Ordinary Differential Equations (noc 24 ma 78)

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Week-05

Lecture 37

welcome students and in this video we are going to continue our study with that loop on of stability so last video we have looked at some loop on of stability right so when it is stable and all that sort of thing now the thing is here what we are going to do is first of all i'm going to start with the definition and then i'm going to explain in that the context of the definition what we did in the last video So definition. So this definition is of something called an equilibrium point or a critical point. Definition. So what is it?

It says that a point, a point B star in Rn is called an equilibrium point, is called an equilibrium point. Okay, for an OD, what is the OD? x prime equals to f of x. Is it okay? If f of b star is going to be 0. Is it okay?

So basically at that point, if capital F, the source term, that is basically zero, then we call this as an equilibrium point. So essentially what is happening is this. See, if f of b star is zero at that point, so x prime at that point is basically zero. That's what it is saying, right? And at that particular point, we call it as an equilibrium point for this system.

So let's look at some example. Let's say for as an example, x prime equals to ax. This is a linear system, right? Linear system.

Constant coefficient, of course. Linear system. Now, you see, what is f of x here? f of x is ax. And for now, let us just assume that a is invertible.

It does not have to be, but let us just assume this. a is invertible. So, if this is the case here, if this is the case, then where... You see, basically, if we want to find what are the equilibrium points, we have to say that when is Ax equals to 0. Since A is invertible, that will only imply x has to be equals to 0.

See, irrespective of A is invertible or not, x equals to 0 is always an equilibrium point. So, irrespective, let me put it this way, irrespective of A invertible or not, or not, x equals to 0 is always an equilibrium point. I hope this is clear. Equilibrium point.

See, if A is invertible, then this system has a unique equilibrium point. If A is not invertible, it has, I mean, of course, 0 is an equilibrium point. But along with that, there are other points also. That is the thing. Now, for nonlinear system, it may happen that 0 is not an equilibrium point or maybe, yeah, it differs.

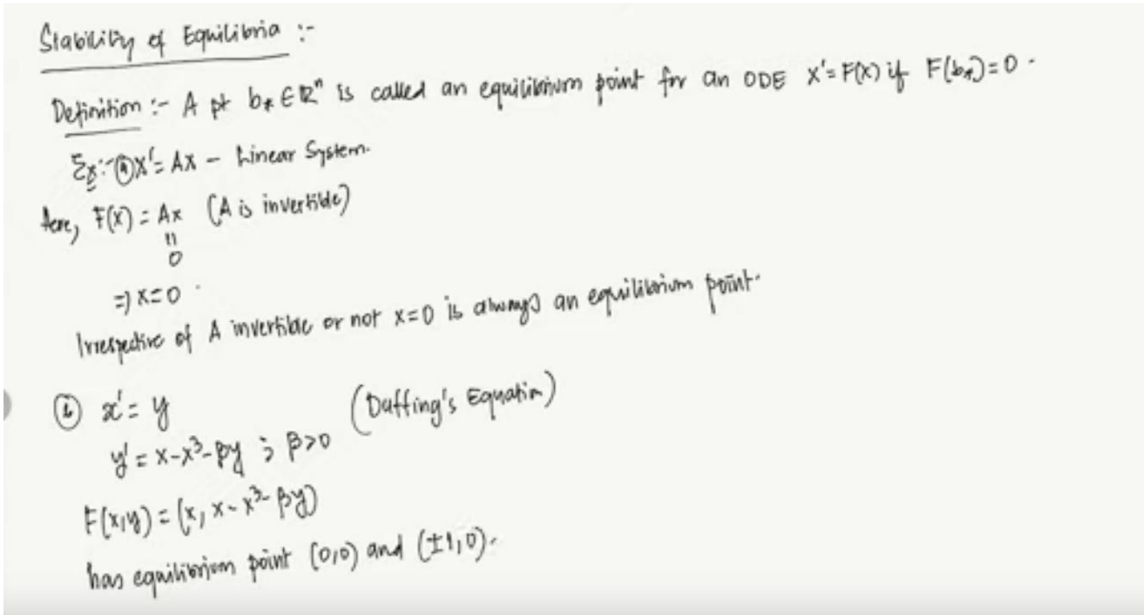
So, for nonlinear system, this is definitely, there can be points other than 0. It can be 0 also. It may happen that there are no equilibrium point which is 0. So, basically, anything can be equilibrium point, okay. So, let us look at one example.

Let us say that the E is thing, x prime equals to y and y prime equals to 0. x minus xs cube minus beta y. Okay. Beta y. So, this is and here we are assuming that beta is positive. Okay. So, this sort of equation, this is called Duffing's equation.

Duffing's equation. Okay. And for this equation, what you can do is you can actually show, you see what is f of xy in this case. It is x x minus xs cube minus beta y, okay?

Now, if we evaluate, of course, x has to be 0. So, any equilibrium, for any equilibrium point to work, see, it is 0, 0, right? So, x equals to 0 and then you have to evaluate y. So, essentially, this has equilibrium point, has equilibrium point. If you just calculate this thing, point 0, 0 and Also, plus minus 1, 0.

So, there are three equilibrium points for this system. Is it okay? So, that is the idea. Now, let us discuss what exactly is so important about equilibrium points. See, essentially, let us say that B star is an equilibrium point.



Let us just assume that B star is an, I am writing it like this, E cube. a Q point, equilibrium point of x prime equals to f of x. Now, b star is what? b star is a point, right? It is some vector in RL, right? Now, note this, that if I am defining a new function x of p, which is a constant function given by b star,

Is it okay? So, x of t is a constant which is v star everywhere for all t, for all t in R. Now, you see what happens to x prime t. Therefore, x prime of t is going to be, it is constant, right, v star. So, that is 0, which is nothing but f of, 0 is nothing but f of v star, right? Okay. So, you see, if v star is an equilibrium point,

Then the function b star. So b star as a point is an equilibrium point. But if you can think of it as a function, so basically it's just a constant function and that also solves the equation. So that is always a solution of the equation. That is always there.

So if you can find the equilibrium point, okay. So if we define a solution out of it using the constant idea that the function is constant. Then what happens is that function also satisfies the equation, yes. So, what is happening is this. See, let us say that it is saying that now for this thing, let us say as, I mean, if you have a function like this, x at the point t is b star, right.

So, in that case, x at the point 0 will also be b star. So, essentially what you are saying is this. Let us say at the point t equals to 0, this is b star, b star. So, what it is saying is this. Let us say, See, this equation, x prime equals to f of x. Now, look at this equation. x0 equals to b star. And b star is the equilibrium point. And b star is the equilibrium point. Okay?

Let's look at this thing. See, what it is saying is this. The first thing, x prime equals to f of x. What it is saying? It is talking about the rule according to which the particle moves. Is it okay?

A particle is moving, right? See, a particle, in this case, let's say, this is... in two dimension right two cross two system so basically capital x is nothing but small x and small y okay now capital x is a function of t so it is basically small x of t and small y of t is it okay now this part this particle it moves right uh so and what is the rule it is x prime of t what is it it is Small x prime of t, small y prime of t. So, basically, it is saying that the tangent is given by the vector field. Okay.

Now, and what is the exterior goes to B star? It is saying that what is the starting point of the particle? Okay. So, where does the particle starts? So, in this case, the particle starts with B star.

Okay. And what is the solution? The solution is also B star. So, basically, what it is saying is this. If a particle starts at B star, it does not, you know, move.

As time moves. So basically it ends up at b star itself. Okay. So that is also a solution. Yes.

Yeah. Do you understand? So basically why it is equilibrium point? Because it is not moving. If it starts there and it ends up there itself for all time t. It's not moving.

Okay. Now you see what is reopen of stability. Let's say we are saying that a particular solution. Yeah. Let's say a solution is stable.

Yeah. So. around a equilibrium point if we are saying a solution is stable so basically uh let's say here b star is a equilibrium point right so we if we are saying it is stable along equilibrium point what we mean by this is essentially if we start very close to b star if we start very close to b star right stability you remember if we start very close to b star okay for all time t we will always remain in a close neighborhood of b star so there is a neighborhood of b star let's say v where the solution will always be contained in that. Is it okay?

It cannot go out of So, basically there will be a neighborhood where you can contain the whole trajectory of the solution. So, basically see this is the original solution. So, B corresponding to B star. We know that corresponding to B star there is a solution which is given by the B star itself.

And let us say you again look at this problem. X prime equals to f of x and x at the point 0 is B star. But in a very I am taking B and such that B minus B star is extremely small. Is it okay? Very small.

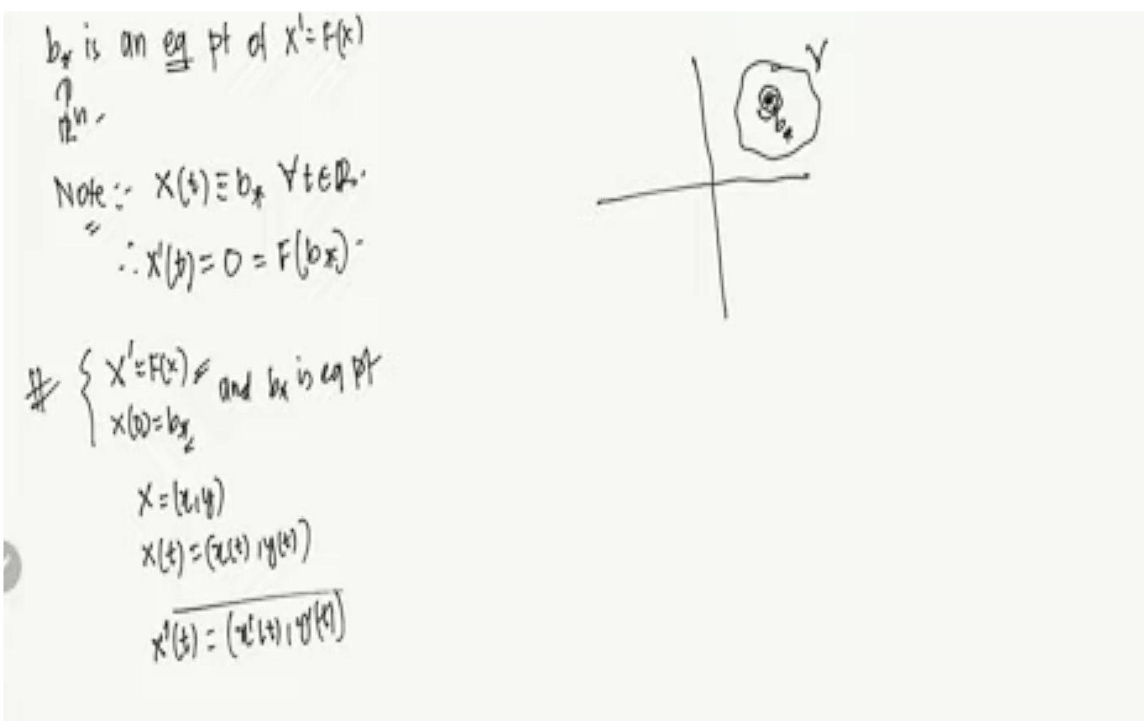
So, I am choosing a B star from a very, very small neighborhood of this thing. Let us say delta of B star. B from a neighborhood of B star. Once I do that, then we get a solution X of T, right? We get a solution.

And that solution for all time T, yes, that must exist. be confined in a neighborhood. So, you get a epsilon. First of all, you fix epsilon and for that you get a data such that what happens is your x of t minus b star should be less than epsilon. So, that is the idea.

Essentially, you start in the neighborhood of b star and you end up Inner neighborhood of B star. Yes. Okay. So the solution will also mimic that.

That's the idea. So that is the idea of stability. And what is asymptotic stability? The solution will actually converge towards B star. You understand?

So it will spiral into B star. That's the idea. Okay. So that's more or less what essentially means what we did in the last video. So in this video, what we are going to do is this.



See, we looked at some certain criteria under which the linear system is stable, is asymptotically stable, that sort of questions, right? Now in this, what we are going to do is this. See, in the last video, we have seen this. So let me put it this way. In the last video, in the last video, we showed that

for the linear system, x prime equals to Ax, okay, if the eigenvalues, if the eigenvalues of A, values of A, okay, satisfies, satisfies the real part of lambda j, So, basically it means that the lambda j of A is the eigenvalues of A. If you look at all the eigenvalues, if you look at the real part, let us say that is strictly negative. For j equals to 1, 2, what can you say? Then every solution, every solution x of t, okay, decays to 0. to zero as t tends to infinity you remember we talked about this right see x prime equals to a of x right and all the idea so for a two cross two system let's just understand that two cross two system you don't have to worry about all these things also

For a 2 cross 2 system, you see if lambda 1 and lambda 2 are both negative. Lambda 2 are both negative. Both are real, let us say. Both are real and they are negative. Then what is the solution?

It is V1 e power lambda 1 t c1 plus c2 times V2 e to the power lambda 2 t. V1 and V2 are the corresponding eigen functions, right? Now, you see if lambda 1 and lambda 2 are going towards 0, what happens to the solution? This solution takes to 0, 0, right? That is what is going to happen. So, that is what we wrote, right?

Okay. Now... See the thing is now we are going to see linear from this thing system is fine. Now the thing is what happens in a nonlinear system. That's the question which we need to answer.

Now see in the earlier case for a linear system what is happening is the solution we are talking about you know it is going towards zero. As t tends to infinity, the solution is, you see, asymptotically stable. And it is basically, while this is moving, it is moving into 0, right? Yeah, it is spiraling into 0. So, but what is so special about 0?

Because in a linear system, you see, 0 is an equilibrium point. Is it okay? That is why. Yes. Now, what is happening is this.

Let us say, if you have a system like x prime equals to f of x. You understand? So, now, consider the system x prime equals to f of x. Is it okay? And we want to look at the asymptotic behavior of solutions of this system near the equilibrium point B star. You understand?

So the question is the asymptotic behavior of asymptotic behavior of the Solutions near P star. Solutions near P star. Is it okay? Asymptotic behavior near P star.

So what is P star? P star is the equilibrium point. Equilibrium point. Okay? Now how do we find it?

See, let's, so what we are going to do is this. See, the theorem which we are going to prove is the following. And this is a very, very important theorem. Okay? Okay.

This is part of a stable manifold theorem. So you can also call it stable manifold theorem. It's not a problem. Okay. So what it says is this.

Suppose V star is an equilibrium point is an equilibrium point point of x prime equals to f of x. That's our assumption. Right. Where what is f? f is C1 of omega.

Omega is any open subset of RL. And assume that the real part of the eigenvalues. So you look at the eigenvalues of D F star. What is D F star? So, it is basically the derivative of the function f evaluated at b star.

Let me write it like b star. Okay. See, f is a c1 function. So, df is defined. df at b star is a linear map.

Right. So, basically it is a matrix. Yeah. And we can talk about the eigenvalues of the matrix. So, those are the eigenvalues.

Let us say lambda j's are eigenvalues. Now, look at the real parts of those eigenvalues. If they are negative. okay, for j equals to 1 to n, okay, then what you can say is this, you see, then there is a neighborhood, there is a neighborhood, nvd, neighborhood, okay, neighborhood, v of b star, v of b star in Rn, of course, in Rn, such that For any initial data B, for any initial data B, okay, in V, is it okay?

The initial value problem, what is the initial value problem? x prime equals to f of x and x0 equals to B, okay? Has a solution, has a solution. For all t greater than or equal to 0 and moreover you can say that limit t tends to infinity x of t is b star. Is it okay?

So basically what it is saying is this. whatever i just explained here so let me explain it again here see the thing is let's say that i know that b star is the equilibrium point of this system f prime equals to f of x c b star is the equilibrium point this is what i am assuming okay now so b star is the equilibrium point right now let's say v is a neighborhood and such that you know you look at d f b star that's a matrix right it's a linear transformation from rn to rn Now, you look at the real part of that matrix. Sorry. You look at the eigenvalue.

Sorry. So that's a matrix now look at the eigenvalues of this matrix lambda j all the eigenvalues and then look at the real part so basically if they are all real eigenvalues that real part is the eigenvalue itself if it is complex you just look at the real part if they are negative okay then for the linear system we know that what happens the solution goes asymptotically towards the towards zero so basically at the critical point here what it is saying is this same sort of thing happens. So, basically, you heard him saying that if this something like this happens and for a linear system, if you start with a B from a very close to B star, that is what I explained here, right? If you start with a B in a neighborhood of B star, you see, V is a neighborhood of B star and if you take any B from B star, then what happens is, the solution starts from here and for all t there is a solution okay and what happens at t tends to infinity it actually goes towards b star you see that's what it is saying it is saying that if b star is a equilibrium then there is a neighborhood v of b star in rm such that for any initial data so if you are choosing an initial data from

v then what happens is the c in the problem f prime equals to f of x x 0 equals to b okay with that initial data the solution exists for all time t and the solution what is the asymptotic behavior of the solution it actually converges to b star yeah it is actually very intuitional if you think about it now the thing is what is the proof Okay, so you see what we are going to do is essentially I am going to start with a translation. See, here I started with b star is an equilibrium point. You can also choose it to be 0. It is not a problem, right?

So without loss of generality, we assume that the equilibrium point b star is 0. Okay, if it is not 0, you just translate everything in Rn. It's not a problem, right? We can do that. Okay.

So, what we are going to do is I am going to choose b star to be 0. That's what I am going to do. So, basically, I am assuming that the equilibrium is located at origin. Yeah. That is f of 0 is 0.

0 is always there. That is the equilibrium point definition. And this is b star, right? So, this I am choosing it to be 0. Is it okay?

Now, what I am going to do is you see... f is a c1 function which is given f is c1 so i can expand f right so basically you see f of x f of x in a neighborhood of 0 essentially in a neighborhood of 0 of 0 what we can do is i can write f of x to be a of x plus r of x r of x right where a of x is df at 0 yeah and r of x is the remainder term r of x is the remainder do you understand what we are doing here we are using taylor's theorem here see first of all f of x is f of 0 if you remember f of x is nothing but f of 0 plus df At the point 0, acting at h and all those things are there. But see, f of 0 is 0.

So, this is gone. So, f of x is df0 acting at x, right? Okay. So, this thing I wrote it here. You see, df0 acting at x. Is it okay?

This is 0. And then you have a reminder term. So, this is first order Taylor theorem. This is what I am using for several variable functions. Yes, r is the reminder term.

Now, you see. We know that, you know, from the earlier, this thing, we talked about it, right? That there exists a constant k and epsilon positive such that we can always do this, right? Mod e power. So, for this a, I will choose this a and I can write it like this.

See, for this a, e power a t. can be made less than equal a constant times e power minus epsilon t if you remember yes for t greater than equal zero we talked about this thing right this is this actually holds while talking about asymptotic behavior of the solutions we use this fact also if you remember of course you can do realize k is greater than equal one it has to be that's the right definition because you know At 0 also it is 0. So it is 1. So essentially k always has to be greater than or equal to 1.

That is just given. Now what we are going to do is I am going to do a little technical jargon. I am just going to write it. So we choose eta positive such that eta is less than epsilon by k. I hope you understand what is the idea of the proof. Right.

Idea of the proof is very easy. So think about it. Here it is saying that I want to talk. I want to have some sort of property of a nonlinear system. Here I equally acquire it.

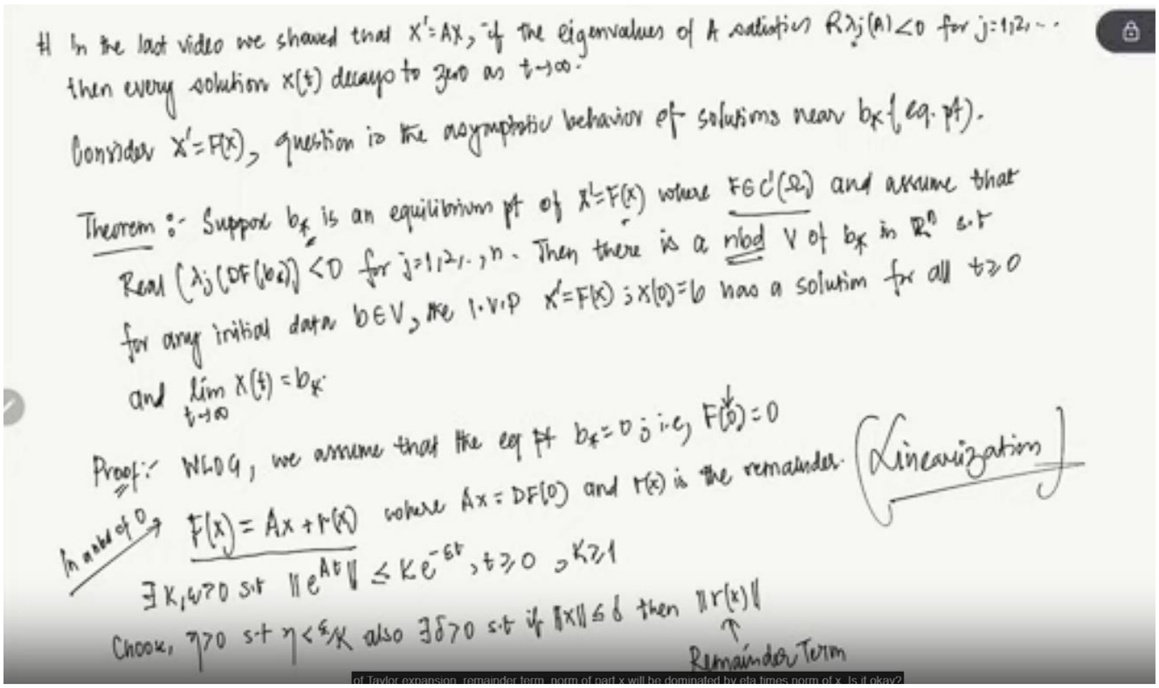
Yes. So, for a non-linear system, there are equilibrium point. Think about it. Since F is C1, in a neighborhood of the equilibrium point, I can think of the non-linear system as a linear system. I can approximate it.

And the behavior of the linear system, I expect it to work also for a non-linear system. That is what is happening here. So, essentially, this is called the linearization. This is called linearization. So, basically what I am doing is I am linearizing the original, you know, x prime equals to f of x with this.

And then if I do that, what happens is this. Basically, I am not talking about the original, you know, equation anymore, but I am talking about a constant coefficient equation. Okay.

Now, let us look at that. So, you see, we choose eta greater than 0 such that eta less than epsilon by k. This is just some technical issues.

So, and you see that r is there, right? Yes, and what is the property of R? That there is also, also there exists delta positive such that if norm of x is less than equals to delta, okay, then norm of Rx, you remember the reminder term, this is the reminder term. of Taylor expansion, remainder term, norm of part x will be dominated by eta times norm of x. Is it okay? That's the definition of the remainder term, right?



Okay. So, we can always do that. Okay. Now, the thing is, see, what we are going to do is this. So, we are going to define a new function.

Define a new set V. What is the definition here? This is the set of all those V in Rn such that mod V is less than, sorry, mod V is less than delta by k. Okay. This is contained in the ball with radius center 0 and radius delta. Why? Because you can see that, you know, mod b is less than delta by k. So, which is again contained in delta, right?

Delta radius ball, okay? Now, you see, the thing is, this problem, where is the problem? x, I am equals to f of x, x0 equals to b. yeah this problem for f e in c1 we have that this problem is not solvable for all t greater than equal zero you understand what i'm saying that's just a we talked about this theorem right that we cannot solve this problem for all t greater than equal zero right so if we can't do that then what does it mean if we see the thing is uh You know, in the last, when we talked about asymptotic stability, sorry, when we talked about maximal interval of existence, what did we say?

That if there is a solution of this problem, right? So the trajectory, if we can bound the trajectory in a compact set, yes, then what can we say? Then the solution exists for all time t. This has to be, otherwise it can't be. So you cannot traffic in a compact set otherwise, right? So let us say that if you have a solution which is not solved, so basically if this problem is not solvable for all t, so basically the trajectory does not exist for all positive t, then this solution, the solution of this problem must leave the ball b0 delta bar.

You understand? So therefore, if a solution exists for all t, in zero infinity okay if your solution does not exist does not see we have to prove it exists right it does exist see we have to prove it does exist for all time t so basically what we are assuming is this let's say if your solution does not exist for all time t okay then you cannot put it in a b0 delta bar right yes then x of t cannot be contained the trajectory cannot be contained Does not belongs to. B0 delta bar.

For all T. You cannot contain the trajectory for all T. So. So essentially what I mean is. X of T. Must leave the compact set. So let me put it this way. X of T. Must.

Leave. b 0 delta bar right that's what it is see this is where we are actually going to find a contradiction so we are assuming that they say that there is a t star positive such that mod x of t is less than delta okay for t less than t star while mod of x t star equals to delta. Is it okay? So, what we are saying is this.

Let us say there is a t star. After t star, what happens is for all t less than t star, x of t is within the ball. Now, see, as the solution does not exist for all time t, that is what our assumption is. See, we have to prove it exists for all time t. So we are assuming that it does not exist.

So what happens is that in that case, it starts from the zero, right? And the thing is, what we are saying is it is not containing that compact set. Because if it contains, it has to be there for all time t. So it has to leave the compact set. Now, if it is leaving the compact set, there is some time t equals t star squared. It has to touch the boundary.

Okay. It has to cross the boundary. And what is the boundary? At the boundary, mod x t star is delta. Okay.

Of course, it has to be. Right. Now, you see what we are going to do is, so from here we are going to get the contradiction. So, we are going to define a new function g of t, which is e power epsilon t, the norm of x of t, yeah, and we use Granville here, okay.

So, what we are going to do is, you see, that equation, this equation, which is x prime equals to f of x, x equals to b, this equation, we are going to write it like this, see, let g of t is this, and we rewrite Okay, see x prime equals to f of x. And what is f of x? f of x is ax plus rx, right? So we have x prime equals to ax plus r of x. Is it okay? And what is x0?

x0 is given to be b. We can rewrite the equation like this. F of x, I am replacing it with x. Now, you see, this is a linear equation with constant coefficient. Of course, there is an inhomogeneous term given by this. So, this equation is nothing but you see x prime equals to, sorry, this is a linear equation with constant coefficient.

A is constant, sorry. A is constant. Right. Yeah. And what is A is dA at the point 0.

But A is a constant coefficient equation. It is a linear equation. Right. Okay. So, we can actually solve this problem.

What is the solution? By Duhamel's principle. So, therefore, what is x of t? It is e power A t times x naught, which is b plus 0 to t e power A t minus s r of x of s ds. Is it okay?

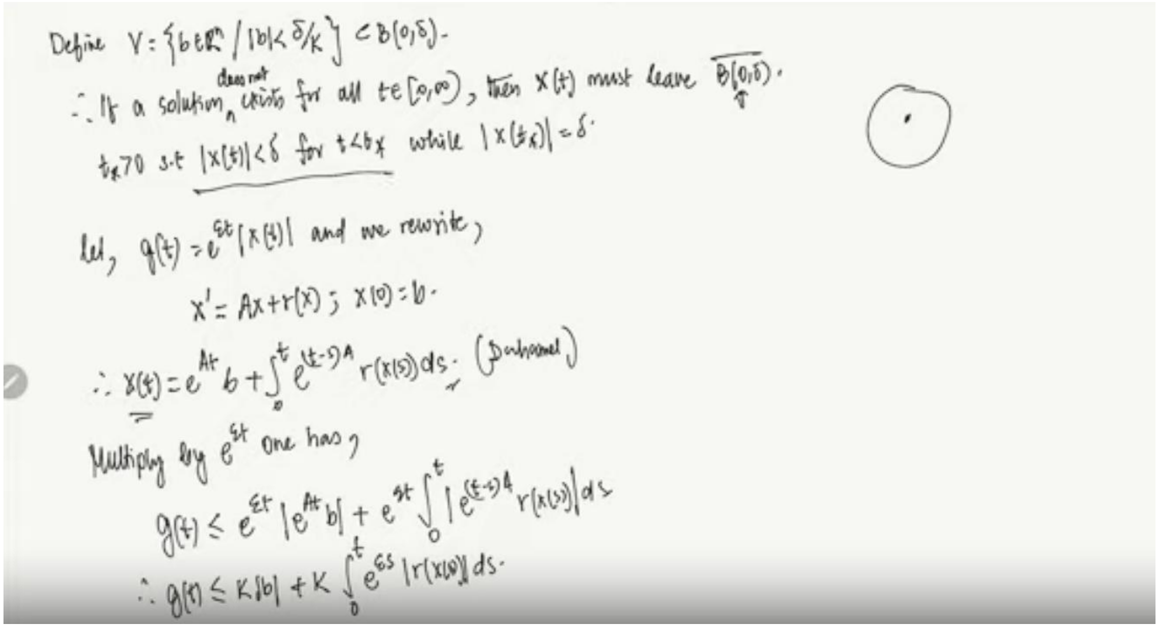
That is what the x of t is, right? By Duhamel. If you remember Duhamel's principle. Duhamel. Why?

Because in this case, what is the fundamental solution? It is e power a t, right? Okay. So, now if we multiply, this particular equation is e power epsilon t. So, multiply by e power epsilon t. One has g of t. Yes.

You see, multiply and take the modulus then. Take the norm exactly. e power epsilon t and then e power a t times b, the norm of that, plus e power epsilon t and then 0 to t mod of e power t minus s a r of x of s ds. That is what we are going to get, right? Okay, what I am doing is this particular expression I am just multiplying by e power epsilon t and I am just taking the modulus or the norm that is.

Now, you see this equation e power a t, yes, what we are going to do is we are going to use the bound on the exponentials. You see e power a t, norm e power a t is less than equal to k times e power minus epsilon t. Okay. So, if you put that, therefore, what you have is this g of t is less than equal k times mod v plus k times 0 to t e power epsilon s epsilon s r of xs ds. Is it okay? Just by using that bound.

So, you see, therefore, For t less than equal t star. See, for t less than equal t star, the solution is in less than delta, right? It stays inside that ball. And at t star, it touches the boundary.



And after t star, it just flows up. That's what we are saying. So, basically, flows outer. Okay? So, you see, for t less than t star, what happens is this.

This term. This term. Okay? Okay. satisfies this particular thing you see we can actually bound it so basically what we can do is this the second term the second term satisfies k times 0 to t e power epsilon s r of

xs ds okay this is dominated by k times eta 0 to t e power epsilon s mod xs ds okay which is nothing but k times eta 0 to t g of s ds i hope this is clear okay Is it fine? I am just writing this part there. See, this particular second expression. Now, we use Grunwald.

So, now, if we use Grunwald, by Grunwald, what happens is, Grunwald lemma, we have that g of t, in this case, will be less than equal k mod v e to the power k eta t okay for all 0 less than equal t less than equal t star okay now if you just look at what is g g of t is e power epsilon see what is g of t e power epsilon t mod x of t so therefore what you have is therefore mod x of t or norm of x of t i should not say mod it is norm of x of t in this case is e power minus epsilon t g of t Now, if we put everything together, it is less than equal to k times mod b e power k eta minus epsilon t. That is what we have. So, see now the exponential is decaying. See k eta minus epsilon.

If you look at this, you see k eta minus epsilon is negative. Okay. So as time tends to t tends to infinity what is happening is this particular thing goes to 0. Exponential this thing this particular thing goes to 0. Okay.

So what is happening is this. So the solution never leaves the ball of radius delta right as t tends to infinity. Is it okay? See if this is true then this can be made less than x delta. right that's that's what we see this is small as t test if this is small so we can write it like it is less than k types this okay uh once again let me just i i should put it in proper

Okay, so okay, fine, let me put it this way. See, norm x t is this, right, and now the exponential is decaying, k eta minus epsilon that goes to, that is negative, right, that we know. Therefore, what do we have is this norm of x t star, let us say, if we take t star, okay, so that will be less than k mod b, yeah. If you look at it, it will be, because this is going to 0, right, as it is infinity. Okay.

So, you can make it less than k times mod b. Okay. And what is k times mod b? This is less than delta. That is our assumption, right? Okay.

So, that is a contradiction. Because k, x of t star, we are saying that this is equals to delta. You see, x of t star equals to delta. So, that is a contradiction. Contradiction.

So what does it say? It says that, I mean, if you have a solution which is strapped, sorry, if it's not strapped, so basically if it leaves the ball, then there's a contradiction. So basically what we proved is if there's a solution which starts in a neighborhood of B star, it stays there. And for all time t, it is actually, you know, defined and it stays in a neighborhood okay and what happens is since you see this is true x of t is less than k times mod v power k eta minus epsilon times t and k eta minus epsilon this is negative right so as t tends to infinity norm of x t must goes to zero okay as zero is equilibrium so basically it says that it is asymptotically stable is it okay so that's how you i mean

I mean prove this theorem. This is okay. Now the thing is this. What I want you to do is this. See the theorem is quite clear.

Okay. Now let's look at the Duffing's equation. Okay. So consider this equation now. So this is an example.

Okay, I am not going to do this thing. You have to do it. So for x prime equals to y and y prime equals to x minus xs cube minus beta y. Okay, we have seen that 0, 0 and plus minus

1, 0 are the equilibrium points, right? Okay, so what we need to do is check that. Plus minus 1 0.

Okay. These. Are asymptotically stable. Okay. So.

This system asymptotically stable near this points. Okay. So you please change this. What do you do? You see what is f of x y in this case.

It is nothing but x. Sorry y. And x minus x cube minus beta y. Just find out what is d f. What is the. b star here, plus minus 0. Find out what is that. Okay. Once you do it, you can talk about the eigenvalues of df plus minus 1, 0.

Right. Talk about the eigenvalues. Write down the eigenvalues. If the real part of those eigenvalues are negative, then you know that you can actually use this theorem to say, control the asymptotic stability. Is it okay?

So, you have to check this part yourself. So, please do it yourself. And with this, I am going to end this video.