Ordinary Differential Equations (noc 24 ma 78)

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Lecture 40: Phase Portrait for Planar Systems

welcome student and in this video we are going to talk about face portraits okay first of all what is the face portrait let's understand that face portrait. Now you have to understand this one. Since we are working with portraits, what happens is, so essentially we are going to draw some pictures. And what happens is, and these are for some particular curves.

So you cannot go to higher dimension because we really can't draw those curves, right? So generally speaking, we are going to focus our attention to a 2x2 system, okay? So first of all, let's say you are given this system, right? x prime equals to ax is given to you. Ax is the given system.

okay and what is a given system with with a being a 2 cross 2 constant matrix right being a 2 cross 2 constant matrix okay we want to see that how are we we know that there are solutions to this right and essentially we want to see how we can you know uh so if there is any way to draw those solutions out so we can see that how what their behavior once i do we do that then we know that even for the non-linear systems also we can do similar sort of thing okay at least in a for certain type of equilibrium points and uh i mean certain systems okay that we already saw for non-linear system so let's look at this okay Now, if you remember, we talked about the canonical forms, right? So, we can reduce all these two cluster system. For two cluster system is not very difficult.

What we can do is we can reduce it into three different forms. So, one is real distinct, one is distinct and one is complex, right? First of all, let us say that you have this system and so this is case 1, right? So, we are looking for real distinct eigenvalues. Real distinct eigenvalues.

Eigenvalues. So, you have a 2 cross 2 system. And essentially, let us say you have both the eigenvalues are real and distinct. And we are assuming that, let us say, this is the case. So, what are the three cases which can happen?

You can have this one, right? Lambda 1. 0 is between lambda 1 and lambda 2. The second case can be lambda 1 less than lambda 2 less than 0. And the third case is 0 less than lambda 1 less than lambda 2.

Now, you see the first case, we want to talk about the first case for now. So, maybe I can start with the canonical forms. So, the first canonical form is C, the first form, form A. So, what do I mean by canonical form? You remember, right, that let us say you are given a constant coefficient matrix A, B, C, D, right?

Let us say 2 cross 2 constant coefficient matrix is given. We can always have a change of variable under which we can reduce this to certain forms, right? You remember? If you do not, please go back to the uh videos where we talked about all this and then come back here okay so what we can do is there are these three cases according to that the canonical forms look like this so first one for the first one

it will essentially be looking like this lambda 1 0 0 lambda 2 okay see why i am doing it because essentially all systems can be reduced to this so basically just to uh i mean studying all the say i mean it's uh you just have to study the canonical form to study all the systems essentially right So first of all, let's say this form is there. This is the most easy or I mean natural form, right? So A is lambda 1, 0, 0, lambda 2. Okay, so that you remember that diagonal form, right?

And with the first part, lambda 1 is 0 is between lambda 1 and lambda 2. Okay, now of course you do realize if you can reduce this problem like this, then what happens is you can of course solve the system, right? Why? Because if you have something like this, it means that you have reduced the problem to x1 prime equals to lambda 1 x1. And x2 prime equals to lambda 2 x2.

Is it okay? Once this is there, we can solve this. This is like unrelated two equations, right? Very easy equations to solve. You can solve it.

And what you are going to get is two different solutions. So, let me just write it down. We already talked about it. So, I am not discussing those part. Let us just look at the solution now.

So, the solution in this case will look like this, right? x t equals to alpha e to the power lambda 1 of t and then the eigenvalue so in this case you can actually show that since lambda 1 0 uh so sorry uh lambda 1 and lambda 2 are two uh this thing the eigen uh same vectors yeah so what we can do is sorry eigenvalues so what you can do is you can show that the for the canonical form the you have two the solutions the eigenvectors correspond that will look like this 1 0 and 0 1 okay beta times e to the power lambda 2 of t 0 1 is it okay yeah and once you do this now you see so that's your general solution is the general solution

is the general solution okay now again i'm telling you if you do not remember how this all of this is working please go back to the um videos where we talked about all these things yes so of course i mean this sort of problems are very easy to solve this is your general solution so i'm not explaining that hit that here and we're just writing it down Now the thing is, see what is face portrait? The main idea behind this video is this. We really want to look at, see this particular thing, let's just look at this. See Xt, if you just think about it, how Xt looks like?

Let's write down Xt. See for a fixed t, Xt is nothing but this thing, right? X1 of t and X2 of t. is it okay i mean you can just break it up like that right and this is nothing but if you write it down you see this is alpha times e power lambda 1 of t and that one is nothing but this is beta times e power lambda 2 of t is it okay and this one for all t so essentially you see you are basically uh i mean looking at a curve right which depends on t so basically this is a parametric curve yes and now what we want to do is we want to look at how are the solutions are so basically how are these curves look like on a two-dimensional plane so these are called planar so that's why it's called planar system right yes uh so how how does this curves look like that's the whole idea of it okay

right so maybe i can put it here let's let's just put it here yeah so therefore this this essentially this is the thing yes hence xt is this for alpha beta in r of course for an arbitrary alpha now let's just draw some pictures again the pictures may not be very nice looking but you will get the idea of it okay so let's draw the two-dimensional this thing plane okay Now again see before I proceed with the diagram so you have to realize one thing that this is very rough. You understand you don't have to be very precise with the diagram but of course I mean it should at least give you a very rough idea but I mean kind of correct idea about how the trajectories of the solution looks like. So, I mean, don't just draw anything you want.

Okay, just you have to give, I mean, so if you are solving a problem like this, you have to write properly that what exactly, why are the curves look like, what they look like. Okay, so you have to explain everything. So, I will do this part here. I will show you how it's done actually. Okay, so first of all, see, if alpha and beta, these are real numbers, right?

So first of all, you have to understand for this curve, there is always an equilibrium. So any constant coefficient 2 cross 2 or 3 cross 3 system doesn't matter. Any constant coefficient system like this homogeneous system, right? And there is always an equilibrium point. So if you consider, so note, there is a small note.

Okay, so if you consider f of x, you remember, let us say this is a of x in your case, right? And you do see that 0, 0 is always an equilibrium point, is always an equilibrium point, always an equilibrium point, right? We have a point. irrespective of whether A is, I mean, you know, invertible or not, doesn't matter, but 0, 0 is also an equilibrium point. And let us say, if there are like two distinct eigenvalues, in this case, it is right, that lambda 1 is less than 0, which is greater than, less than lambda 2.

Essentially, in this kind of case, you do realize that you have a, I mean, a So you may have this sort of thing that A is invertible and then what you have is 0 is only equilibrium point. So in this case you see the only equilibrium point is 0, 0. okay so equilibrium point is basically now the question is this we want to see how the trajectories behave near the equilibrium point that's the whole idea of this course right yeah so how let's let's look at that see what is happening here is this first of all alpha and beta is any real right so let's say if i am assuming alpha equals to 1 and beta equals to 0 you are basically looking at e power lambda 1 t 0 so basically you are looking at this sort of thing right x t is e power lambda 1 of t and 0.

You understand? Now, you see, please do not start drawing anything. You see, what is happening is this. This is a curve, right? So, xt and yt.

And what does this doing? It is e power lambda 1 of t and 0. And this t is varying in r. So, t is going from minus infinity to plus infinity. Now, you think about it. See, for a fixed t, let us say t equals to 1, let us say.

just a motivation you don't have to write all these things i mean this is how you start you see if lambda t equals to 1 what happens what is e power lambda 1 t is basically a real number right so with this this particular point let's say so let's say t equals to 0 let's say t equals to 0. This point is nothing but 1, 0. So, see, it is somewhere here, 1, 0. Let us say this is 1, 0.

Now, you see, if t, say, lambda 1 is negative here, you see, it is given lambda 1 is negative. So, if lambda 1 is negative, since lambda 1 is negative, you see, as t tends to infinity, what is happening is this particular point is going towards, this particular thing is going towards

0. you understand so essentially this vector approaches the equilibrium point zero zero you understand so essentially as t tends to infinity you see From minus infinity, this is the arrow which will indicate where are the trajectories going. So, basically, this is one of the trajectory, along this line only.

So, what does it mean? One solution look like this for a particular choice of alpha, beta. And you see, it goes towards 0. You understand? Going towards 0.

So, basically, as t tends to infinity, this goes towards 0. You do not have to draw this particular line. I just draw it for the purposes. This is 1, 0. okay at t equals to zero it is somewhere here and after that as t tends to infinity it is going towards here at t tends to minus infinity you see this is actually blowing up going towards minus infinity zero so basically it is coming so it is it is coming from here minus infinity let's say this is a large n zero right as t tends to minus infinity

So it is extremely large. It is going towards minus infinity. So it is coming from here. The trajectory is coming from here and it is pushing towards 0. You understand that's the idea.

So the first trajectory looks like this and really to be very frank with you, you really can't draw it properly because it is merging in the y axis. So y axis is the solution in this case. Sorry, x is a solution here. Similarly, you can also see that let us say if alpha equals to minus 1 and beta equals to 0, you have similar sort of thing, but in the other way, this way. So, this is another trajectory.

You understand? Let us draw it like this. Maybe I can use a different pen. Let me do it this way. I hope you understand what is going on.

You see? See, all of these are going towards 0, 0. It cannot cross 0, 0. Please understand this. This is one trajectory.

This is one solution, one solution curve. The other solution curve is this. Yes. yes okay right now the thing is what about see similar thing is if you are taking alpha is 0 beta 1 so you see if alpha equals to 0 and beta equals to 1 yeah what happens is you are exactly having the similar sort of situation but in the y axis right so you see you have similar sort of situation but in y axis you see this is what is happening right see here and where is it converging again same now see here lambda 2 is positive here lambda 2 is positive right so you see in this case x of t is alpha, first I am taking alpha to be 0, right? So, basically this is 0 and this one is e power lambda 2 t, okay? Lambda 2 is positive. So, as t tends to infinity, okay, this actually blows up, right? And as t tends to minus infinity, this goes to 0.

So, you understand, We will look at the trajectory as it starts at minus infinity and we will look at what happens at t equals to infinity. You understand? So you see, in this case, the trajectory, they are moving away from origin, right? So, it is starting from very close to origin as t tends to minus infinity.

From minus infinity, it is going away from the origin in the y-axis, along the y-axis, right? Because all these points are in y-axis. You do realize that, right? As t tends to infinity, since lambda 2 is positive, e power lambda 2 goes to infinity. So, basically, this is going in this direction.

Is it okay? And similarly, here also, this is going in this direction. This is going in this direction. Is it okay? You do not have to draw all these double arrows.

This is an example. I want to emphasize what is happening here. And that is why we are doing it. But generally speaking, when you do this sort of thing, you draw at least one arrow just to signify what is happening. And the arrow will actually be in the direction from minus infinity to infinity.

We are going to look at what is happening to the trajectory from minus infinity to infinity. So, arrow points this way. Is it okay? Of course, you can put double or such sort of problem, but this is just an idea. okay now the thing is this see how do you what see this is for a particular choice of alpha beta one zero zero one that sort of thing right and but alpha and beta both can be i mean non-zero right yeah in this in that case what do you do that's the question okay so first thing first see that in any other case so if let's let me put it this way if alpha beta

both are non-zero, okay? Then, all other solutions, they tends to infinity. Also, okay, by the way, I forgot to mention this. You see, this line, this line, all the trajectories are moving towards the equilibrium, right? So, we actually call that line, since they are going to go moving towards the equilibrium, we call that line as a stable line,

stable line, okay? Please remember this thing. And this line, where it is moving away from the equilibrium, we call it an unstable, unstable line. Is it okay? Now, you see, if alpha and beta is not equal to 0, you can see that all solutions, let me write down, all solutions

We are looking for what is happening to the solutions as t tends to infinity. So, tends to infinity in the direction of unstable line. Direction of unstable line. What do I mean by this? C, let us write down, xt is e power lambda 1 of t and e power lambda 2 of t. Now, let us say alpha, beta, non-zero.

Let us just look at one trajectory. We are not really interested in all the trajectories, just one trajectory. Let us just look at what is happening. If alpha and beta are 1 and 1, so x of t should look like this, right? and what is happening is this so please understand this all of these trajectories are individual solutions of this system it's okay right now you see x t equals to e power lambda 1 p and e power lambda 2 t lambda 1 is negative lambda 2 lambda 2 is positive lambda 1 is negative right right

And we want to see how does this trajectory looks like, okay? Now, as t tends to infinity, you do realize, sorry, when you start at minus infinity, very close to minus infinity, that is, okay, since lambda 1, you see, since lambda 1, this particular, lambda 1 is negative, right? So, at minus infinity, this particular coordinate blows up, right? Yes. And what about this coordinate?

This coordinate, if you just look at it, you see, at minus infinity, very close to minus infinity, that is when you are starting out, this goes to 0. You understand? Okay. So, see, the trajectories as t tends to minus infinity starts very close to, see, the y axis is 0. You understand?

yes the y axis is zero okay and when so when this t tends to infinity you see these solutions are getting very bigger and bigger but this is going towards what you see this is towards So, as t tends to infinity, what did I say? This is minus, so this is going to a 0 infinity kind of thing, right? As t tends to, you understand? So, basically it is converging towards this line, stable line.

You understand? Okay. So, what is happening is this, for a large n of t, okay, the solutions are starting to very close to, so as t tends to, see again, let me clarify again what I am trying to say here. See, as t tends to minus infinity of, so what I am saying is t is negative, extremely large.

Let us just look at this, where is this point? So, where is this point? Well, let us have a pie here. lambda 2 is negative right so it starts from very close to uh if this is negative this is negative very close to infinity right this is converges to infinity and this particular thing converges to zero lambda two is positive t going to one minus infinity okay so this starts very close to i mean minus infinity zero so here you understand that t tends to minus infinity And you see at t tends to infinity, what is happening?

t tends to infinity, this goes to 0 infinity. So basically, you do realize it will be somewhere here, you understand? So it is very, I mean, kind of, it is merging towards the unstable, this y axis. So the solutions look like this. Okay, you understand what is happening?

And see, this is minus infinity to infinity. So, the arrows will point here. I hope this is clear. See, as t is very large in the negative direction, so t tends to minus infinity, these points, if you can see, they are tending towards infinity 0, right, kind of. So, what does it mean?

They are actually very close to the x-axis, right, the x-axis. And what happens, so very close to x-axis, see. And after that, as t tends to infinity, it is becoming asymptotically, I mean, going towards the y-axis, because it is going to 0, infinity, you see. And that is why the arrow points this way, okay. So, once you are done this, you can just draw another curve, does not matter.

Now, you do not have to be very precise with it, do not worry about it, okay. It is just to give you some idea, you see. okay so another curve like this yeah again my drawing is not very nice but you do realize what's happening here okay and similarly how do you feel so once you are done this you see for uh now you can take alpha negative beta positive similar sort of thing will work so essentially you have this sort of thing okay you have this sort of thing and the arrows will point like this here also you will have similar sort of thing okay And the arrows will point this way. I hope this is clear why it's happening.

And again, here also, you can actually guess which side the arrow is. It's not guessing. I mean, you can just see symmetry, right? So, essentially, it will go this way. okay so this is for different choice of alpha beta if one let's say alpha is always negative beta is positive alpha both are negative okay so in that case you are here in this particular quadrant you understand so all these solutions are there now you just you just draw those solutions that's the thing you just draw it out and it doesn't have to look very prettier you can you can just give you if

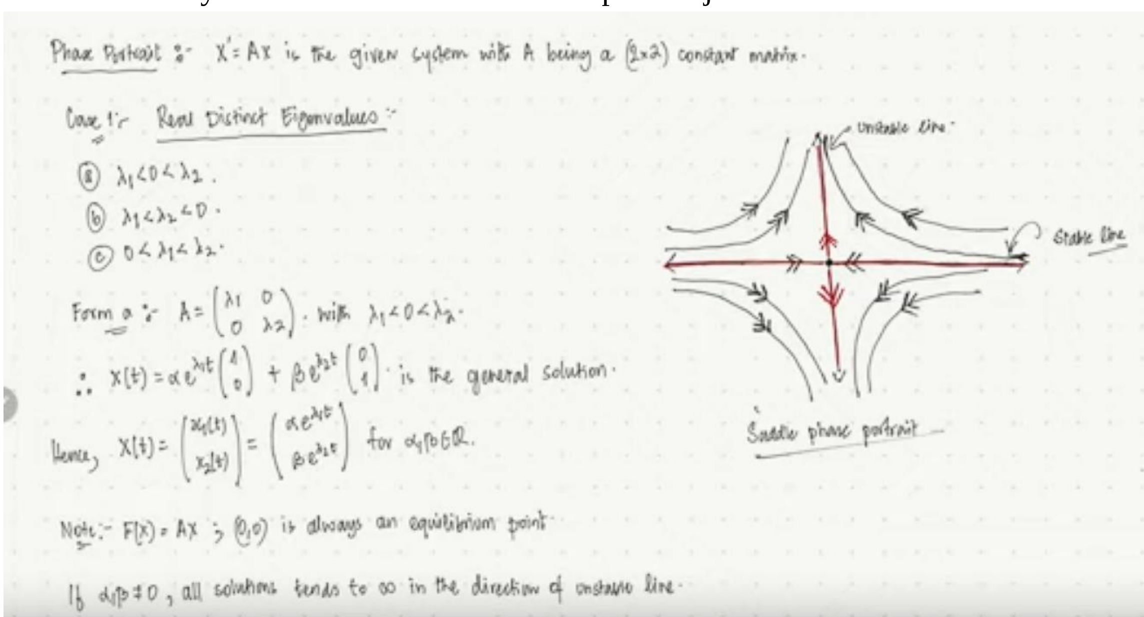
It should just give you some idea. Okay. Now the question is this. See why we are calling this sort of situation. See origin is there, right?

This is the origin and you have this situation that along this direction. along some direction, you have, you are approaching origin along the other direction, you are actually going away from the origin. This sort of situation you already know from several variables. We call such a situation as a saddle, right? So, you see, 0 is a saddle point in kind of thing, right?

Situation. So, this particular picture, the portrait which we draw is called a saddle face portrait. Is it okay? It is called a So let's write it down.

This is called a saddle face portrait. Saddle face portrait. Okay, why saddle? You have seen the horse saddle, right? So what happens is along some, you know, direction you are reaching the maxima kind of the critical point where you are sitting essentially, yes, point of contact.

And the thing is along another direction you are essentially reaching a minima kind of, right? So that's why it's called saddle, right? Now, where do we have? Yeah, so this is one situation. Now you can of course take an example and just draw it out.



But I mean that is for assignments. So I am not doing that part. So let us look at the other case. So this is the form A. There is a different form. Let us just call it form B.

okay so what happens is if both are non-zero but there are you know different signs then you have a saddle okay now we have another situation this situation we are going to look at this see you have two signs lambda 1 less than lambda 2 less than 0. so basically two eigenvalues are there both are negative Is it okay? And then your solution x of t, again I am not explaining this. You know that this will look like this, alpha times e power lambda 1 of t. Again, 1, 0 plus beta e power lambda 2 of t, 0, 1. Is it okay?

Now, we want to see how the trajectories looks like. Let us just draw some pictures. Let us just draw some pictures. So, again I am explaining you one thing. do not worry about how the pic i mean don't draw whatever you want okay what i'm trying to say is try to draw the best you can it doesn't have to be precise the idea is not have to be it should not be a precise diagram i mean it's not possible right but the thing is you can just give you i mean just by looking at the portrait you can just understand that what is the behavior of the solution so how is the flow moving you understand see think about it

In this picture, if I am saying that if I am putting a point here, you understand. Now, I want to see what is the evolution of this point. Essentially, what it is saying is this. See, this particular thing and with this initial condition. So, our initial condition is this point.

Let us call that point x naught, t naught. So, let us say it is x naught, y naught, that point. So, x of t at t equals to 0. Let us say this is your initial position of the particle. What this is saying is this is actually giving you a kind of vector field because if you can think of and what is happening is if you put a point here, think of your magnetic field, that sort of thing.

If you put a point here, that point will be carried out. So there is this flow lines going. You understand this. Think of an invisible magnetic field. They are like flow lines, right?

These are also flow lines. See, essentially what happens is if you put a trajectory, a point here, that point will be carried along this trajectory. You understand? It will be carried along this trajectory and so it will go from this side to it will go along and it will, I mean, converge towards the unstable. If you are starting with a point here, what happens is that point will be forced to go towards the origin.

If you are starting from a point here, that will be moved away from the origin. It will be thrown away from the origin. You understand? So this is kind of the evolution, the flow. We want to look at what is the flow of a particle.

If you put a particle anywhere in this thing, see all of these points are You want to see what is the behavior of the particle? How is it evolving with time? So that is what face portrait does. okay so now let's again look at this particular thing i mean you don't draw this x y and put this comma this arrows here don't please do don't do it otherwise it will get confusing just do it like here also i did it but you don't do this arrow thing don't do it okay so just put this x just write it as x and y don't put that all these arrows it will get confusing okay now again what we have is just let's say

If you have something like this, this you can write it as alpha e power lambda 1 of t and beta e power lambda 2 of t. Now, you do realize again, depending on whether alpha beta is positive or negative, of course, lambda 1 and lambda 2 are both negative. We actually have to draw our curves. So, first thing first, let us just understand this. You see, if both alpha, see, first of all, alpha is positive, beta is negative. Or let us say beta is 0.

Alpha is 1, beta is 0. Let us just understand that. Then what you have is the first coordinate is the second coordinate is essentially 0, right? And as t tends to infinity, if lambda 1 is negative, you do realize that the points will converge towards the 0, 0. So essentially, see here, since both are negative, you do realize that all solutions, all solutions

tends to 0, 0 as t tends to infinity. Note. So, I do not have to explain to you this part. You do realize that t tends to infinity. Since lambda 1 and lambda 2 are both negative, all these vectors essentially they are pushing towards the origin, the equilibrium point.

Equilibrium point is this. So essentially what it is saying is this, if you are drawing a vector face, so basically if you are drawing the face portrait, it actually says that if you can put any particle, so it will actually resemble that particular vector field where if you put any particular particle, that will get forced into, I mean that will actually, you know, it will go towards the equilibrium point. You understand? So kind of a, you understand, it's basically attraction going on. And everything is coming towards the origin.

So we call this sort of thing as a sink. You understand, we call it a sink. sink piece you do realize everything every trajectory is sinking towards the origin okay so that's why it's called a sink okay so first of all the first solution you do realize it will look like this so I'm not using a different color you do understand that this is one solution of course the solution cannot cross zero zero you understand that right so it will be very close to zero this line is our solution again similarly the other part will also be there Okay, and what about the y-axis? See, initially this particular thing, y-axis is going moving away, right?

But here, this is not the case. It has to move in, into the equilibrium point because all the solutions that go to n equals 0, 0, right? Because both are negative, see? This is happening. Okay?

Now, so these are the solutions at least for one coordinate 0, another one is 1. Right? So, that is there. Now, the thing is this. Let us say that if alpha and beta are both non-zero.

Yeah? How do you draw all these other curves? Let us just understand. So, what is general trick? So, how do you do that?

What you do is first of all, you want to see how the question is this. How do They approach the origin. You do realize that the face portrait won't look like this. See here what is happening is this.

In one direction it is moving towards the origin. In other direction it is moving away from the origin. So even the linear combinations when alpha beta non zero they looks like this. Depends on which one is dominating based on t. Which which coordinate. here what is happening is this every solution doesn't really matter where you are starting so basically it may happen that you are starting from here right and what happens is the trajectory can move like this you understand it can move like this it can go towards the origin it has to right again it can start from here and it can go like this you understand and go towards origin you understand now the question is how is what is happening so in which direction then this direction

like this something like this or something like this you understand we want to understand or it can go towards this okay yeah so or you know in a straight line also it can go like in a straight line right towards the origin so we want to uh figure out what is happening yeah how how should it how is it moving towards the origin okay so do that you see what we are going to do is we are going to write down d x, sorry, dx2 by dx1. Let us just look at it. So, basically given any point terms of the curve, we want to look at the slope. You understand?

So, what is the slope of the solution? So, that is given by dx2 by dt divided by dx1 by dt. Is it okay? This is your general usual thing.

So, I am just writing it like this. So, that is nothing but lambda 2 beta e power lambda 2 t by lambda 1 alpha e power lambda 1 of t. So, that is given by lambda 2 by lambda 1 beta by alpha, beta by alpha e power lambda 2 minus lambda 1 times t. Clear? That is what, so the slope, what it is saying is this, given any trajectory, the slope will look like that. Yes?

Now, since lambda 2 minus lambda 1 is positive, right? It is positive. See, lambda 1 is less than lambda 2, lambda 2 minus lambda 1 is positive. Since this is positive, So, all these slopes approaches plus minus infinity.

Why plus minus infinity? Because it also depends on the sign of alpha beta, right? That will also change. See, lambda, this particular thing, lambda 2 by lambda 1, this particular term, this is always going to be positive. Both are negative.

Lambda 1, lambda 2, both are negative. So, that sign is always going to be positive. Now, depending whether beta and alpha, they are positive, negative, it can have any sign, right? Because they are like, how do I put it? Alpha, beta, non-negative, sorry.

But they can be negative, positive, right? So, it can be negative, positive. So, the thing is as t tends to minus infinity, see, this is positive, lambda 2 minus lambda 1. So, as t tends to, sorry, infinity. So what is happening is e power lambda t goes to infinity and depending on the sign of beta and alpha it will actually the slopes will approach plus minus infinity.

Is it understandable? So that is what I wrote. I hope this is clear. So this is a sink. Let me write it this way.

approaches in plus minus infinity okay now the thing is this how is it approaching see again we we understand this is approaching to is approaching plus minus infinity but how is it doing that okay see these solutions if you look at this these solutions these solutions okay tends to origin tends to the origin origin tangentially to the y-axis. So, essentially what it is saying is this, it is tangential to the y-axis. So, when it moves c, As lambda 2 minus lambda 1 is positive, t is, how do I put it, t is starting from, so we are starting t. So, at minus infinity, this particular term e power lambda 2 minus lambda 1 times t at minus infinity, this is going towards 0, right.

If you think about it, so the slope is kind of like this. You see, parallel to x-axis, kind of, at minus infinity. And you see, as it tends towards infinity, t tends to infinity, the slope is parallel to this. You understand? It is parallel to y-axis.

So, you see, the curve should look like this. You understand? So, this is the idea. See, the curve should look like this. And similarly this.

And similarly this. And similarly here. Similarly here. This also, this and this. here here you see if it is coming towards origin everything is going towards origin so the arrow looks like this i hope the idea is clear here what is happening okay see this and this is okay i mean the picture is not very pretty but you do realize what's happening here okay so that's the idea

Okay. So, now you see what, so this is called a sink because every solution is basically moving towards the origin. You can put a particle anywhere and you know that the particle will end up, sorry, you know that the particle is going to end up, going to end up near the origin. Okay. As it gets to infinity.

So, that is the situation sink. Okay. And there is another thing from C, let us say. What is form C? If lambda 1 and lambda 2 are both positive, they are distinct and positive.

In that case, what is going to happen? You see, I mean this, please check this part. You have to do it yourself. Check it. You have similar sort of similar face portrait, right?

Similar face portrait. Face portrait. But let us just understand what is happening. You see the similar sort of situation. Whatever we did for the same case.

Same sort of thing works. But the thing is in that case. See lambda 1 and lambda 2 are positive. So this particular thing will be positive. Depending on beta alpha.

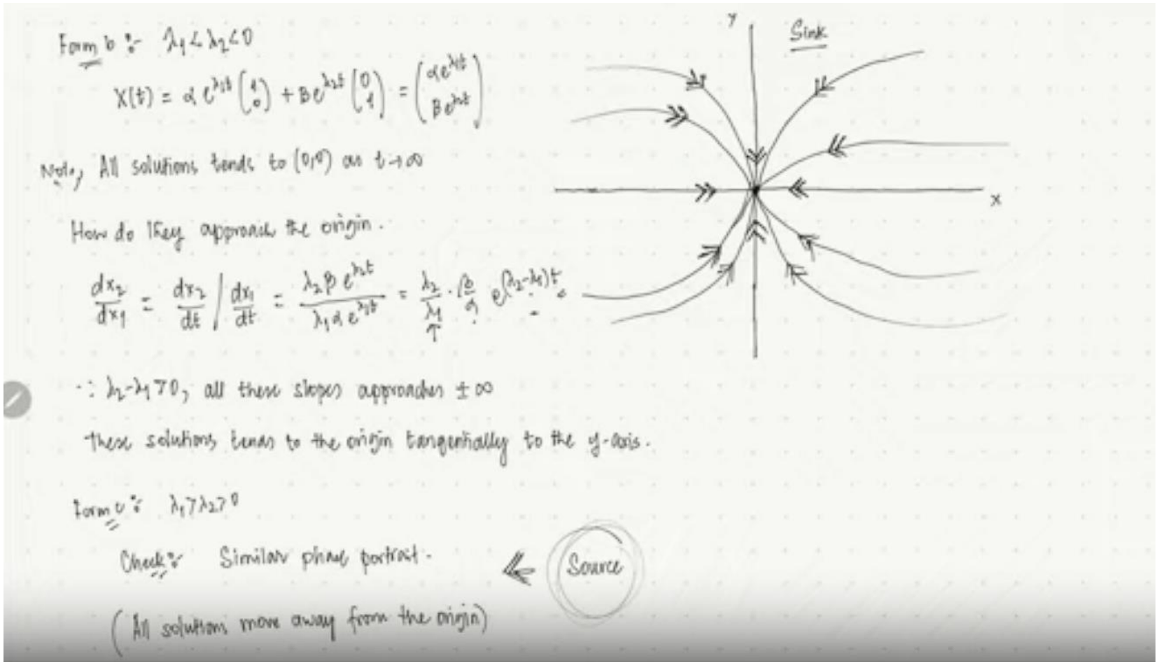
This is also similar sort of thing. Exactly same. So the face portrait. This trajectory will look similar. Nothing changes.

But the thing is this. See, as t tends to infinity, if lambda 1 and lambda 2 are both positive, then the solution is moving away from origin. You understand, all solutions moves away from origin. Here, you have to note, all solutions move away from the origin. from the origin.

I hope you understand what is happening. In the earlier sin case, everything is moving towards the origin and here every selection is moving away from the origin. Is it okay? So, if that is the case, see then origin, if you can think of everything is originating at the origin and moving away. So, basically we call such a situation as a source.

You understand? The initial situation is sink. In this case, it is source. So, what you do is this. This is part of the checking part.

So, what you do is please go to the drawing board and please draw some portraits of this particular case. See how it works. Exactly it is a similar sort of thing, but the direction of the trajectories are reversed. Right? okay so the more or less i i hope the every the idea is clear so let's say if you're given a general two cross two system and if you have two distinct real roots and what you do is you can of course change it to a canonical form that's like very easy to do right please do that and once you do it then you know that this will be either either in this form or in this form so this is one of those at least um it cannot be anything more right.



okay now let's look at the complex eigenvalue case eigenvalues okay now things are very easy see this is that's the general idea let's just draw some curves now and understand so the first case is this for me for me You have this system x prime equals to x. And your a in the first form will look like this. 0 beta and minus beta 0. If you remember. And of course beta is non-zero.

Okay, now you see the characteristic polynomial in this case. So if you want to just write down the what is the solution, let me just recall. First of all, you have to look at the characteristic polynomial and you get the eigenvalue which is plus minus i beta, right? And then corresponding to that you find the solutions. So please check this part.

Check. That you can show that the solution here x of t the complex solution complex solution will look like this e power i beta t 1 i is the complex solution is the complex solution. Now, if you remember, you see the thing we told you that here we are working with real systems. So, we are not going to look at complex solutions. Now, this is a linear system.

So, you understand that if a solution, let us say here x of t looks like this, x1 of t plus i capital X2 of t. If this is your solution, that's a linear system, you can show that both x1 of t and x2 of t are going to be your solution and they are real value. So, all these theories we already did. So, I am not going to do all that. I am just going to write down what the general solution is.

Again, if you don't remember all this, please go through it again. So, what is the general solution in this case? x of t. is c1 times x1 of t, which is, let me write it, it is cosine beta t minus sine beta t plus c2 times sine beta t and cosine beta t. Okay, that's your example.

It's okay. Now you see what I need to do is this. We need to draw some curves. We need to draw these curves. Now you see these curves are really easy to draw.

You do realize that these are nothing but circles, right? These are nothing but circles. So let's look at this. See, setting is first remark. Let me just put some remarks here before we go to the drawing part.

Remark. You see, each of this trajectory, each of these solutions or trajectory, whatever you want to call it, each of these solutions is a periodic function, right? Why periodic? Because it basically is sines and cosines, right? So, it's periodic.

And what is the period? It is 2 pi by beta, right? So it is period with period 2 pi by beta. Now you see, so if you want to draw the pictures, let us draw the curves. How does it look like?

Okay, so see this as I told you, I mean, but you also know, right? If let's say c1 is 1 and c2 is 0. Yes. So what are you going to get? You are going to get a circle.

So you are going to get some circles and please excuse my drawing, but you do realize these are circles. Okay, these are circles. Okay, circles. Now, what happens if c1 is 0, c2 is 1? Similar.

Same thing. You see, this is a circle. Again, if c1 and c2 are both non-zero, then also you can see that they are going to be a circle. I mean, nothing changes. So, essentially what is happening is this.

All these trajectories are essentially ending up becoming just circles. Now, the thing is circles is fine. So, this is the easy case, right? Just circles. Now, the thing is...

See, again, 0, 0 is the equilibrium. 0, 0 is the equilibrium. That is always there, equilibrium. And all the trajectories are just circles. So, what does it mean?

It means that if you put a point here, So, it will move radially around the origin. You understand? It will maintain the same distance from the origin. It will move around indefinitely.

As it is, it doesn't matter. So, this is moving around. Okay. So, now the point is this. But it can move in a counterclockwise direction or in a clockwise direction.

Yes. How do we get that part out? How do you know that is it moving in a clockwise direction or counterclockwise direction? Okay. See, I have a point, right?

And this point is on this vector field. And I want to see that what is happening from minus infinity to infinity. Now, I understand that they are moving in circles. But how? Whether it is counterclockwise or counterclockwise.

Now, so please check this part. This is, let us put it this way. Hash. Let us put a hash. See that the circles are transversed.

Are transversed. in a clockwise direction right in clockwise direction wise direction if beta is positive i hope this is clear i will explain to you why this is okay and counterclockwise direction clockwise direction direction if beta is negative is it okay now uh so why is it true but you can just take one example and just see huh you don't have to worry about everything because it's basically i mean after change of variables basically the same thing so it's essentially a circle now let's say that you look at this cosine beta t and sine beta t let's look at this particular thing okay Now you know how to draw this but let me just go through it. You see at pi by first of all let us just look at t equals to 0.

See the trajectory once you look at what happens at t equals to 0 you will now look at as t equals to something which is more than 0. So you understand in which direction. I mean direction is moving. At t equals to 0, this is 1, 0. So at t equals to 0, it is somewhere here, let us say.

Maybe here. This is a unit. So 1, 0. Somewhere here, t equals to 0. Now, I hope so, t equals 0 is 1, 0, right?

Yeah. And t equals to, let us say, beta is there, right? So, pi by 2 beta, okay? Beta is positive, beta is positive. So, this turns out to be cosine pi by 2 and sine pi by 2.

Cosine pi by 2 is 0 and this is 1. So, this is here, okay? Pi by 2 beta. So, this is cosine pi by 2 and sine pi by 2. What is cosine pi?

What is cosine pi by 2? Cosine pi by 2 is 0 and sine pi by 2 is 1. So it is 1. Am I doing it correctly? Let me just see.

Yeah, so as t tends to infinity, what is happening is this, as t tends to infinity, this is going from 1, 0 to, I hope everything is correct, right, whatever I am doing. As t tends to 0, sorry, as t equals to 0, this is cosine 0 is 1, 0. So, at t tends to 0, it is 1, 0. And at t equals to pi by 2 beta, beta is positive. Yes, beta is positive.

So, this becomes pi by 2, cosine pi by 2 is 0, I hope. and sine pi by 2 is 1 so this is coming from 0 to see as t tends to infinity so basically we want to see increasing t t at 0 is 1 0 and t at pi by 2 beta is 0 and 1 here so you see this is moving in this direction you understand yeah so we we can draw this particular thing so this is in counterclockwise direction okay maybe i wrote something this is mistake right one second let me just look at it ah sorry there is a minus here i missed that minus sorry I am really sorry, we missed that minus. So, it is cosine beta t minus sine beta t. So, you do realize that here it will be 0 minus 1.

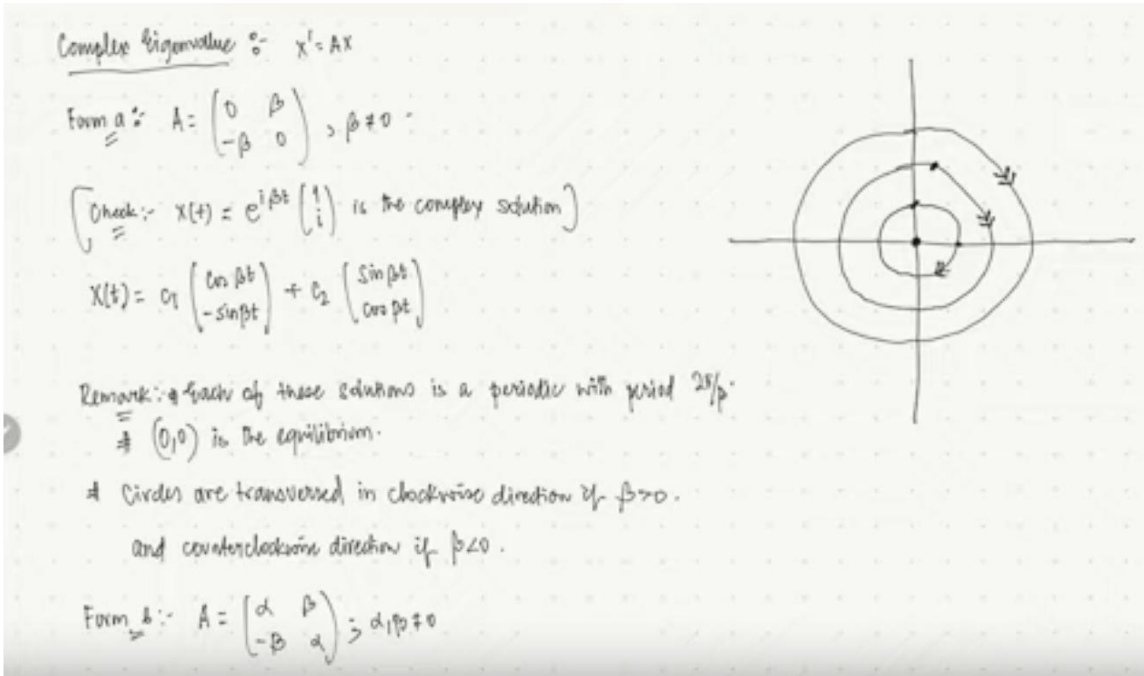
So, you see it is, so it is a circle, it starts from 1, 0. At t equals to 0, it is this. So, as t is increasing, t equals to pi by 2 beta, beta is positive. So, it goes here, right, it goes here. So, you see it is moving towards this.

That is why it is counterclockwise. Sorry about it, sorry for the confusion. So, this goes towards So, please look at the signs, otherwise you will get confused. So, this is clockwise direction, beta positive.

And similarly, all other curves also will follow the same thing. So, you just have to draw it like this. Now, you do realize that if beta is negative, you can just do this thing, but in opposite direction. Now, let us look at the second form, form B. This is a little complicated.

So, what happens is in this form, you have your A, which looks like this. So, this is, please understand, these are all canonical forms, which I am writing. This is not any form, but canonical form. If you have a problem initially, which has complex eigenvalues, you can actually have a canonical, you can reduce it towards canonical form, which will look like either A equals to 0 beta minus beta 0, or it will look like alpha beta minus beta 0. It's okay.

Z. Now, here alpha beta is not 0. Yes. Now, we want to draw the curve. So, maybe I can do it in the next page. It will be better for me.



Next page. Let's do that. Okay. So... Ksp. So, what do we have is C. In this case again, what is the general solution? No, again I am not solving it here. I am just writing the general solution. That will look like this. C1 times e power alpha t and cosine beta t minus sine beta t plus C2 times e power alpha t

sine beta t and cosine beta t. Is it okay? Now, you see, without the e power alpha t term, this is the case, right? If alpha is 0, you do not have e to the alpha t and this is the case, right? So, we have this center. This is the center.

We call it a center, okay? Because everything is like center, right? Now, in this case, what we have is this, but if alpha is not 0, Here, if alpha is not 0, then e power alpha t, what it will do is, it will turn these same solutions, yes. So, the solutions with alpha equals to 0, what is happening?

The solutions are moving in circles, you understand. Alpha not equals to 0, the e to the power alpha term, they will turn this solution into spirals, yes. So, the solutions, this, please again check this part, check this part. It is good for you. In that case, you will get a very good idea what is happening.

The solutions, they converts the solutions trajectories. Sorry, maybe I can write down the trajectories and it is better. the trajectory spiral into the origin into the origin okay So when alpha is negative, you understand that alpha negative, you see what is happening is this. Initially the points are just every point, whatever they are, they are always on the circle.

Now here what is happening is this, the points they will start somewhere here, you understand. See as t alpha let us say is negative. So, as t goes from minus infinity and towards infinity, if alpha is negative, t goes towards the 0, right? You understand what is happening? But kind of a circular because of this cosine sign thing, they are very much like in a circular fashion.

So, something like this is happening at minus infinity. see, since alpha is negative, let us say, then e power alpha t, that is minus infinity, right? So, it is coming somewhere from minus infinity. Now, how is it coming towards origin? That we will discuss later.

But for now, you understand this is coming from minus infinity. And it is, you know, springing, kind of a spring. So, it is going towards the origin like this, okay? Now, Going towards the origin, there are two cases.

See, it can go in a clockwise direction. Sorry, it can go in counterclockwise direction. It can go in clockwise direction. You understand? Towards the origin.

Go towards the origin like this. Towards clockwise or counterclockwise. Okay, first of all, again, since alpha is negative. t is negative, let us say, towards minus infinity. So, e power alpha t is positive.

So, basically, sorry, not positive, it is towards infinity. So, basically, it starts from infinity somewhere. And what is happening is this, as t tends to infinity, alpha negative, so e power alpha t is going towards 0. So, you understand all these trajectories are moving towards 0. Yeah, but the thing is they can't move like this.

You understand they can't move like this. They can't move like I mean in a straight line. Why? Because this this particular terms are there. Alpha cosine sine thing that's periodic terms are there.

Now what these terms are doing is this essentially they are actually pushing it back towards kind of it into these particular terms. see of course you can just draw it out and write it but the thing is initially speaking what cosine and sine is doing is you know it is actually uh i mean forcing the solutions to stay very circular as circular as possible right depending on alpha but the thing is so but alpha is forcing it to you know move towards or away from 0.

So that actually if you want to draw it, please do that part. Check that part. You can see that there are basically trajectories which are moving towards 0 if alpha is negative.

Now the thing is again as it is moving, it can be counterclockwise or clockwise. We are just trying to figure that part out. So what are the things? So it is spiral into the origin when alpha is negative and away from origin. from origin as alpha is positive i hope the idea is quite clear to you right okay so let's draw the pictures here now again as i told you you don't have to be very precise about it but i mean at least try to give a general idea of how the solutions may look like okay so see the thing is

this is this sort of situation this this particular situation we will call it a spiral you see everything is coming towards the origin so it is sink this is called a spiral sink is it okay and this particular situation this is called a spiral source source why because everything is moving away from it is basically a source the equilibrium is uh here also the origin is the equilibrium right so it's basically a spiral source means that it's moving away from the origin okay source it's basically a source okay so uh how does this drawing looks like let's destroy it so essentially uh again At minus infinity, it is infinity. So, it is just moving like this. Again, my drawing is not very good, but you do realize what is happening.

So, it is coming like this and moving towards the origin. Moving towards the origin. I hope this is fine. And it is moving towards the origin. See, towards the origin.

Like this. okay so here in this case i draw it in i drew it in a counterclockwise direction okay so see Why is it counterclockwise? It doesn't have, sorry, I drew it in a clockwise direction. It is moving towards the origin, going, converging to origin but in a clockwise direction, okay.

This is the drawing which I did, doesn't have to be, it can be in a counterclockwise direction also, okay. Now the question is this, how do you actually gather in which direction it is moving, okay. So what you do is this, let's say you take a pointer, okay. on this uh i mean how do i put it plane okay let's just call one zero point one super okay see if you are putting it this xt alpha beta this depends on on what alpha beta is right see if you have a point here The point will move depending on the tangential direction, right?

And that is given by the equation. See, x prime equals to ax. So, it means x prime of t equals to a of x of t. You understand? See, what is happening is this. Let us say t equals to 0.

yeah if you just look at t equals to 0 here so at t equals to 0 what is happening is e power alpha t is 1 and you can just write it down what that point is okay see what is happening is this let's say that the point the trajectory you know this thing sorry that matrix is this one And let's just say beta is again 1 and alpha is 1, beta is 1. So this is one form. This is the matrix for which that is negative. Let me give you an example of how to calculate these particular things.

So that's your way. Now the thing is you see. I want to see that what this matrix does to the point 1, 0. You understand? 1, 0. See, let us say that I start my particle at the point 1, 0.

So, basically x0 equals to 1, 0. You understand? Now, in that case, I can see what is x prime of 0. It is nothing but A x of 0, which is A times 1, 0. In this case, what is A times 1, 0?

It is 1, 0. And this minus 1. Is it okay? So, in which direction is it? I mean for this particular day, this is actually looking at 1 minus 1 direction.

You understand? 1 minus 1 direction. The vector is pointing towards that whatever direction that is. Now, so you do realize that. Depending on that creation, you can actually see that whether this is, I mean, clockwise or counterclockwise at this direction.

Let's say it is, I mean, for example, let's say here you see, since alpha beta, I did not write what alpha beta is. I mean, not a particular example. So I really can't draw the direction here. What you do really is in this case, what is happening is at the point 1, 0, the direction it is showing is 1 minus 1. Okay.

So basically this sort of direction. So you see if a trajectory starts from here it has to move like this. You understand? So depending upon the direction it will actually show which way it will move as t increases. So that will actually give you whether it moves in a clockwise direction or a counter clockwise direction.

I hope this is clear. Put a point on the face plane and see what is the tangential direction. That will actually show you that in which direction the tangent is. So, essentially that will give you the idea of in which direction the curves is moving. Yes.

Okay. So, I hope the complex case is fine with you. Now, we have the difficult case. What is the difficult case? The difficult case is the repeated eigenvalues case.

Okay. So, repeated eigenvalue. What is the repeated eigenvalue? So, basically in this case, let us say your matrix A looks like this, lambda 0, 0 lambda and lambda of course not 0. Now, the thing is we want to see how the trajectory looks like.

So, now I really do not have to explain to you how all of this works like the general solution because we have already talked about it. I am just writing down what is the general solution. So, in this case, the general solution will look like this. x of t is e power lambda t 1 0 0. plus, sorry, c1 times plus c2 times e power lambda t, if you remember, t1.

So, this is a vector 1, 0, but this vector is t1. This is a little different. That is the general solution. Yes. Again, if you do not remember, please go through it again.

Very, very important that you know how these things work out. Plus C2 times T e power lambda T. So, if you write it down, it looks like this. And then it is C2 times e power lambda T. Okay. This is the thing. Now, you see what is happening is this.

If I want to draw the, I mean the face portrait for this particular thing. Let us just draw that. I am not going to draw everything here. I am just going to give you an idea. See, if C...

2 is 0. If c2 is 0, and let us say lambda is negative or lambda is positive, it does not matter, lambda is negative, let us say. It can be negative also, right? If C2 is 0, what is happening is this particular term and this term is not there. So you actually end up with C1 e power lambda t and 0.

You understand? So essentially depending on C1, you can actually draw the peaks. And since lambda is negative I am assuming, so you understand that that will actually push the trajectories towards the early. For lambda negative, just drawing one part. I hope this is clear.

Let me just write it down. Otherwise, you may get confused. So, note for lambda negative, every term of the solution tends to 0, right? Each term of the solution tends to 0. tends to 0, 0, okay?

Tends to 0, 0, yes? And of course, it is, see, c1 e to the power lambda t, c2 e to the power lambda t, it is quite clear that it is going to a 0, 0, yes? Lambda is negative. As t tends to infinity, it goes to 0, 0. But what about t e power lambda t?

See, this is t e power lambda t also goes to 0. You have to understand that. Why? Because You see, lambda is negative.

t tends to infinity. e power lambda t is the dominating term, right? Dominates t. So, this also is true as t tends to infinity. How do you prove it? Use L'Hopital.

L'Hopital. Okay? Please use L'Hopital to show this. So, essentially what is happening is we can show that for lambda negative, every solution is tending towards 0, 0. Okay?

And of course, if lambda positive... Similar sort of thing is happening but it is moving away from 00. So check for lambda negative. Solutions move away from 00. from 0 0 is it okay now the thing is this uh we want to see that uh what happens to the case when c1 and c2 are non-zero okay so in this part i want you to check yourself up please check this part check if c1 and c2 are both non-negative sorry not non-zero then the solution stands the solutions

solution, tends towards, tends towards or away from origin, or away from origin. Why tends towards and away from origin? See, C1, C2 is both non-zero, let us say. But lambda can be positive, negative, right? If lambda is negative, of course, they tend towards 0, 0.

If lambda is positive, sorry, if lambda is negative, they tend towards 0, 0. If lambda is positive they move away right so that is always there but the thing is since c1 and c2 are both non-zero how the solution how are they moving away or going towards the origin say they do it in the direction tangent to the vector eigenvector 1 0 origin inner direction inner direction inner direction tangent to tangent to the eigenvector 1 0 vector 1 0 okay how am i getting this thing you remember we did this analysis initially in the first case you see we did this analysis right the the slope analysis okay you also do this slope analysis here yeah please do this part you do this slope analysis and show that the solution basically what happens is uh they actually uh the the direction yeah they are tangent to the eigenvector 1, 0.

So, basically tangent to this particular the x-axis. So, what is happening is this. See, the solution will look like this in this case. Sorry, my drawing is really bad, but the Okay.

And similarly here also it will look like this. Yes. I mean you see moving in moving away. So let me just do it for lambda negative. So this is for lambda negative.

Yes. Okay. This is all moving in towards the origin. Okay. Moving in towards the origin.

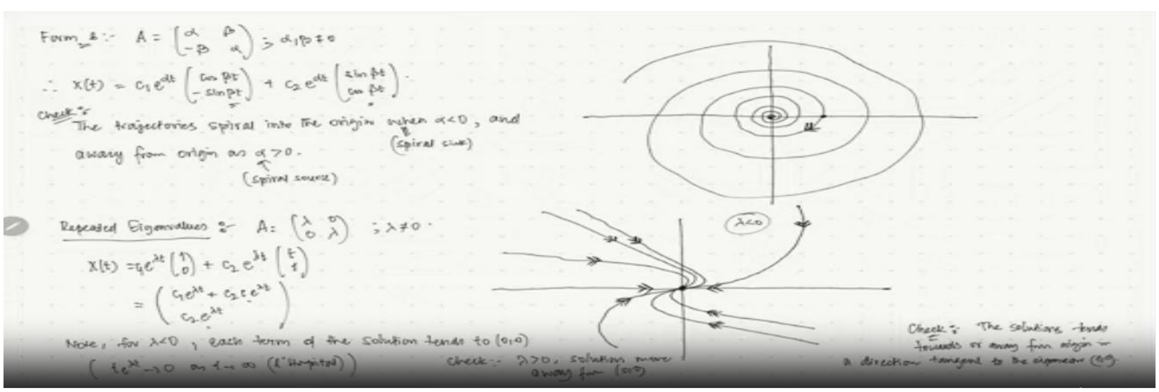
Yes. See, as you can see, the direction, so basically how is it tending? The tangential direction is in the direction of 1, 0. You understand? So you see, it is tangential towards 1, 0.

So it looks like this. That's the idea. And of course if you want to just look at what some initial some initial points what they are doing you just put t equals to let's say 0 and let's put some t positive some number and see how the trajectory moves if you really can't determine what exactly should the trajectory be. So, you do that. Now, this is general idea, but you do realize that the whole curve will be, the whole plane will be, I mean, filled with this sort of trajectory.

So, basically, let us say, if you are starting somewhere here, how do you move? So, essentially, you move tangent to the, you see, you move tangent to the eigenvalue 1, 0. So, basically, you go like this and after that, You come here. You understand?

And similarly here also if you start somewhere here, you end up doing this thing. i hope this is clear okay so please check this part it is not very difficult and of course you do not go on drawing the whole lot of thing just draw two three curves in either side of the plane and you are more or less done just to give an idea okay so what happens is here you see since uh for if you have a non If you have a matrix which is invertible, 0, 0 is the only equilibrium point and then you just draw everything around 0, 0. You understand? We just want to look at what is happening around 0, 0.

Now let us say if you have a nonlinear system or a very complicated linear system, what happens in that case is there may be different kind of equilibrium points. And in that cases, we talked about this stable manifold theorem, you remember? In that theorem, we talked about how trajectories behave near the equilibrium point for certain matrices, right? For certain particular matrices. And the thing is, so they are basically linear systems, right?



So once you know how linear system trajectory works, you can actually draw a complete idea of how the trajectory works in a nonlinear system. So around... the trajectory points how are the vector fields working is it okay you understand okay so that's the more or less the i mean content of this and please go through some different different examples and try to figure it out yourself i think this will be i mean it is not a very difficult thing to do you can of course do this thing okay so with this i am going to end this video