Ordinary Differential Equations (noc 24 ma 78)

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Week-05

Lecture 38

so welcome students and in this video we are going to talk about the direct method of the open up to show stability of a system okay so first of all we are considering this system the nonlinear system consider the nonlinear system What is the system? x prime equals to f of x. Is it okay? Now, the thing is this. See, we want to talk about the stability of this particular system.

Yes. So, let us just suppose that for now, suppose b star is the equilibrium point of this system. So, what do I mean by this? I mean equilibrium point. of this system with this system if you remember we talked about this in the last video also what is an equilibrium point so basically f of b star is zero yes okay so that is there so we are supposing that b star is an equilibrium point of this system of course b star is in rn right so where where we are assuming f is from omega subset of rn

Let's say some, I don't know, maybe RT. Yes. Okay. Let's just suppose that. So basically B star is in omega.

We are assuming it like that. Okay. Now the thing is this. See, the question is we want to analyze the stability of the equilibrium. Okay.

So the question is this. Question is, is the equilibrium point stable? equilibrium point so in this case we start stable okay now So see, when I say equilibrium point, you do realize that let us say xt, if I am assuming it to be b star, okay, for all t, yeah, then what happens is, so basically the constant vector xt, yeah, for all t. Then what happens is, what is x prime of t? This is 0, yes, because this is constant and that is nothing but f at the point b star.

okay so you do realize that b star actually is a solution of this problem if b star is an equilibrium point that actually turns out to be a solution of the problem also okay so essentially the question is this we want to see that whether it's a stable equilibrium or not b star b star that is here okay so here uh we looked at uh in the earlier uh you know videos we have talked about stability under certain conditions we know that there exists stability okay liubanov has a different idea and this is called liubanov's direct method okay there are some you know advantages of using this method to show stability so what are the advantages so first of all the method which i am going to show okay so what are the advantages So first of all, you see, this particular method, okay, it is, I mean, you know, if you want to, so it is very useful, very useful if one wants to derive global solutions global uh sorry results okay okay Okay.

And also, you see, in the earlier cases, you have seen that it depends on the, I mean, the asymptotic stability of some equilibrium that depends on what is the eigenvalue. So, basically, let's say if eigenvalue is 0 and all, we really can't say anything. Right. So, here also, the thing is the asymptotic stability, the asymptotic stability, let me write it like this, asymptotic stability. Okay.

Of an equilibrium of equilibrium okay uh in which one or more okay so maybe i can write it or more eigenvalues eigenvalues of the jacobian of the jacobian have zero real part have zero real part so the thing is just see in the earlier this thing video we have seen that if you have i mean if the thing is if you look at the jacobian matrix and one eigen value or more is has a zero real part then we really can't say anything about the stability of the system right okay and the stable manifold theorem also we know that it does not hold so what we can do here is this see in this sort of situation also if you can find so essentially here the whole point is this I will define something called a Lyapunov function if you can find a Lyapunov function then I mean it's very easy okay like it's very easy you can actually show that it predicts stability okay so now the thing is i'm going to define something called a leopold function okay right so first of all you see we are going to so definition definition okay okay so first of all let l i'm defining a function let l okay uh from omega

okay maybe i can do it in a different way okay well let me do it this way omega 2 r be a real value function be a real valued function omega 1 valued function which is defined defined on a open neighborhood neighborhood of b star okay so what is b star b star is the equilibrium point you can remember b star is the equilibrium that is what we are assuming okay and uh we this this neighborhood we are calling it as an omega which is containing omega 1 which is containing omega so you understand what i'm trying to say see Of course, b star as I told you, b star is contained in omega, right? Because otherwise f of b star is not defined, right? Okay.

Now what I am doing is this, since omega is open, that is assumed. So we will assume that omega is open. So since omega is open, what happens is around any point you can actually talk about the open neighborhood, right? So we talk about a neighborhood. Let's say omega 1 is a neighborhood.

Okay. And neighborhood of what? Which point? B star. Yes.

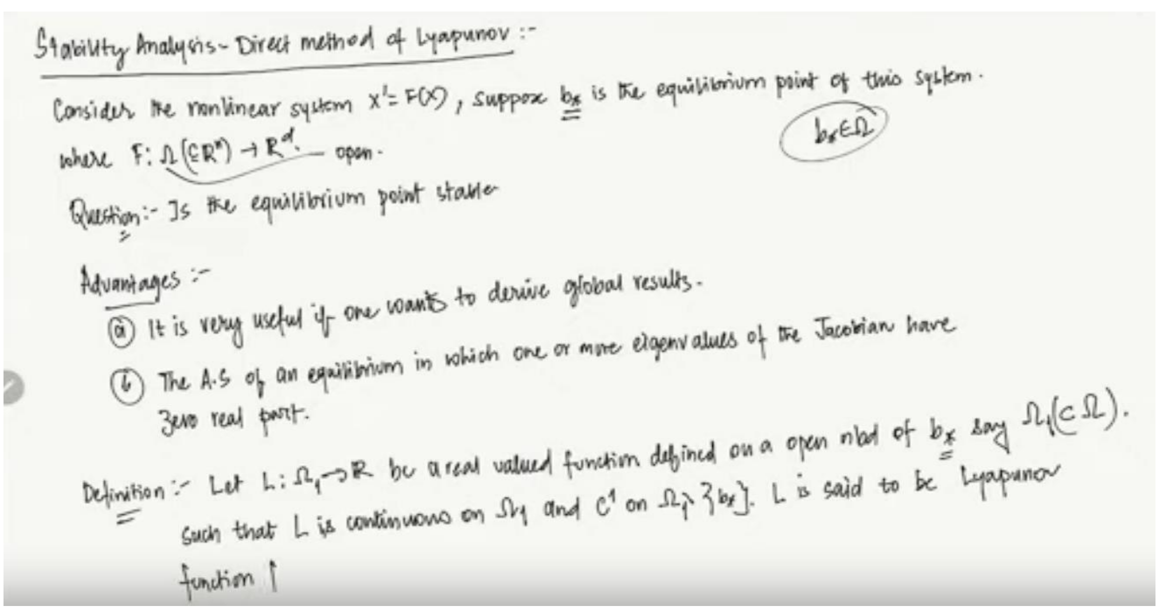
And we are defining the function L, which is defined on that neighborhood. Okay. Omega 1. And it's basically a real value function. Is it okay?

And what are the properties? So we want, let there be a defined on an open neighborhood of B star, say omega, such that, such that, L is continuous on omega 1 and it is C1 on omega minus 0. Is it okay? So, basically, I want continuity, sorry, not 0, be stuck.

So, I want continuity everywhere and then I want just the continuous differentiable property. everywhere in omega 1 minus the b star point okay now l so i am starting out with this such a real value function which is continuous in some neighborhood of b star and this c1 in some you know the How do I put it? The neighbor root minus the b star. So, in that neighbor root minus b star, everywhere it is c1.

So, L is said to be Lyapunov function. It's said to be Lyapunov function. Lyapunov function, if it satisfies some properties. If, okay, right. It satisfies if, okay.

Okay, Lyapunov function, maybe I should write it like this. Lyapunov function for the system, let's just call this system 1, for the system 1 near the equilibrium, near the equilibrium b star. Okay, if The following holds. Let me write down the following thing.



The first point is this. For all x in omega minus b star, we want gradient of Lx acting at f of x. This should be less than equal 0. Is it okay? And I will explain what all of this means. Let me first write it down.

And the second part is this. For all x in omega 1. L of x is dominated by. L of x dominates L of b star. With equality.

Equality. Only at x equals to b star. Is it okay? So, essentially what I am looking for is this. This property, the b property, it basically says that b star has to be a strictly minima of l. Is it okay?

So, see the thing is this. What we are trying to do is this. We know that there is an equilibrium point, right? Let us say b star is the equilibrium point. Yes.

Now, so let us say b star is the equilibrium point. okay and this is in some omega yeah so first of all we identify a neighborhood omega 1 which is containing omega right now we have a p star now in this neighborhood i am defining a function v yeah i am defining a function v sorry l this function we are calling it as a leucon function and this is defined from this omega 1 to r yeah What are the properties? It is continuous everywhere here and it is c1 everywhere except at the point b star. Now the thing is this, you see, we will call such a function as a Lyapunov function when first of all, you see a property is there.

I will explain what a is, but let us look at b. b basically says that The B star at the point B star, the value of L should be least. You understand? So B star has to be a strict minima of L over omega 1. So let's just put it this way.

What are the meaning of these particular two properties? The B implies that B star, the condition B implies that B star, the point B star is a strict minima. is the is the strict minimum not a is the strict minimum is the strict uh minimum minimum of l okay uh over omega one is it okay yeah so that's your b now the thing is this is the this so this is very important strict minima we will show you i will show you that why it is important we cannot I mean omit this strict part. And what does A imply?

What does A imply? Let us look at this. See, it says that the gradient of Lx acting at f, the dot product of that has to be less than equal 0. What does it mean? So, let us look at this.

See, d dt, let us look at d dt of L of xt. Is it okay? See, xt is somewhere in omega 1, right? Okay, let us say xt is in omega 1. So, I am defining L of xt and I am taking the derivative.

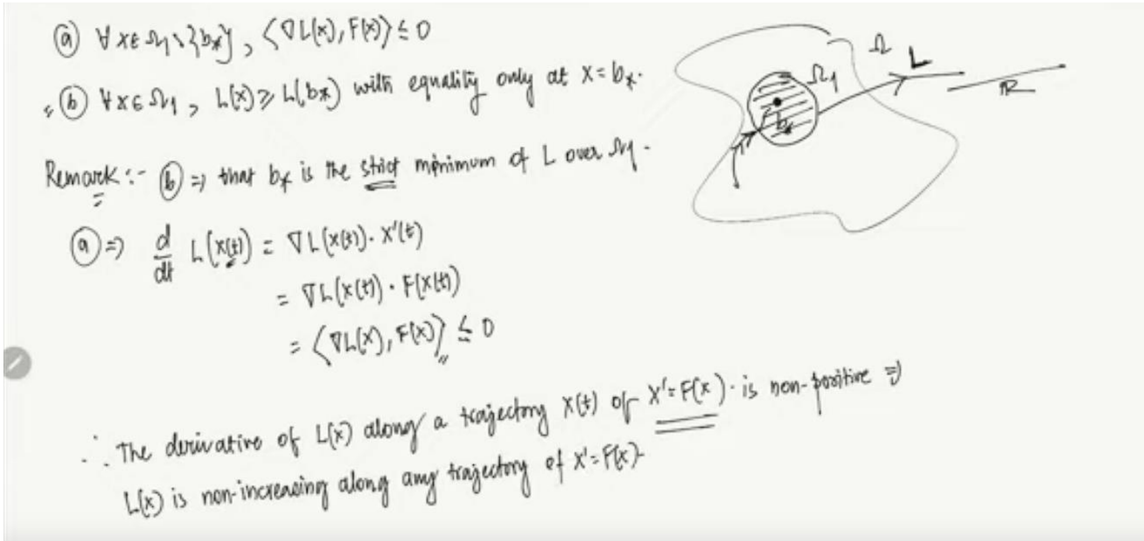
Let us just do that. See, L is c1, right? L is c1. So, basically, if I take the derivative by chain rule, that will be gradient L acting at x of t and then by chain rule, the derivative of this thing, which is x prime of t. Is it okay?

Now, what is gradient L at the point x of t? This is nothing. This is fine. And what is x prime t? x prime t, you see, x t is a solution.

So, of this problem. So, x t is f of x t, right? x prime equals to f of x t. So, it is f of x of t. Is it okay? So, this is nothing but this is what we are writing as gradient of Lx. acting at f of x. Now, you see this d dt of l x t is essentially this and we showed, we are assuming that this is less than equal to 0.

It means that the derivative of l is decreasing along this trajectory. So, basically it means that the derivative of l is x along the trajectory you see it is saying that along this x of t is not for every point if you go see let's say there is a x of t okay which converges towards b star right so if you move along that you know root around that curve okay towards b star then and what does l do so l is taking minima at b star okay and it is decreasing around that along the path okay so along a trajectory trajectory xt of x prime equals to f of x i hope i can make you understand what's going on here so essentially you see d star is a equilibrium point right so let's say you have a curve okay which solves so basically which is the solution of x prime equals to f of x right and the thing is we want to we are constructing a function l such that at the point b star l is minima and while it is approaching b star the curve approaching b star l along that curve is actually non-increasing is it okay that's what it is saying yeah so it is uh derivative along is

is non-negative essentially, non-positive, sorry, non-positive, which actually implies, which implies that Lx is non-increasing along any trajectory. So, which means L of x is non-increasing, non-increasing along any any trajectory of x prime equals to f of x i hope this is clear to you now okay fine so the thing is this what we did is we defined a particular kind of function which has some properties okay now the question is this what is so special about this why what can we do with this so this is the theorem which will actually guarantee that once if you have a system with a Lyapunov function, it is going to be Lyapunov stable. So basically, this equilibrium is going to be stable.



Stable in that sense, the last video we talked about stability, exactly in that sense. So if x prime equals to f of x admits a Lyapunov function, Lyapunov function. Okay. L of x. Near b star.

Is it okay? Then. The equilibrium b star. In the equilibrium. Vm.

Is Lyapunov stable. Lyapunov stable. I hope this is fine. Yes. Okay.

So let's look at the proof of this. Okay. See. So first of all, so, okay. Again, what thing is very, this is very easy.

Okay. What it is saying is this. If you want to show that the equilibrium is Lyapunov stable, all you need to do is you just have to find out a Lyapunov function. That's what it is saying. Okay.

Very, very simple statement. Okay. But very powerful. So, how do I prove it? So, let us start with let V be a neighborhood of B star.

V be a neighborhood of B star. B star is our equilibrium. Now, we are just assuming that it is given. Now, you see, you choose a radius delta small. such that the ball with center at b star and radius delta, the closure of that ball, okay, is containing omega 1 intersection v. Is it okay?

I can of course do that, right? It's very easy. Okay, now you see the thing is why I am interested in this ball because what I can do is since this is compact, I can talk about the minima of this L of x over this x minus b star. equals to delta over the boundary.

You see this ball, there is an interior and there is a boundary, right? Boundary is a compact set. Of course, the whole ball is a compact set, but the thing is the boundary is also a compact set. What we are doing is basically we are taking the minimum of L over the boundary, right? On the boundary, let's just call that L, alpha.

Why does it exist? This exists because L is assumed to be continuous, right? So continuous function on a compact set assumes a minimum. So that's your alpha. Right.

Now, once again, I hope this is fine. And of course, alpha will be greater than l of b star. I hope this is clear. Why? Because b star is the minimum of l on that omega 1, right?

Omega 1. And since this ball is on omega 1, okay, so alpha has to be greater than l of b star because l b star is the minimum. Okay, now let, I will define a new function, set, sorry, v1. This is a set of all those x in b, the ball, the ball which you are talking about, such that L of x is strictly less than alpha.

Let us just look at that set. Is it okay? V1 is a set of all those x such that the ball with center b star and radius delta such that Lx less than alpha. This is our V1. Is it okay?

Now, if b is in V1, if b is in V1, Let I will define a new function x of t which is phi of tb be the solution of x prime equals to f of x. Yeah, we just assume. Now, since L is decreasing, now you see. So, what I am doing is this.

I am just starting with the trajectory. Trajectory is given by x of t. Here, I am writing it as a phi. It does not matter. But basically, it is accepting. Okay.

So, since Lx is decreasing, is decreasing along the trajectory, decreasing along along any trajectory right that's what is given along trajectories trajectories of one okay what do we have l of x of t gets dominated by l of x of 0 which is dominated by alpha what is l of x of 0 see l is decreasing along the trajectories of one non-increasing essentially not decreasing it should be non-increasing okay so you see at the point x 0 it is b star right see uh this this thing oh sorry i have to write x it equals to b star essentially so at the point x 0 it is b star okay l Sorry, what am I doing? One second.

See, at the point x0, see, it starts from 0, right? And t is increasing, t is increasing. So, at the point 0, this is x0 and let us say this is x of t, okay? So, along the trajectory, as the particle moves along this trajectory, L is decreasing. So, at this x0 point, the value of L should be, you know, less than, greater than the value of L at xt, right?

Yeah. So, that is what I wrote. So, L of xt is less than equal to L of x0. Now, what is L of x0? L of x0 is nothing but this.

b star right okay so x0 x0 is the so this now i should write it you see this along this trajectory i have to write it here i forgot to write it see the thing is trajectory the equilibrium point is b star right so x at the point zero has to be b star that's the starting point okay so and l at the point b star is dominated by alpha so l at the point x2 is dominated by alpha yeah is it okay right Now, you see, the thing is, this actually is a problem. You see, thing is, if this is true, alpha is the minimum of Lx on this compact set. Again, L of xt is strictly less than alpha. Is it okay?

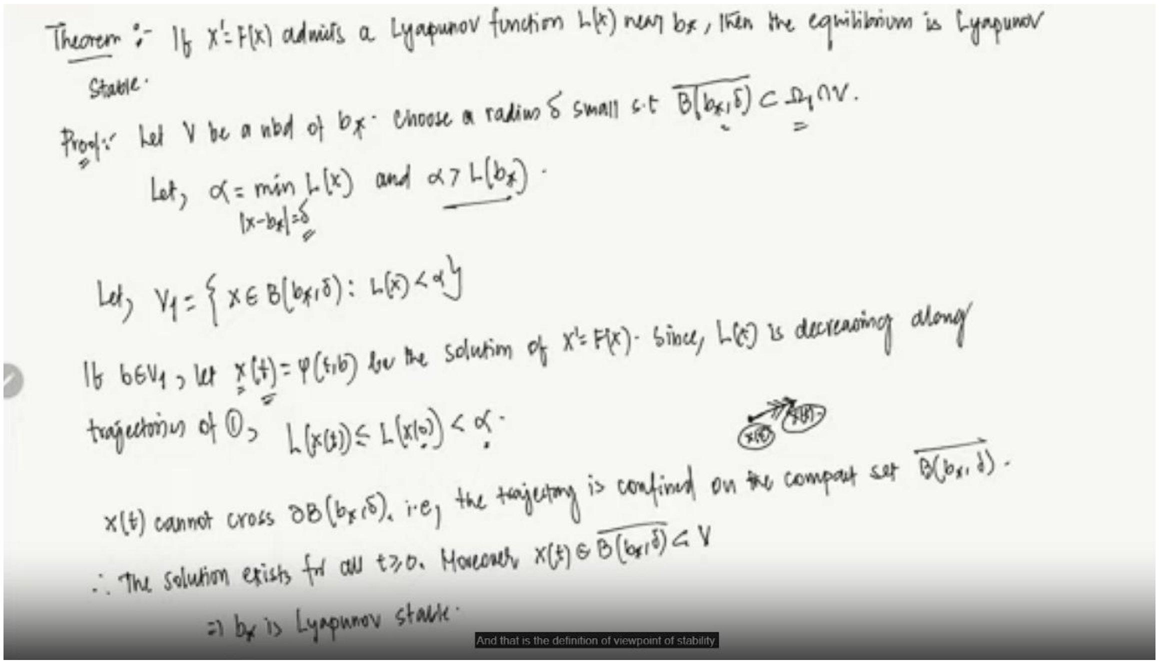
So, the thing is this, x of t cannot cross the boundary of the ball with center b star at radius delta that is clear right see l of xt is always strictly less than alpha right and alpha is the minimum of l of x so if it crosses somewhere if it touches that wall if it touches this wall in that at that point it has the value of l has to be greater than equal alpha because the minimum is alpha okay but here we have the xt for any trajectory xt l of xt strictly less than alpha so basically what i'm saying what we actually showed is this the trajectory gets trapped inside this ball right that is trajectory so if I am starting with the trajectory you see this is some trajectory okay the trajectory is confined is confined confined on the compact set on the compact set what is the compact set B sorry the ball with center b star and radius delta, the closure of that, okay?

But you remember we talked about this, I mean, property long time back, right? While talking about the, you know, the maximum interval of existence, right? That if you have a trajectory which is confined on a compact set, okay? Then what happens? The solution, therefore, the solution,

the solution exists for all time t greater than zero okay moreover we can say that x t is containing the ball with radius b star sorry center b star and delta radius part which is containing v That's what we assume. Right. And then this actually shows that B star is Lyapunov formula. So this implies that B star is Lyapunov formula.

Okay, I hope this is clear to you. You see, you remember the theorem we did, right? We proved that if there is a trajectory and I told you this is a way at that time also I told you it is a very very important theorem which we are going to use later also. So the thing is if you can actually bound a trajectory in compact set then what happens is the solution that will actually imply that solution has to exist for all time t. Is it okay?

So, since it exists for all time t, so x of t is in that compact set which is again containing v, right? Okay. And that is the definition of viewpoint of stability. Then that will actually imply that p star is viewpoint of stability. So, if you start with any



you know, you see, if you start with any solution, what is happening is this, for all time t, the solution exists, right, in the neighborhood of b star. So, this is the equilibrium. Fine. Now, the thing is this, see, here, as I told you, strict minima is very, very important. Okay.

So, let's take this example. See, remark. Strict minima of L of x is necessary. It is a very essential requirement.

So basically, if we have to show that this is necessary, we have to give a counterexample or an example. So let us say, consider this system. X prime equals to Y. Y prime equals to minus X. This system. Okay. Clearly.

The origin is a equilibrium point. It's the only equilibrium point. 0, 0. Is the. Only equilibrium point. Equilibrium.

point is it now you see l of x y if you want to define as y square okay that satisfies condition a conditioning you see this condition is which is conditional that decreasing is this one yeah so you can easily check this part that's trivial there's nothing to do here okay so please check that part it is there but the thing is yeah uh and and has a minima at zero minimum at zero right okay but you can check what the equilibrium is unstable okay so what happens is and how do you prove it please look at the eigenvalues you can check so check this part it's a constant coefficient linear system right you can of course write down what the matrix a is and check the eigenvalues and then you can see that the equilibrium is unstable right so uh if both are negative then you have a stable equilibrium so basically it is unstable equilibrium that you can check and that is why the thing is even you see here l x y equals to y square okay that has a minimum of zero but the minimum is not a strict minimum okay so and what happens is this in that case the theorem does not hold and the equilibrium is unstable is it okay Right.

I hope you understood why it is a strict minima, right? It's a two-dimensional thing. So, y squared, so it is zero on the whole thing. y equals to 0, so xx is right. Right.

So, now, the next theorem which we do is C, the next theorem. See, first of all, what did we show? We showed that if you have a Lievenhoff function, the system is Stable. Is it okay?

Just finding a Leuphana function is fine. It's stable. Okay. Now the thing is this and we have assumed that, okay, strict minima is required at that equilibrium point. But we also assume this, you see, it is less than equal zero.

So along any trajectory, the Leuphana function is non-increasing. That's what we have assumed. Now what happens if it is strict? If it is strictly decreasing, is it okay? So basically we say in that case we call L is a strictly Lyapunov function.

So maybe I can write it here. L is strictly, we say this is strictly Lyapunov function. Lyapunov function. If all conditions hold, if all conditions... Conditions of Lyapunov function holds except the condition a. Except a. So except one has to change.

Change the following. So what is the first condition? Gradient of L of x acting at f of x. That should be strictly less than 0. Is it okay?

Once this is there, then we have, you see earlier this was less than equals to, right? So, it is non-decreasing. So, non-increasing, sorry. But now, I want this to be strictly decreasing. Yes, sir.

Now, the theorem says that if the equation x prime equals to f of x admits a strictly open-off function. Strictly open-off function. You can actually guess what is going to happen. Near b star.

Then the equilibrium is asymptotically stable. is asymptotically stable. Asymptotically stable. Okay? So, just Lyapunov function, just the existence of Lyapunov function guarantees that it is stable.

The existence of a strict Lyapunov function guarantees that it is asymptotically stable. Okay? So, let us look at the proof. See, the thing is, proof.

Now, by the earlier theorem, what we have seen is this. See, if the trajectory x of t starts near b star, then for all positive time, it stays in that compact neighborhood of b star, right? That's what we proved. So, by the previous theorem, theorem, one has that If the trajectory, any trajectory, doesn't matter.

Trajectory x of t. It starts near b star. Near b star. Okay. Then it exists for all time t. All t greater than or equal to 0.

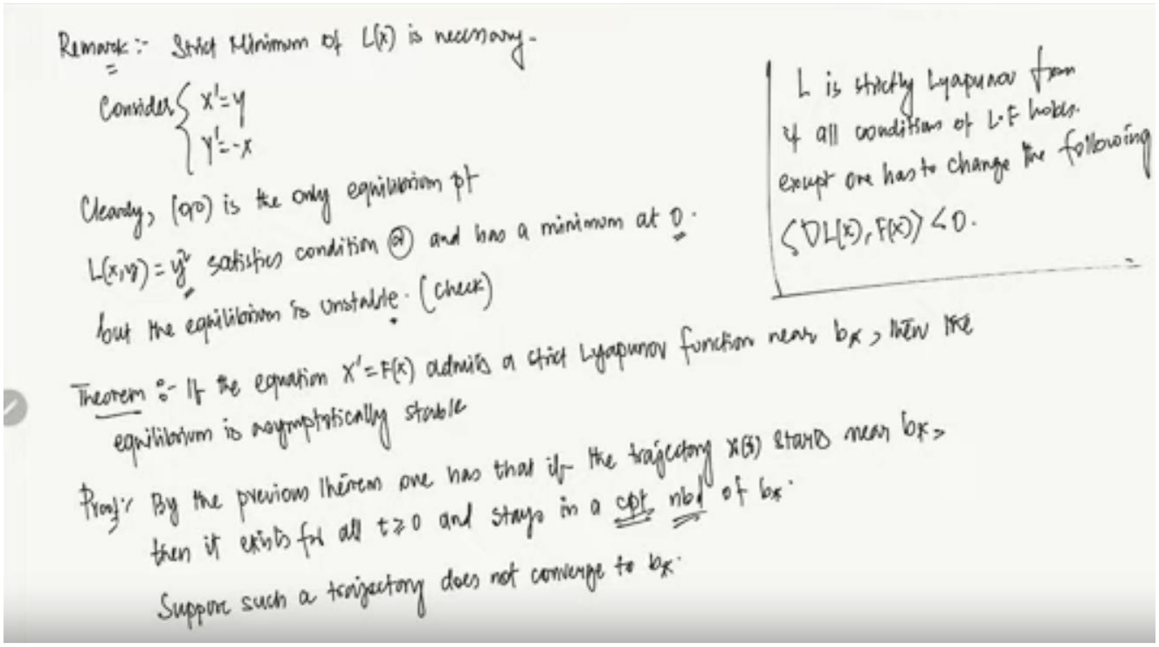
It's okay. And stays in the compact neighborhood. And stays in a compact CPT, compact neighborhood, NVT neighborhood, okay, compact neighborhood of B star. Remember, this is what we proved here. You see, this is what we proved.

Xt cannot cross this. So if the trajectory is confined on the compact set this and this set is contained in V. Right. Okay. So that's what I wrote. Okay.

See, now what we do is this. Let's say that. See, for stability, it can just have to be in the neighborhood of Bistad. It cannot move away from Bistad. That's the point.

For asymptotic stability, it has to converge towards Bistad. Is it okay? Right. Now, let's see that let... So, suppose it does not converge to Bistad. Okay.

So, suppose... such a trajectory such a trajectory okay does not converge to vista does not converge to vista is it okay now then what we have is we can have they then there exists a sequence sequence Tn, okay, which tends to 0, sorry, tends to infinity. So, you have a sequence with tending to infinity such that x of Tn is bounded away from b star. Is it okay?



So, basically you have a sequence here which tends to infinity such that x of p is bounded away from b star because if it cannot converge to b star, it should be away from b star. So, we can write it like this. So, now what we are going to do is we are going to invoke the compactness and we will pass to the subsequence if necessary. So, without loss of generative, what we are going to say is this. Let

x of tn, okay, has a limit, okay, because it is bounded, right, x of tn is bounded. Bozano-Weinstein theorem, bounded sequence, convergent subsequence, okay. So, uptree subsequence, I am just writing x of tn, I am not writing uptree subsequence. Let us say that converts to b, okay, for some b which is not equals to b star, okay. See, it is away from b star, right?

So, it cannot be b star if it converges, okay? Now, you see l is continuous given. The function capital l is continuous, okay? So, l of x of tn, the limit n tends to infinity, okay? This will be l of b continuity, right?

Continuity of l. If x of tn converges to b, L is continuous, so L of x of t has to converge to L of b. And this is nothing but limit t tends to infinity L of x of t. This is definitely true because x of t is a decreasing function. Since x of t is a decreasing function. Sorry, L of x of t. So basically, L is decreasing function. Is it okay?

Right. Now, the thing is, see that now consider the initial value problem. So consider x prime equals to f of x and x at the point less than 0 is b. Okay. And let's write the solution.

The solution to be, how do I put it? Let phi is be solves the problem. Let's say, let's call this problem as 2. Solve the problem 2. Is it okay?

Now, C. since l is a strictly lupin of function yeah so for any for any s positive see s is the parameter yeah for any s positive l of phi of s is dominated by l of phi of zero v yes see l is strictly decreasing so for a positive s what is happening is this L will decrease. Clear? Okay.

Now, the thing is, and that, what is phi at the point 0 b? It is nothing but phi at the point 0 b is b. So, this is nothing but L of b. It is okay. Again, if you look at this, see, phi is continuous. See, phi solves this problem. So, phi is continuous.

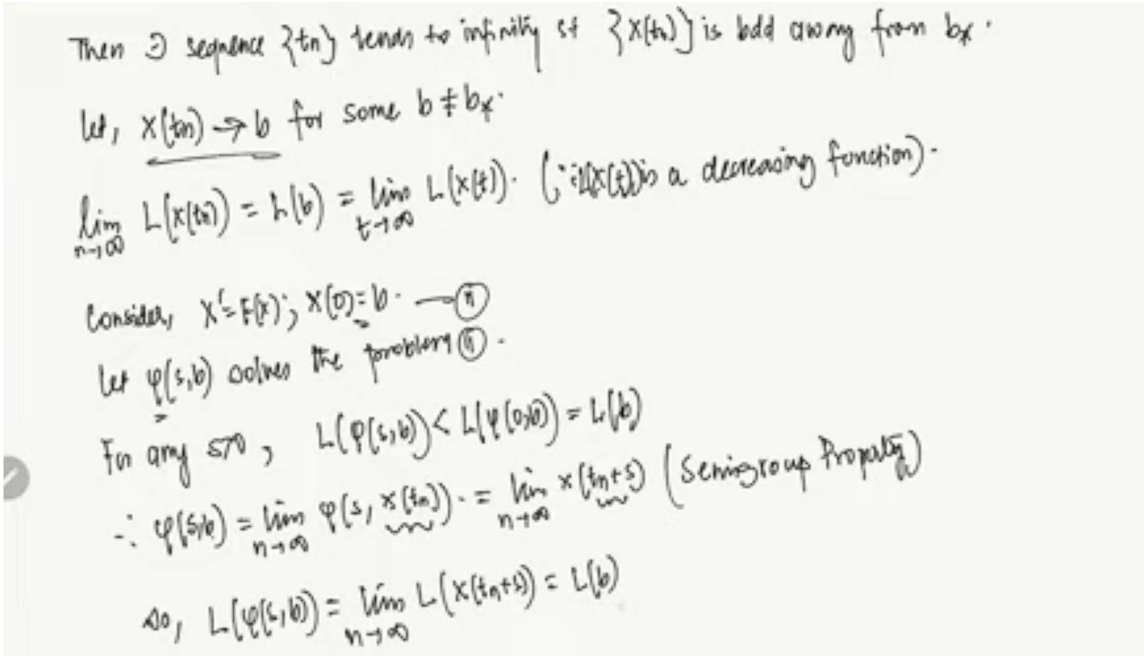
Okay. Therefore, phi of s b is nothing but limit of n tends to infinity, phi of s, x of tn. I can write it like this. See, x tn converges to b. Since phi is continuous, I can write it like this.

Right? Yes? Now, the thing is this. See, these, you have to understand this thing, what is happening is this. Phi of x s tn is nothing but x of tn plus s. Right?

Okay. So, So, this is nothing but limit n tends to infinity x of tn plus s. Is it okay? So, what is happening is this. Limit n tends to infinity x of tn.

You understand what I am saying? See, if you start from b. So, basically it is saying that x of tn as n tends to infinity x. that goes to b, okay, so if you are starting from b, so basically, basically means it is starting from tn plus s, so basically, you see, s equals to 0, what happens is, you are at x of tn, which at the limit goes to b, so that is why its limit enters to infinity, x of tn plus s is nothing but phi of x, x of tn, is it okay, right, so, so what happens is this, L of phi of sp, I can write it like limit n tends to infinity L of x of tn plus s. This property is also called the semi-group property. This is also called semi-group property.

Group property. So, we are just changing the initial data. So, the point from let us say x is 0 to x at the point t and that is what I am doing. So, that is limit n plus 2 infinity L of x of t s which is nothing but L of b. As you take the limit inside because L is continuous, x is continuous, you can take the limit inside. So, if you take the limit inside, it is nothing but L of b. Because of this, see.



eSo, if you take the limit inside, it is nothing but L of b. Because of this, see. Because of this, we can write it like this. Is it okay? But you see, this actually contradicts this one, right? L of phi of Sb is strictly less than L of B. And here I am saying that L of phi of Sb is equal to L of B. That's a contradiction. What does that imply?

It implies that it has to. So basically what it is saying is it cannot stay away from B star. It has to converge to B star provided the strict equality. See here the strict inequality is important. That's why we have used it.

Is it okay? Now what we are going to do is look at one example of how to calculate Lyapunov. So basically how to construct a Lyapunov function. Let's look at that example. Okay.

So first of all, so the question may look like this. We investigate the stability of the following system. of the following system okay what is the system uh let's say x prime of t equals to minus 2x of t and y prime of t equals to x of t minus y this is system right okay now the question is this we need to see whether this system uh the zero solution of course zero is the solution of the system right If you look at it, you see the equilibrium point is 0, 0. So 0 is the solution of the system.

Now the question is this, whether 0 is a stable solution or not. The question is, is the 0 solution, let me put it this way, 0 solution you do realize is 0, 0, right? 0 solution is stable.

Okay, zero solution what I mean is x t y t is identically equals to zero zero. That's the solution, this solution.

Whether this solution is stable or not, that's what the question is. Okay, now we have to find a Lyapunov function. What you do is this. Generally speaking, most of the times the Lyapunov function has a very, I mean, structure looks very much like same, like this one. See, the thing is,

We will define L of x, y to be alpha times x square plus beta times y square. In some cases, it can be alpha times x to the power 4 plus beta times y square. have to you know how do i put it you have to play around with these coefficients and see what works for you okay right now the thing is this you see uh of course you do realize that this is greater than equal zero okay and it is only taking zero at origin with zero only at origin okay so you see at the equilibrium point an origin is equilibrium point of the system so basically at the equilibrium point this uh function uh the the function has a minima okay which is essentially zero okay uh and everywhere else this is positive provided alpha and beta are positive yeah so provided alpha and beta are positive is it okay now the thing is this we have to find what alpha and beta is yeah now you see to do that i have another property right that it has to be strictly uh non-increasing okay so i need to have dt of l of x t y t this property is here right strictly less than zero if you remember the second this is the second property right or the first property

I don't remember which one I wrote first or second. One of the properties, first property. The first property. This property has to be satisfied, right? Okay.

So, how do I show this thing? So, see, it is nothing but, this is nothing but dL dt, right? dL dt. I don't want to write x dy all the time. So, this is nothing but general dL dx dx dt.

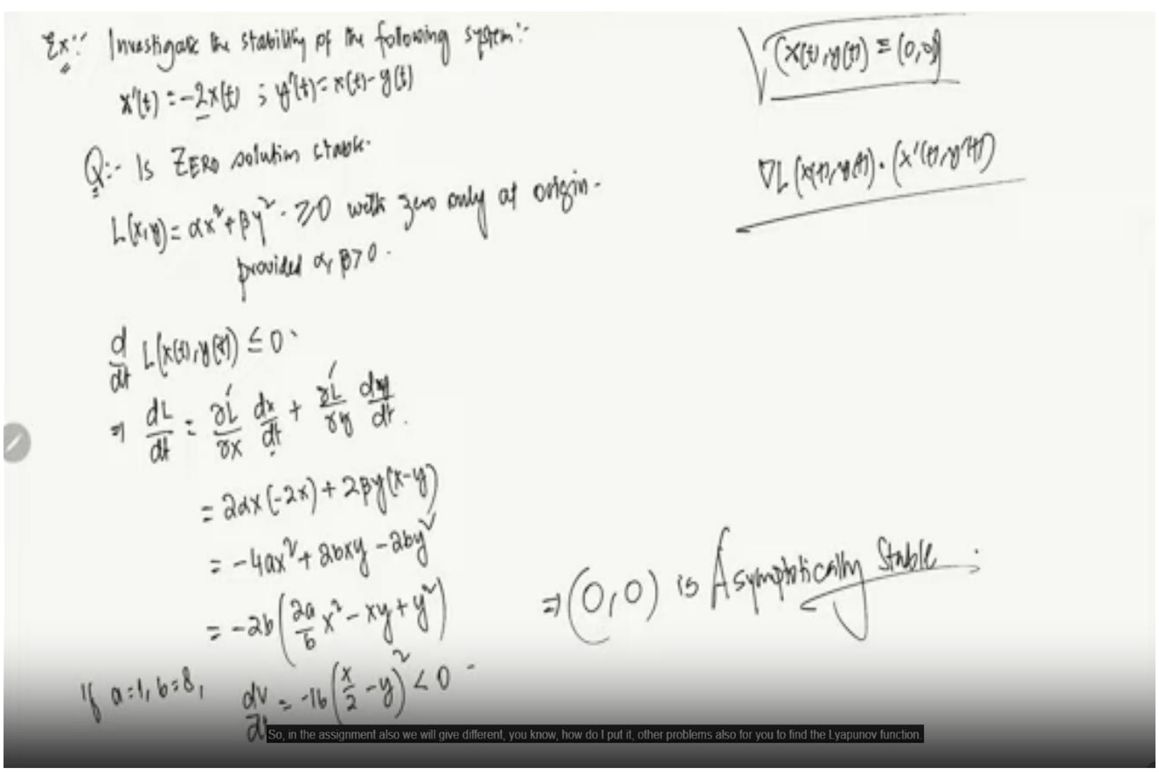
plus, I should write it as del R, because this is a del L, del X, see, A is a two variable function, so basically, by chain rule, it is gradient of L, at the point X, T, Y, T, and then Y prime, and then X prime, T, Y prime, T, that's what it is, right, you understand, so, but I can write it like this also, Y, T, X, sorry, D, Y, D, T, So, this is the same thing right dot gradient of L is this and this and then x prime and y prime is this and this. So, if you take the dot product this is what you are going to get. So, you see if we write it everything properly it is 2 alpha x minus 2 x right. x prime t is minus 2x plus 2 beta y and then y prime t is x minus y. Okay.

So, this will actually give us that minus 4ax square plus 2bxy minus 2by square right that's your del y del dl del t and these you can write it like this you see minus 2b let me write it like this it is 2a by b x square minus x y plus y square Is it okay? Yeah. Now, you see, if we choose, see a and b is on us, right?

Yes. So, if we choose a equals to 1 and b equals to a, this is not a, I mean, you do realize that you can change it also. If you do that, then dv dt, okay, will look like minus 16 times x by 2 minus y square. Okay. You just put everything together, this is what is coming out to be.

And then this is strictly negative. So if it is strictly negative, that will imply that 0, 0, this solution is asymptotically stable. Not only stable, it is asymptotically stable. Is it okay? So, basically the whole point is you have to find the Lyapunov function.

So, in the assignment also we will give different, you know, how do I put it, other problems also for you to find the Lyapunov function. So, what we are going to do is we are going to finish this particular video with the last part which we are going to call the Lasals in various properties. invariance principle okay so what does it say let's just understand this thing see what it says is this if you let's say let's look at this particular system so consider x prime equals to y And y prime equals to x minus xsq minus beta y. And beta is positive. Let us consider this system.



Let us just call this system as 3. This equation is called the Duffing's equation. Duffing's equation. okay please check this part that these are the equilibrium there are three equilibrium 0 0 1 0 and minus 1 0 okay these are the equilibriums are equilibriums okay yes and and i mean you know It is possible for you to show that plus minus 1 is asymptotically stable.

You can do that. So, you know, and plus minus 1 are asymptotically stable. Maybe what you can do is use stable manifold theorem. To show that plus minus 1 is asymptotically stable. That you can do.

Yes, please do that. Now the thing is this, we want to use d upon f. Let us just see that since it is asymptotically stable, we can also use d upon f to do that, right? Okay, so if you do that, so what can we do is we can propose a function, let us say L of x, y, okay? What I am trying to say is this. C plus minus 1 is asymptotically stable.

How do we know it? You can actually just check this part. Check this. So we can actually use the stable manipulative theorem to do that. But for now, let's say that we can also look at this thing.

We open up, right? Let's do that. So let's construct this function L of x, y. And for now let us just propose it like this. This is the energy function as they say.

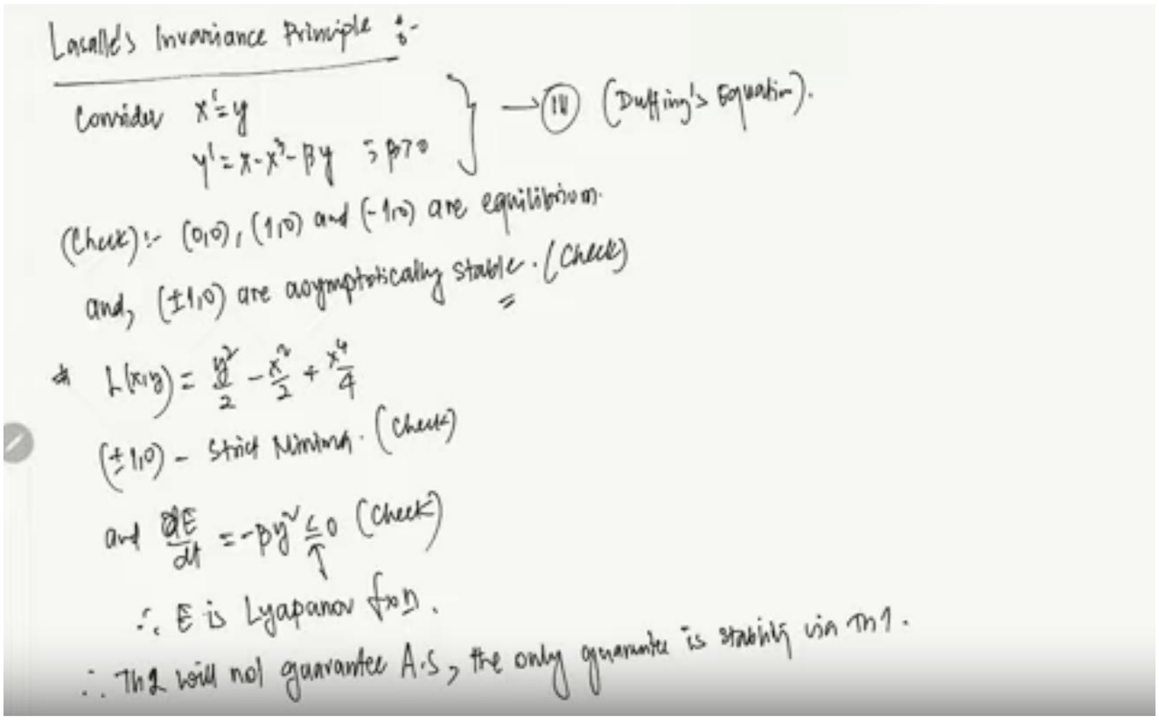
Minus x square by 2 plus x to the power 4 by 4. This particular function. Yes. Now you do realize that plus minus 1 0. This is a strict minimum.

Strict minimum. yeah i hope you know how to find i mean try i mean how to show that so please check this part also it's very easy to check nothing special here and you see d e by dt if you consider this thing this is nothing but minus beta y square right yes uh so again you have to check it so this is again strictly less than equal zero so e is indeed a leap on a function okay therefore e is lyapunov function okay but the thing is still d dt is strictly less than equals to zero so basically what happens is if you want to use the lyapunov theory it can actually give you up to stability it will not tell you that it is asymptotically stable you understand for asymptotically stable you need to have this one as strictly negative is it okay so therefore theorem one theorem two let's say theorem two will not guarantee asymptotic stability the only guarantee only guarantee which you can get guarantee is stability t via theorem 1.

Is it okay? But the thing is, we already know that it is asymptotically stable. So, we are missing something. You understand? We are missing something.

So, what can we do? Let's see. So, these sort of situations are very, fairly common actually. And we can use something called a new one. So, this is the LASA's invariance property principle.

So, this actually, I mean, how do I put it? If you have somewhere that a loop on of inequality that it fails to be strict in that case you can use Lassana. So what is this? It says that let us say that define a set S. What is this set? It is a set of all those X in omega 1 minus B star.



Okay, such that gradient of Lx acting at f of x. is equal to zero okay see in a loop on opting it is strictly less than equal zero right i i need strictly less than zero for asymptotic stability so the zero where it is taking will become zero okay so basically the trajectory is not strictly decreasing but it is monotonically decreasing okay so specifically it means what does this means it means that no trajectory trajectory that starts in s starts in s okay remains in s in s for all positive t all positive time t this is what it means so sorry this is what we assume now for all positive time okay So, let us just assume this thing. This is our assumption.

Okay. See, yes, we define this set, right? And now, under this assumption, we have this theorem. So, this is LaSalle's invariance principle. What it says is this.

And the proof, I am not going to do the proof. If you are interested, please look at the proof yourself. You try to do it yourself. Okay. But the thing is, the theorem says this.

That if... Near an equilibrium, near an equilibrium B star, equilibrium B star, okay, x prime equals to f of x has a Lyapunov function, Lyapunov function, okay, L. That satisfies satisfies this property the earlier one let's just call it star star okay then b star is asymptotically stable okay so it's it's not a very difficult thing to see see basically what i'm trying to say is this You just have to show that the thing is if there is no trajectory. See the set S is there.

S is where the trajectory is not decreasing basically. It is basically constant. It moves constantly. So basically, if it starts in S, it will remain in S for all time t because it is constant, right? It cannot go down, right?

L is not decreasing along the trajectory. So basically, it is fixed kind of thing, right? So if along the trajectory, see, if it starts in the trajectory, if it starts at that point, okay? It remains in that point. It remains on that set for all time t. If something like that happens, if you have a trajectory for which something like this happens, then we can say that B star is asymptotically stable.

Is it okay? So, basically, what we want is this. You cannot have a trajectory, no trajectory. You cannot have a trajectory which starts in S, remains in S for all time t. Yes.

See, it has to somehow leave S because until and unless it leaves S, it cannot be, we cannot have the energy going towards zero. See, we know that at the equilibrium point, what is happening? If it is asymptotically stable, it has to approach the equilibrium point along any trajectory. You see, if you, let us say P star is the equilibrium point. If you move along any trajectory, the energy of the system is decreasing.

And at the point B star, that is why the minima. We need the energy to be minima. The system along any trajectory if you move, the system has a minima at B star. That's what it is saying. Is it okay?

Now, the thing is, that's what the idea of Lyapunov's table is. So, basically, you see, along this trajectory, it has to somehow decrease, right, to reach B star, the minimal state, energy state. Now, the thing is this, if it gets trapped in S, right, there is no way it can reach B star. Is it okay? So, basically, what you need to do is, there is no trajectory, yeah, if you can show that there is no trajectory that starts in S, remains in S.

OK, for all time t, then we study the technique. It's very easy. The idea of, I mean, the intuition behind this is very easy, right? OK, so the proof is not very difficult, but the thing is, it's kind of complicated. For now, what I'm going to do is, for this course at least, we are not going to look at the proof of this.

OK, so please remember that you can actually use the Russell's invariance principle to... I mean, you know, compensating the deficit of Lyapunov stability direct method, okay? So, here, so with this, I am going to end this video. I will give you some problems where you will see how to use Lasalle's invariance principle, okay? Thank you.