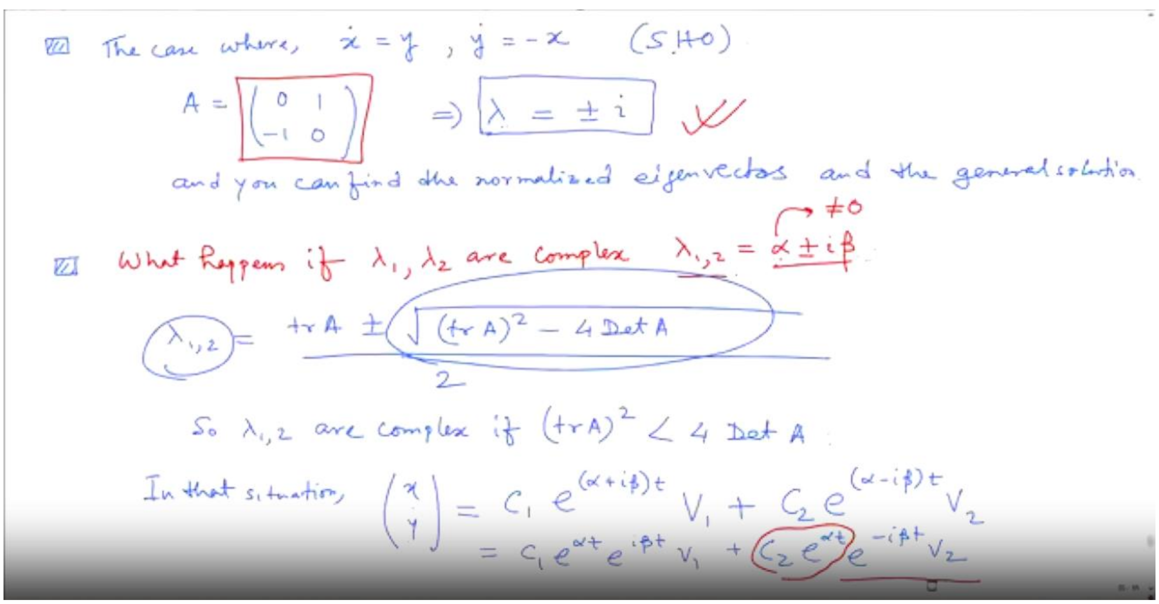
So, this is the case of discrete or distinct eigenvalues. So, you can find the normalized eigenvectors and the general solution. So, general solution you will always find a trigonometric sinus or cosinus type of function ok. what happens if lambda 1 and lambda 2 are complex and of the form of lambda 1 2 is equal to some I can simply write alpha plus minus i beta.

ok. As I said that the mean tree of the coefficient matrix they are always real. So, even if they are the eigenvalues are ah complex they should be complex conjugate ok. So, alpha plus minus i beta and alpha is not equal to 0. If alpha is equal to 0 then of course, you have this case of simple harmonic oscillator.

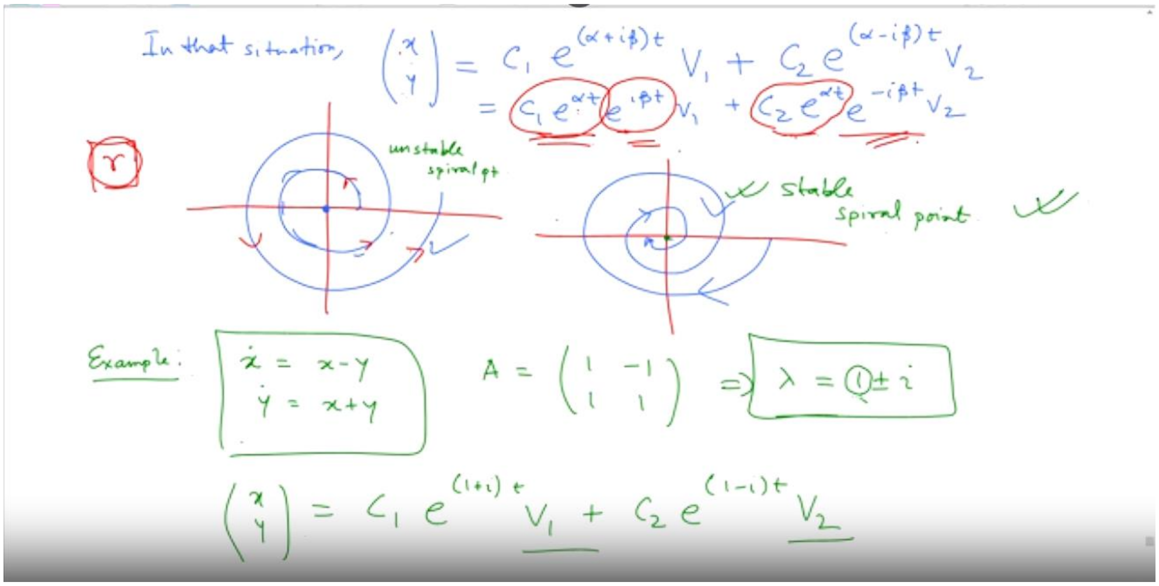
If this is not equal to 0 ok then what happens? Then of course, In this case, you just see a little bit that when this type of case arises, okay, what happens? That lambda 1, 2, we all know that this is trace A plus minus root over trace A whole square minus 4 determinant of A by 2. And this is complex only when this part is imaginary

and that means that so lambda 1 2 are complex if trace A whole square is less than 4 determinant of A ok. In that situation what we will have in that situation Of course, the general solution will still have the same form C1, then you will have e to the power alpha plus i beta times T times V1, the corresponding eigenvector plus C2 e to the power alpha minus i beta T times V2, okay? Now, you see I can do the proper analysis, but that will be cumbersome. I would just try to find the nature of the phase space trajectory here.

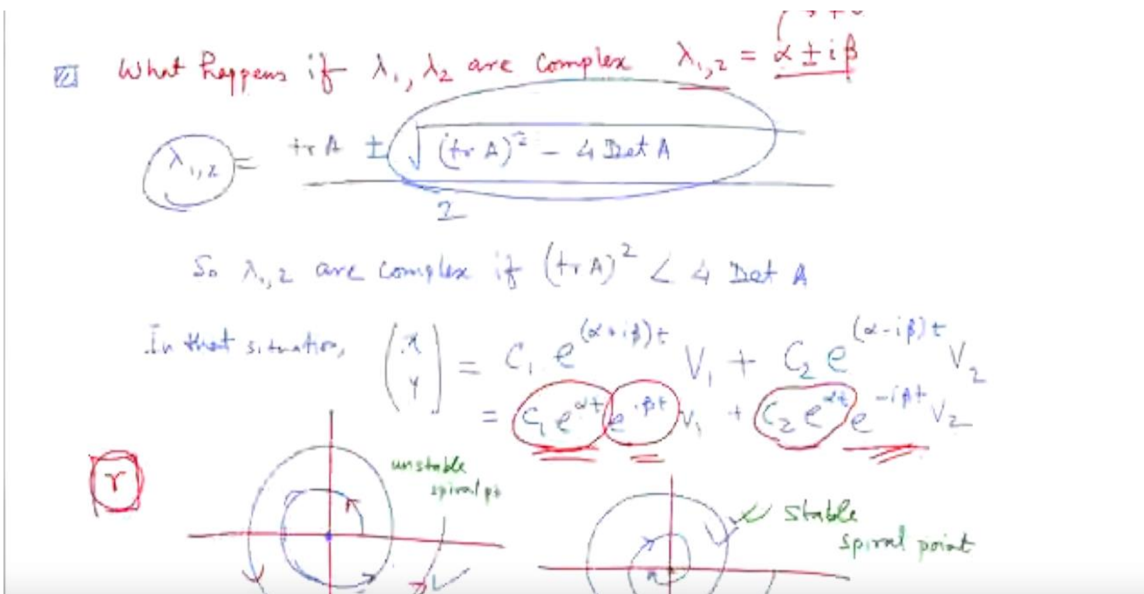
So, you will see that both X and Y they are having some part. So, you can actually write like C1 e to the power alpha t and e to the power i beta t V1 plus C2 e to the power alpha t e to the power minus i beta t V2. So, this part is simply giving you the oscillatory part okay and this part gives you okay sorry I mean this part of course, giving you the oscillatory part and this part is giving you an amplitude which is exponentially increasing with time same here if alpha is greater than 0 or exponentially decreasing with time if alpha is less than 0 ok.



So, it means that you will have and if for both x and y you have this that will also be reflected in your okay r the polar r coordinate so actually you can think that in phase space diagram you will have orbit type of phase space trajectory due to these functions but this r is no longer a constant and that is why you will have some situations where you will have structures like this or So, where this is actually growing outward ok or you have a structure



like this where this is actually decreasing and coming inward ok. In both cases we talk of a fixed point here of this system which is known as the spiral point ok. the you see that the spiral point is somehow repelling the phase space trajectory which is going outward.



So, it is an unstable spiral point and here the spiral point is somehow attracting this. So, asymptotically the phase space trajectory will come into this point. So, that is why it is called stable spiral point okay. And in the next lecture, we will actually see that the concept of this spiral point are fundamental and pivotal to understand also another concept which is known as the concept of limit cycles, okay? So, finally, here I have just given you some qualitative description.

I give you a small exercise and which you can calculate by yourself, okay? So, if I give you x dot is equal to x minus y and y dot equal to x plus y then you will have actually a is equal to 1 minus 1 1 1 and from that your lambda will be equal to 1 plus minus i using that ok. You just try to understand how it gives a spiral ok. And of course, you understand that here this 1 is positive.

So, it will always be an outgoing spiral outward spiral ok. And what will be the even if you do not do anything further what will be the general solution? The general solution will simply be equal to C 1. e to the 1 plus i T V 1 plus C 2 e to the 1 minus i T V 2. You calculate V 1 and V 2 and then you are done.

If in addition you have the initial conditions, you can also evaluate C 1 and C 2 and you have the total solution, ok. Then finally, till now, We have discussed only the case where the eigenvalues are distinct. What happens? if lambda 1 is equal to lambda 2.

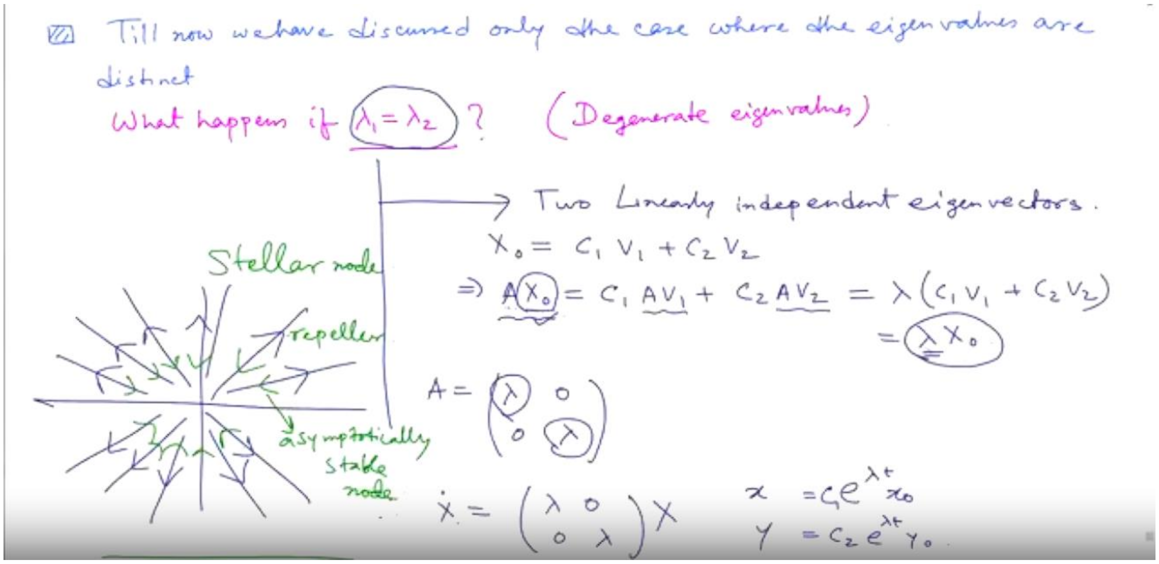
So, this is called the case of degenerate eigenvalues. If I have equal eigenvalues, then the case actually becomes tricky and actually I have then two possibilities. The first possibility is that although the two eigenvalues are similar, still the system has two linearly independent eigenvectors, okay? And if it is like this, then what happens?

Then I can actually again write the initial value of the state vector as X 0 is equal to some C 1 V 1 plus C 2 V 2. But now, if you just do A X 0, that will simply give you C1 just a constant. So, AV1 plus C2 AV2, but now both AV1 and AV2 gives you lambda times V1 and V2.

So, you will have lambda times C1 V1 plus C2 V2. So, you will have lambda times X0. So, if you simply apply this a on this x 0, you will see that it will simply be dilated by a factor lambda, ok. And that simply says that a is nothing but a diagonal matrix lambda 0 0 lambda, ok.

And x 0 is its eigenvector, ok. And if you have this, then you can actually write x dot equal to lambda 0 0 lambda, x and from that you can easily integrate and you can find that actually you will have both for x and y e to the power alpha t times some x 0 and some of course some c and here also some c 1 and c 2 e to the power alpha t sorry not alpha t. e to the power lambda t sorry yeah and here also you will have e to the power lambda t ok y 0 and in phase space diagram you will have straight lines like this. You remember and depending on the value of lambda actually it can be outward ok.

or inward okay and this is known as the case of steller node okay a repeller. So, here in this case this is a repeller star repeller or steller repeller and here this is asymptotically stable node okay. So, this we all know what happens if for degenerate degenerate eigenvalues we have only one linearly independent eigenvector.



What happens? If for this we have only one then we have something called degenerate nodes. I will not discuss about this. this is just ah vocabulary i introduce here if you are interested you can go through some internet resources or stroberts book ok so yeah but then this system a little bit ah tricky ok but you still have fixed points which are called degenerate nodes ok finally we have seen that basically depending on the nature of the eigenvalues of the coefficient matrix a we actually found different systems, different type of fixed points, okay, which actually influences the nature of the phase-phase trajectory near the fixed points and also the global dynamics in the two-dimensional phase-phase flow, right.

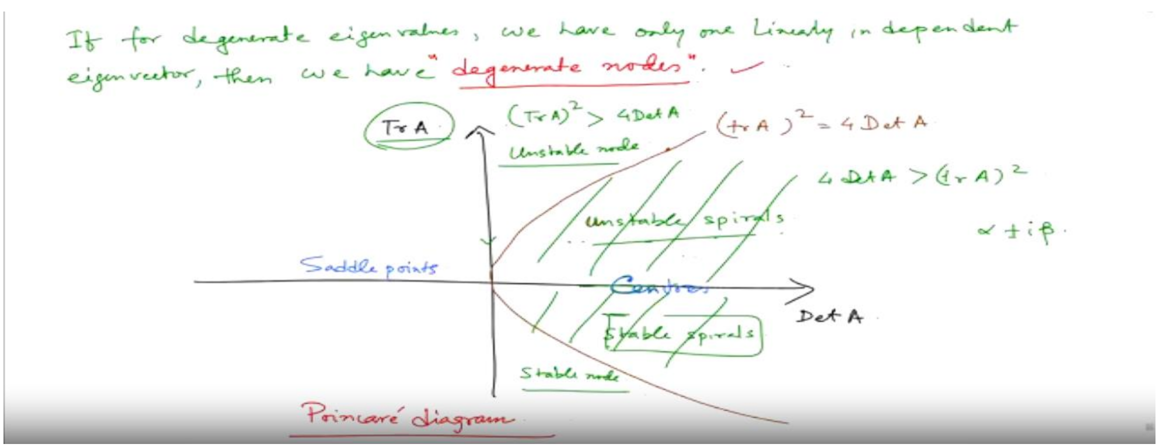
If I want to now represent this usual type of fixed points in a diagram, then the most appropriate parameters or the most appropriate intrinsic variables which can actually represent all possible fixed points are simply the trace of the matrix A and determinant of the matrix A. And if you do such a diagram, this is called a Poincaré diagram. So, a Poincaré diagram is a diagram where you can ah draw determinant of a along let us say x axis there is no hard and first rule you can do the other way round as well and the trace of a ok as y axis for example then you will actually you can actually draw a parabola on which the trace a whole square is equal to 4 determinant of a that means the discriminant of lambda 1 and lambda 2 is 0 ok and if you have this so on this line of course you will see whenever you have this discriminant so on this line you have degenerate roots ok.

But if you now think that where the other roots will be there, then I just write it down and it will be my request to all of you to check one by one that whether they are really exactly I mean put in the proper place in this diagram. So, here I am just writing. So, unstable unstable So, this is the region which is the region where 4 determinant of A is greater than trace A whole square and here the case if you just remember this case little bit for yes not here, yeah.

So, here you see that the eigenvalues are complex when 4 determinant of A is greater than trace A whole square, okay? And that is why you will have finally in this part the unstable spirals and in this part the stable spirals because Just for unstable spiral the trace A is positive that is the alpha part in alpha plus minus beta ok. If you remember and for stable spiral they are negative ok. Here I put unstable node and this is the case where you have trace A whole square is actually greater than 4 determinant of A ok.

Here you have stable node. Again, this is the distinction between the case of positive trace A and negative trace A, okay? Here on this line, you have trace A is 0. If you have trace A is 0, then the eigenvalues will be purely imaginary, okay?

And you have centers I can draw right you have centers on this line ok and here you have saddle points ok. So, you see that by this way you can also put in this picture the stellar nodes ok stellar repellers and also degenerate nodes, but degenerate nodes I have not discussed. If you are interested, you can go through it.



And this picture is known as the Poincaré diagram and gives us a unified view of all sort of possible fixed points which we can get for a linearized two-dimensional phase space flow. Thank you very much. Bye.