Ordinary Differential Equations (noc 24 ma 78)

Dr Kaushik Bal

Department of Mathematics and Statistics

Indian Institute of Technology, Kanpur

Week-05

Lecture 36

so welcome students and in these last two weeks what we are going to do is concentrate on the you know a general linear nonlinear equation okay so consider this problem let's say consider the problem of x prime equals to f of tx so we have a system here yes and x at the point let's say 0 equals to x naught this is okay i mean i am writing 0 you can substitute it by t naught but anyways let me let me do this do it otherwise you may get confused so t naught some point t naught t naught can be zero also no problem okay so we are looking at this equation now the thing is you see we know that let's say we are assuming f is smooth yes So, we know if f is c1, then around t0, there is a unique neighborhood where you have a unique solution, right? So, essentially what, let me put it this way. So, Picard's existence, Picard's theorem, right?

Existence uniqueness theorem that is. what does it say it says that there exists epsilon positive such that so in t naught minus epsilon and t naught plus epsilon okay so there exists a unique so maybe i can put it this way such that There exist unique y or x let us say x which is from C1 of T0 minus epsilon to T0 plus epsilon to Rn. Of course, this I wrote it as a open, this closed set, but it is generally as a maximal interval of existence. We talked about it.

It can be open also. So it will be open if you talk about that. So there exists an epsilon greater than 0 such that there is a unique solution satisfying 1. satisfying one this is we know okay now the thing is this we talked about well post is a problem right now the question is this question when is this when it is well post okay so when is one well post when is one well post okay so in this next few videos we are going to talk about that sort of thing so basically existence uniqueness is guaranteed by Picard's theorem now the question is this when is it well post means the third part when is the solution stable okay what do i mean by that okay let's just understand this okay so with an example let's just understand this with an example see let's say we are talking about an problem like y prime equals to a y

This is the equation given to you. Now, with the initial data, y0 equals to, let us say, y0. Now, for this problem, we know that there exists a unique solution, unique solution y of x. I hope you can prove uniqueness. which is y naught e power ax. So, please check how do you prove uniqueness.

You have to check that part here. We did it. We did the same while we were proving, you know, Menier theorem. So, basically, you do not need Picard to do this here. So, please check this part.

Check why the solution is unique. Yes. So, we know that there is a unique solution yx, which looks like y naught e power ax. And this holds for all x in R, right? This holds for all x in R.

Please don't get confused between this capital X and this small x. This capital X is a function of t. t is in R. So please don't get confused here. That is our variable. Basically the curve here is just a point. Now, the thing is this. Now, let us look at this.

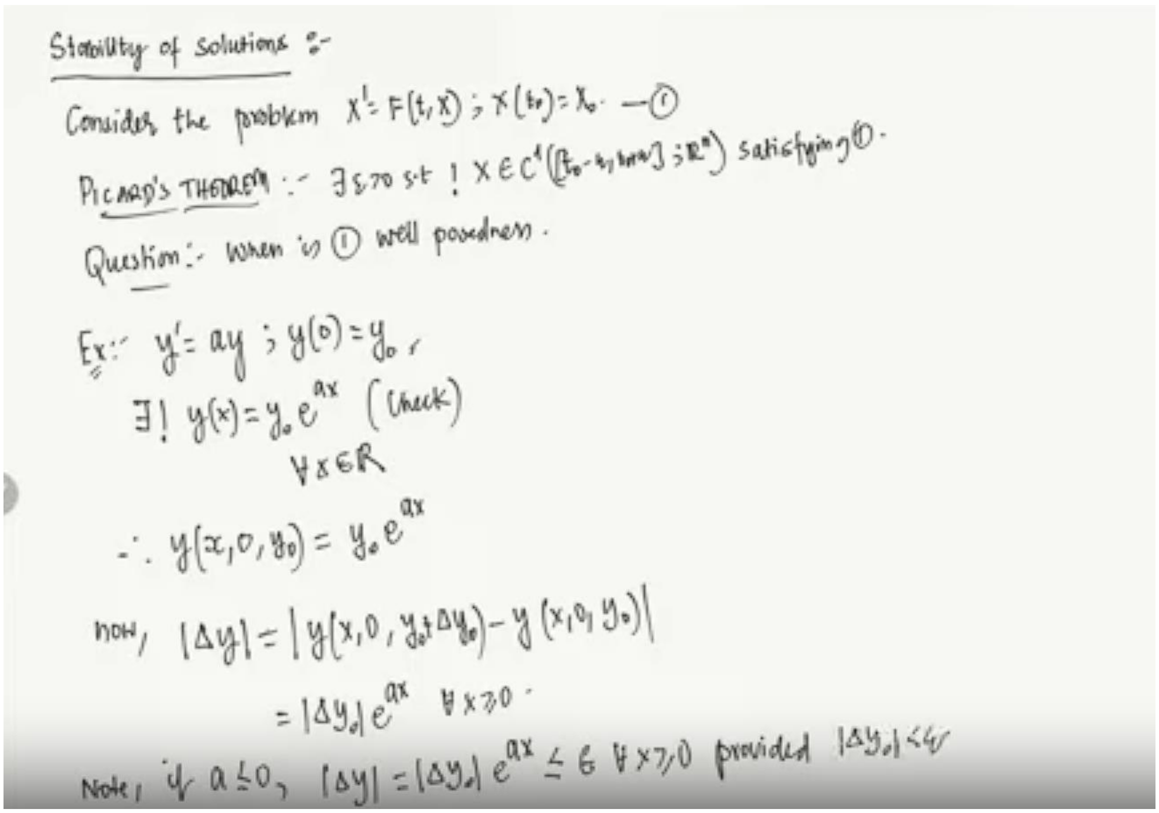
See, therefore, let us say that if you define, you know, if we write like this x, y as a function of x, initial data is 0 and at the point 0, it takes y naught. Let us write it this way. So, in that way, if we do it, we can write it as it is y naught e power ax. Okay, this is the solution. I can write it like this.

Yes. Now, you see, now, We can do it like this. See I will define delta y. What is delta y? Delta y is the change of y when y naught is going to be changed from y naught plus delta y naught.

So basically you see if I change y at the point 0 to y naught and I change it to y naught plus a small increment on y naught. What happens? you want to we want to see how much is the solution changing right that's what stability is basically we don't want our solution to fluctuate too much with this initial date change of initial data right okay so that's the idea so basically what we have is this see what is delta y naught the mod of delta y naught we are defining it as the function y at x 0 so variable x at the initial data 0 and what is the what is y does at the point 0 it is y naught plus delta y naught okay minus y x 0 and y naught so we want to see how much is the solution changing if you are changing the initial data now you can of course you can see that since this is linear that will give you delta y naught the modulus of that times e power x is it okay this holds for all x greater than equal 0. Now, this is a very simple equation.

Now, the question is this, if I am saying that it is well posed or not, that is a tricky question. Why? Because see, note that if a is less than equal to 0, then you see e power a x, this actually is basically decreasing. So, basically what happens is delta y, that is equals to delta y naught times e power x okay and is if a is negative i can make it less than epsilon right for all x greater than equal 0 provided

provided delta y naught is less than epsilon. Is it okay? I can of course do that. If a is negative, e power a x, that can be made less than equal to 1 and hence all of this is true, right? So, you see, I can make delta y as small as possible.



You understand? So, basically, whatever the initial data, however I am changing, the original solution is also changing proportionately, okay? But now, see that if a is positive is it okay if a is positive then delta y okay less than epsilon this holds if and only if if and only the increment of delta delta y naught the modulus of that should be less than epsilon times e power minus ax okay now this is possible only for finite values of x right so you see this is possible only for for finite values finite values of x

Is it okay? Because you see the thing is x if it goes, I mean you can take arbitrarily large x, this will actually push delta y naught to be towards 0, right? Okay, so I mean no matter how small mod delta y naught is is it okay yeah so you do realize that in this case you cannot really say that this problem you see in zero infinity this solution is in zero infinity you can think of it like that right so this solution in this case it is stable but in this case there is an issue right we cannot talk about this sort of stability okay right so now let us put it this way see the thing is uh Without, you know, going into exact definition.

Yes. So basically what we are trying to do is this. When do we say some solution is stable? If a small change in the initial data brings only small changes in the solution. Right.

That is what we mean by stable. And if it is, of course, if it is not stable, it is unstable. So that sort of thing is there. Okay. So in this case, you see, this case, what can you say?

That it is unstable. okay it is unstable what is unstable the solution is unstable right yes or i mean since this equation has only one solution in some author may say the solution is the equation itself is unstable in that sense that the solution is unstable okay right so with that let me give you the definition of what stability means yeah stability what does it mean a solution a solution of what, of 1, yes, this problem, okay, and a solution of 1, how, where is defined x, okay, so a solution x of t, our variable is t, yes, and how do we write it, it is x t, and the initial data in our case is t naught, okay, you can take it to be 0, it is not a problem, and then our initial data is x naught, this is okay, you see. We are writing it like this just to make everything clear. Otherwise, you see initial data is changing, right?

So, you know, I have to write x and another different function. So, we are just writing it like this. So, a solution this of the initial value problem 1 is said to be stable. So, please remember this is not anymore a linear equation, but this is a nonlinear equation. Is it okay?

It is said to be stable. To be stable. stable when is it stable if for each epsilon positive for each epsilon positive okay there exists delta this is delta which depends of course on epsilon and the initial data zero so you know the initial point in there sorry not the data the point t naught Of course, this is positive, delta positive, such that if your initial data, if you perturb it a little bit, so basically the change on of x0, if delta x0 is less than delta, that will imply that u, sorry, not u, tx, sorry, x of t, t0, x0 plus delta x0. okay so basically i am looking at the solution corresponding to the initial data x naught and i am taking the difference of those that solution with t t naught x naught we are saying that that's that should be less than epsilon is it okay so given epsilon greater than zero we can find a delta basically continuity kind of stuff so if you are changing your initial data a little bit you also expect your solution to change

Likewise, right? So, that is your stability. Is it okay? Now, let us look at the definition of unstability. Definition.

Now, please remember, this is, I am saying the solution is stable, not the equation or something like this. We are saying the solution is stable. This is the nomenclature which we are going to use. Now, a solution, a solution x of t, which is given by x of t, t naught, x naught, is it okay?

This is of 1, of 1, is said to be unstable, is said to be unstable, right? Unstable, okay? If, of course, it is not stable, if it is not stable. Quite simple, right? When is it unstable?

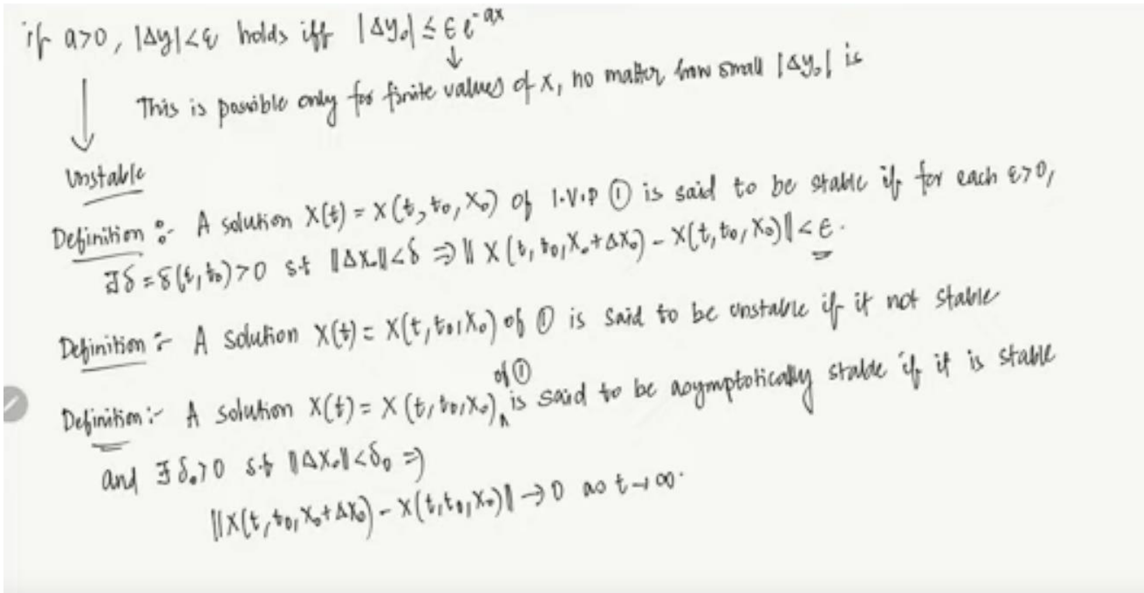
If it is not stable. So that is why the earlier definition doesn't hold basically. Now with stability, we have another beautiful definition which we are going to use. Stability has different aspects, right? Now this is another aspect.

What is it? A solution. A solution x of t, which is defined by x, t, t0, x0 is said to be, so please remember this thing. The first one is stability. The second one is unstable.

Third one is said to be asymptotically stable, okay? Asymptotically, asymptotically stable, stable, okay? of one sorry the solution of one is said to be asymptotically stable if of course it has to be stable you see the by in the name itself we are saying it has to be asymptotically stable so basically it has to be stable so if it is stable if it is stable of course that is true and there is the asymptotic part so basically what i want is this and then and there exists delta not positive okay such that for the change in the data okay so delta x naught less than delta naught implies what implies this thing x at the point t t naught x naught plus delta x naught minus x at the point t t naught X naught, okay?

See, what we are saying is this. First of all, it has to be stable. So, basically, this is true. This has to be true. Moreover, we also want as for large n of x, okay?

Sorry, for large n of p, this must go to 0 as t tends to infinity. Is it okay? This is what our definition of asymptotic stability is. Basically, what we are saying is this difference between is always less than epsilon that is fine it has to be for stability so we want stability but more than that what we want is for sufficiently large t okay we also want that the solution the the part of solution should converge to the original solution is it okay so let's look at some of the example with some examples what i mean by all of this okay so first of all let's look at this example example one okay so



look at this equation y prime equals to x i'm not putting any initial data right now okay y prime equals to x now has a solution has solutions of the form solutions of the form you can easily solve this problem right what is it so let me write it down y x is let's say um y at the point x naught okay, minus x naught square by 2, okay, plus x square by 2. So, basically, you know, I want at the point x naught, this should pass through y, you know, y at the point x naught should be y at the point x naught. So, basically, yeah, that is the solution, yeah, we want that, okay. So, you see, this is the solution, right, any solution will look like this, right.

Now, you see that this is stable, of course. So, the above solution, DevOps solution is stable. Okay. Can you prove this?

It is very easy to prove, right? I mean, of course, see, basically stability is nothing but like a continuity property. It is basically a continuity property, right? And you can see by this that this is just a function of a, it is a polynomial x square by 2 kind of thing. So, of course, you can show that this is going to be stable.

It is not a big issue, okay? So, please check this part. Check it, yeah? Yeah? So, I want you to do it.

It is very easy. You can do it. So, this is a form of equation where every solution is basically stable, okay? So, therefore, every solution is stable, okay? Solution is stable.

Okay. Now, let us look at some other examples. Okay. Now, you see, let us say that every solution, every solution of y prime equals to 0, let us say. How does it look like?

Is of the form, is of the form, y x equals to y x naught, right? At any initial data, whatever the point is, y x equals to y x naught, that's your solution, okay? Now, is it stable? Of course, it's stable.

I mean, there is nothing to prove here. By definition, it's stable, right? Epsilon equals to delta. I mean, you do realize, right? It's basically a constant, right?

It has to be stable, yes? But is it asymptotically stable? See, the thing is this. If you look at it, y at the point x here let us say x naught y naught sorry y naught plus delta y naught minus y at the point x x naught y naught.

Let us look at this. See this is nothing but y at the point x naught sorry y at the point x Once again, this should look like this. y at the point x0 is y at x0, right? So, y with the initial data, so this is y0.

This is y0. So, with this, you see, you have y0 plus delta y0 and minus y0. This is what we have. This is what we have. Now, you see, the difference is delta y0.

okay the difference is delta y naught but you do realize that what is delta y naught you see why so once again i did some mistake somewhere no y at the point x naught so basically it is let's say it is So, let us look at this. You see y prime equals to 0 and y at the point x naught is y naught. This equation we are looking at. So, what is the solution?

The solution is y x is y at the point x naught. That is the solution. Again, you see y prime equals to 0 and y at the point x naught is y naught plus delta y naught. So, what is the solution? y x equals to y at the point x.

Sorry, this is y0. And this is y0 plus delta y0, right? That will be the solution. Yes, it's fine. That's what I wrote here.

Yes. So, you see, so basically this is the difference, right? Now, if you are taking delta y naught less than delta, let us say, yes, less than delta. So, basically epsilon equals to delta works, right? It is not a problem.

If you choose an epsilon, epsilon equals to delta works. So, basically this will be less than epsilon and which is equals to delta. So, epsilon equals to delta works. So, this is stable. This is stable.

Okay? But, but the question is this, but You see, as t tends to infinity, does anything change? Nothing changes because you see the difference is always delta y naught, right? So, it is independent of x. So, in this case, t is x. So, as x tends to infinity, you see this difference.

Let us just say this is a. So, norm of a is always constant, which is norm of delta y naught. It is independent of x. And so, this is not going to 0. So, this is not asymptotically stable. Not asymptotically stable. Stable.

I hope this is clear. I am sorry. I got confused a little bit. So, I hope this is clear. So, you see this is an example of a solution of an equation where every solution is basically stable but not asymptotically stable.

So, you do realize that those two situations are different. Now, let us look at another example. Let us clarify more what happens. So, you see every solution of this equation. Every solution of y prime equals to Pxy, Pxy, okay, is of the form, it's a linear equation, right, is of the form.

So, basically, you find the integrating factor and you solve it here. And once you solve it, what you are going to get, y at the point x0, exponential integral x0 to x, our initial data is at x0, okay. p of s ds this is what you are going to get right now you see that if you just think about it this way yeah We are not now talking about all solutions, yeah? Just one solution.

Let's just see. See, we talked about whether it is stable or asymptotically stable, whether a solution is stable or not, okay? So, here we are saying that, you see, 0 is always a solution, right? If you think about it properly, 0 is always a solution for this problem. You see, for this problem, 0 is always a solution.

So, you see, trivial solution is always there, okay, for the equation. Now, the thing is this. The trivial solution, the trivial solution, is it stable? Do you think it is stable? It is of course stable.

Trivial solution is of course stable. I mean, there is nothing to prove it's less, I mean, it's trivial, right? I mean, what I mean, if the trivial solution is stable, checking that part is trivial, yes? So, it is trivial. So, the trivial solution is stable, is stable, okay?

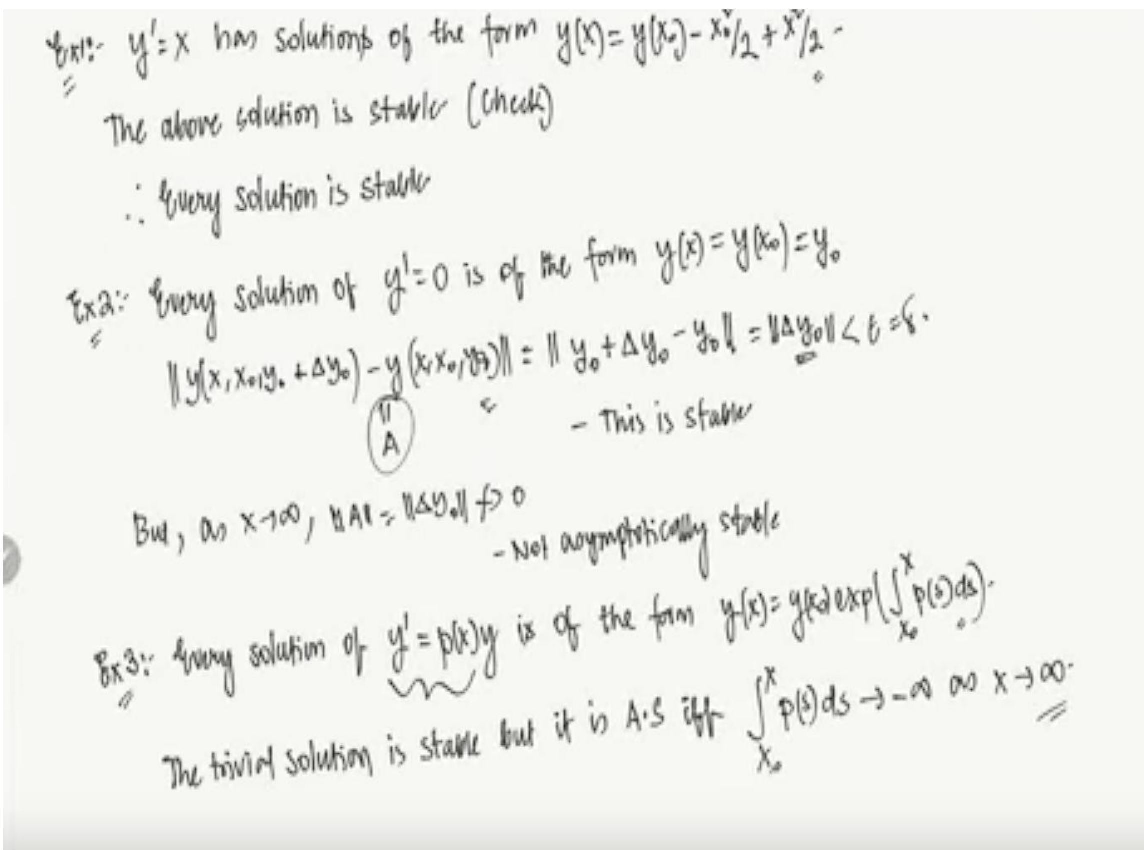
But what about asymptotic stability? But, It is asymptotically stable. Let me write it this way. A is asymptotically stable.

If and only if. When do you think it is asymptotically stable? See, if you write down this difference here. Okay, what is going to happen? It is delta y naught exponential of this particular thing, right?

Now, you see if the exponential of this particular whatever inside that goes to 0, okay, sorry, if that goes to minus infinity, if the exponential of this whole thing goes to 0, then we are in business. So, basically what we want is this, this is asymmetrically stable if and only if x naught 2x P of s ds, okay, this is going to minus infinity as x goes to infinity. I hope this is clear. See, if the inside part of the exponential goes to minus infinity, so exponential of that thing goes to 0, right.

So, basically, I, and then. i have that the difference a the difference a this goes to 0 this is what we want it has to go to 0 as x tends to infinity okay so what we proved is in this case the trivial solution is stable and it is also asymptotically stable if and only this particular condition holds this is okay right Okay, so now let us look at some theorems which will actually guarantee stability. Okay, some theorems. Theorem 1, let us just call it theorem 1.

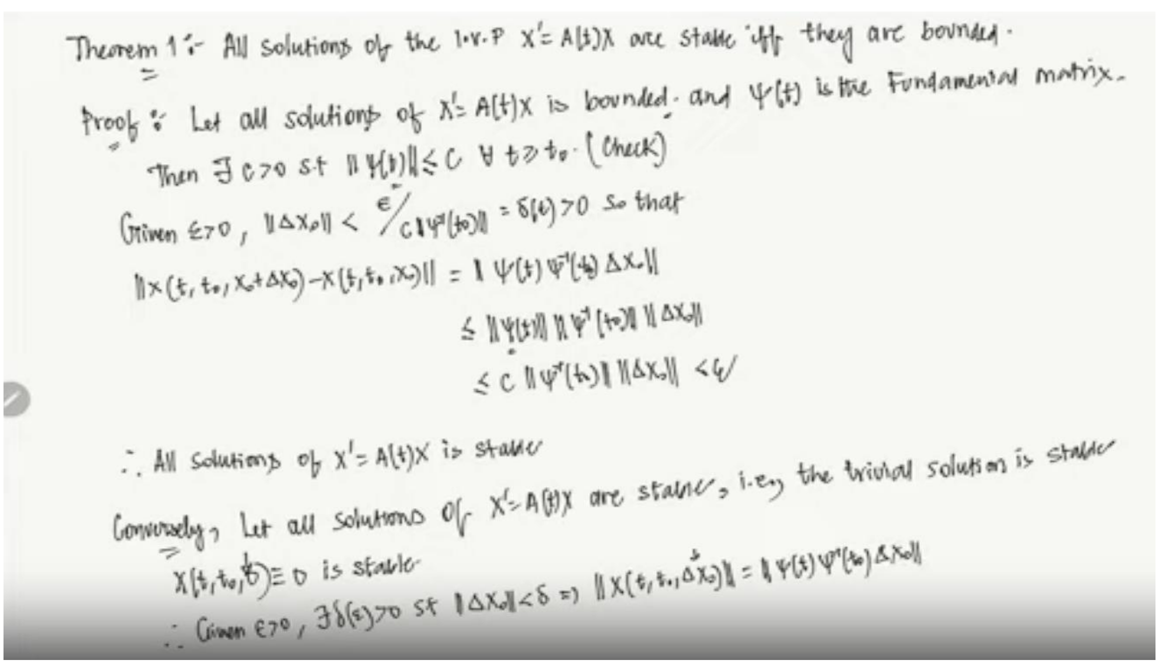
Okay, so what is this theorem? You see, it is saying that all solutions, okay, all solutions. One second, I think it is okay, right? What I wrote? So, let me write down.



You see, all solutions of the differential system of the initial value problem. Let me just put it this way. Initial value problem. What is the problem? x prime equals to 80 times x okay so we are in the realm of linear systems now okay so for a linear system what can we say they are stable because you see here all the examples which we gave are of linear systems essentially yeah so x prime equals to 80 times x are stable when when is this table let's just understand that first okay if and only if if and only if they are bounded

they are bounded so beautiful theorem right so you see the thing is you just have to find out the solutions yes and you just have to make sure that the solutions are bounded yeah if they are bounded they are stable is it okay right so what is the proof so you do understand what i'm trying to say let's say if you have to find which is basically bounded then you can say that their solution is stable and what we hear this theorem says that every solution is stable if and only they are bounded essentially okay right so first of all let's just start with this let's say that i have to show that the solutions are stable right so basically i start with all solutions are bounded so let all solutions solutions of x prime equals to 80 times x is bounded is bounded is it okay now if that is the case we have of course if you you have a linear system right now you remember the fundamental solution right okay so we have a fundamental matrix of this differential system right let and let's say phi of t Is the fundamental matrix. Fundamental matrix of this system.

So, I am not writing all that. Is it okay? I hope you understood. See, if all the solutions are bounded, that is given to us, right? And phi x is fundamental matrix, then there exists, we can say that there exists a constant C positive.



such that the norm of phi of t is bounded by c right less than equal c doesn't matter less than c is less than equal c okay this holds for all t greater than equal t naught okay t naught is our initial data or any point some point t naught so essentially what we are saying is this that if all solutions are bounded phi t is a fundamental matrix so we can actually put a bound on fundamental matrix right why is it true c to be what is phi of t every column of phi of t is a solution of is basically this thing right x prime equals to 80 times x and x at the point what is it t naught let's say is ei yeah so any there is a unique solution xi of t which is defined in whole of r in this case and then i can write your fundamental matrix that's x1 xn right that's the thing which is x1 is the full column right now you see what is the norm of phi You can just look at the Rn norm also. It is not a problem.

The Euclidean norm, exactly the same thing will happen. So basically or you can also talk about the matrix norm. You calculate the matrix norm. You can of course check this part that if each x i's are bounded, then the matrix norm is also going to be bounded. Okay.

So, please check this part. This we also used it in earlier also if you remember while talking about asymptotic behavior. Okay. So, please check this part. This is very easy.

Yes. This is very easy. So, what did I use? The fact is this. Every column of this matrix is bounded.

Yes. It is given. That is the thing. You see it says that every solution is bounded. So, every column is bounded.

You just have to show that the matrix is bounded. Yes. I hope you can do it. Now, you see, so given epsilon greater than 0, given epsilon greater than 0, okay, we can choose delta x naught. Okay, how do we choose?

So, this is just a trick which I am using, but you will see why we can use it like this. See, epsilon by c times norm of phi inverse t naught. This is what we are using. Okay, let us just call it that is a delta which depends on epsilon positive. Now, why we are choosing it?

So that, see, the point of choosing is this. x of t, t0, x0 plus delta x0 minus x of t, t0, x0. If you look at this difference, this difference is nothing but, if you remember, it is a homogeneous problem, right? The solution is phi of t, phi inverse of t and then t naught sorry phi inverse of t naught and then delta x naught that will be the solution right

I hope you remember. Again, let us do it. x prime equals to 80 times x. x at the point 0 is x0. What is the solution? The solution is x0 equals to the fundamental matrix xt times c.

Now, if you calculate it, you see x0 equals to x at the point t0, which is phi at the point t0 times c.

so what is c c is nothing but phi inverse t naught times x naught okay so this difference is phi of t phi inverse t naught so what is x of t x of t is phi of t phi inverse t naught times x naught i hope this is clear now you just calculate this thing this is what we are going to get right okay ah so this is clear now see the thing is this particular thing i can use the matrix norm you remember in the first week itself we talked about this matrix norm so phi of t is bounded by c so i can write it as norm of phi of t and then norm of phi inverse of t naught and then norm of x naught so please remember this is rn norm and these are all the first two are matrix norm this is the rn norm okay right now you see this is bounded by c right so this is less than equals c yeah and i have norm of inverse t naught and norm of delta x naught so you see with this choice of delta x naught if we choose it to be less than epsilon by this particular term then we can show that this is less than epsilon is it okay and therefore what we can say is all solutions all solutions of x prime equals to 80 times x, x is stable. So, I want you to understand one thing, you see here I am using that all solutions has to be bounded.

see if you have this idea that there is some solution which is unbounded we can't do this thing because i in that case we do not know whether the fundamental matrix will be no bounded or not okay so if we can do that of course if you can show that the fundamental matrix is bounded everything can be done it's not a problem okay so essentially what it is saying is we are assuming that all solutions are bounded and then we can say that then all solutions are stable Now, let us look at the converse. Converse is what? Let us say that all solutions are stable. So, let all solutions of x prime equals to 80 times x are stable.

okay so what does it mean it means that the trivial solution is stable that implies that is the trivial solution because you see this is a homogeneous problem right so the trivial solution is also stable trivial solution is stable right it's stable right okay so what is the trivial solution so basically that is x of t t naught and corresponding to the zero data right that is your trivialization so that is zero is stable stable okay therefore what can you say therefore given epsilon greater than zero given epsilon greater than zero there exists a delta okay which depends on epsilon positive such that if i am changing the data the initial data x delta x naught a little bit lesser less than delta that will imply that will imply that x of t t naught delta x naught yeah see this change yeah See, the first initial data is corresponding to 0. What is the same initial data?

0, I am partnering it to 0 plus delta x0. So, the only difference is this, right? So, this difference is nothing but phi of t, phi inverse of t0 and then delta x0. This is what we have. Is it okay?

Now, you see, this is of course less than equal c times, sorry, sorry, what am I doing? Yeah, this is fine, yeah? This one. No, this is not less than, this is the earlier case less than c. Okay, so this is what we have, okay? So, we have that this particular thing is less than epsilon.

Yeah, it is given, right? It is given that it is trivial. So, basically, for epsilon, there is a delta such that this happens, okay? We have this. Okay, now you see this delta x naught, this choice is on us, right?

We can make it as small as possible and we can take any vector we want, right? But sufficiently small that is, okay? So, you see, now choose delta x naught, okay? This we are choosing it to be delta by 2 the e j. What is e j? e j is the jth unit vector,

vector in rf is looking yeah so we are choosing it see if we choose it like this what is delta x naught that will be delta by 2 which is you see which satisfies this right so i can choose this now once you choose it what do you have then we have x of t x inverse of t naught and then delta x naught this yeah we can write it as This is nothing but, let me just write it and then I will tell you what I mean by this. phi j of t times delta by 2. So, here where, what is phi j? Where phi j of t is the jth column of phi t, phi inverse t0.

Is it okay? See, this matrix, we can just multiply it and let us say there will be a jth column. Let us just call that jth column as phi j of t. We will just call it. And delta x0, I can just bound it by delta by 2. So, I can just write it like this.

Now, you see this is less than epsilon. That is given to us. You see, this is given to us. So, we are just writing this part like this. So, we are choosing a particular vector delta x naught such that this is happening.

So, we have this epsilon here. Therefore, what we have? Therefore, it follows that norm of phi t phi inverse t naught okay this will be given by the maximum of phi j of t j lies between 1 and n. That's the matrix norm, right?

So, basically, if you look at this matrix, phi t times phi inverse of t naught, then that matrix norm is given by maximum of these columns. Yes? That's just the definition, right? And that, that can be written as 2 epsilon by delta. Yes?

Is it okay? You see? norm of phi j t is less than equal to epsilon by delta. So, the maximum is also bounded by 2 epsilon by delta. Yes, this is what we can write.

Now, if that is true, then you see, therefore, therefore, any solution any solution x of t t naught x naught of 1 of 1 Looks like this, right? Can be written. Can be. Is of form.

Is of the form. Let me write it this way. Is of the form. How does it look? Okay.

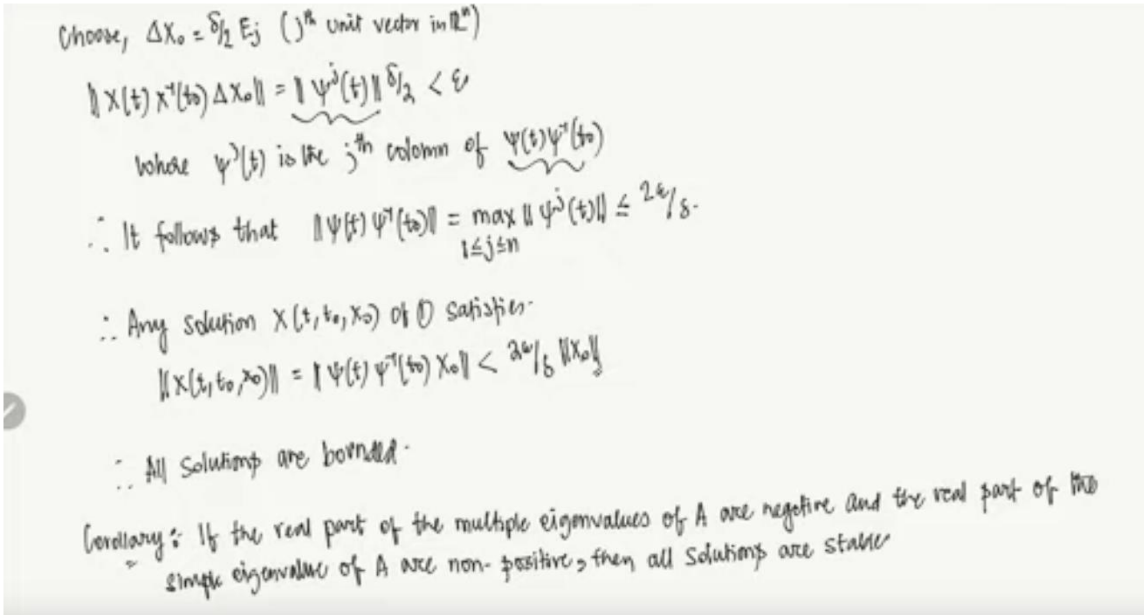
Maybe I can write it as satisfies. It will be better. Satisfies. satisfies norm of xt t naught x naught. This norm is nothing but norm of phi of t, phi inverse of t naught and then x naught.

Now, that the first thing you can take this matrix norm that definition is there. So, we can use that matrix norm inequality and we can write this is definitely less than 2 epsilon by delta times the norm of x0. Is it okay? I can write it like this and therefore, of course, see delta x0 that is bounded. This is some vector.

So, therefore, all solutions are all solutions are bounded are bounded okay so please remember the thing you know which we did here now is this we are saying that for a variable coefficient equation if you have that all solutions are bounded then all solutions are stable and vice versa okay so both of them are true is it okay now you see the thing is there is a small corollary which you can write here See, if the real part of the multiple eigenvalues of A are negative, negative okay so basically when we have this is the case of constant coefficient equation okay and the real part real part of the simple eigenvalue simple eigenvalue okay of a are non-positive are non-positive.

Okay? Then, all solutions are stable. All solutions are stable. Okay? See, first of all, you start with a matrix.

Let us say, or 2 cross 2 matrices and you have two distinct eigenvalues, both of them are non-positive, okay. So, basically negative let us just say, yeah. Now, what happens is you can see that the solution you can write it as C1 times e power lambda 1 t plus C2 times e power lambda 2 t v1 v2, right, with the corresponding eigenvectors. Now, you can see that the difference, if you look at the difference of two solutions with the part of this initial data, you can see that you can make it as small as possible.



So, basically this goes to 0, right, in that case. So, in this case, it is stable, right. Now, what we are going to do is, we are going to give a necessary and sufficient condition. Necessary and sufficient condition for the system x prime equals to 80 times x, this system. To be asymptotically stable.

Asymptotically. Stable. Okay. So let's look at that. So the first thing we proved is when it is stable.

So basically if they are bounded. Now what about asymptotically stable. So the theorem says. Theorem 2. Theorem 2.

Let. phi of t be the fundamental matrix of the system be the fundamental matrix I am not writing the whole thing fm a fundamental matrix of x prime equals to 80 times x is it okay then all solutions let's call this system as star it will be better then all solutions of star are asymptotically stable. Let me write it as asymptotically stable. If and only if norm of phi t, this goes to 0 as t goes to infinity.

Is it okay? We can prove this. So, basically what I am saying is this. When is it asymptotically stable? If the fundamental matrix, the norm of the fundamental matrix that goes to 0.

So, proof. First of all, let us say that this system, star, any solution, how does it look like? So, any solution, solution of star is of the form, of the form x of t, t naught, x naught is phi of t phi inverse t naught times x naught okay now see since phi of t this we talked about it right this is continuous this is continuous right

Why it is continuous? See, phi is what? Phi is the fundamental matrix and every component, so every column of the fundamental matrix that is basically a continuous function. It is basically continuously differentiable, right? Curve.

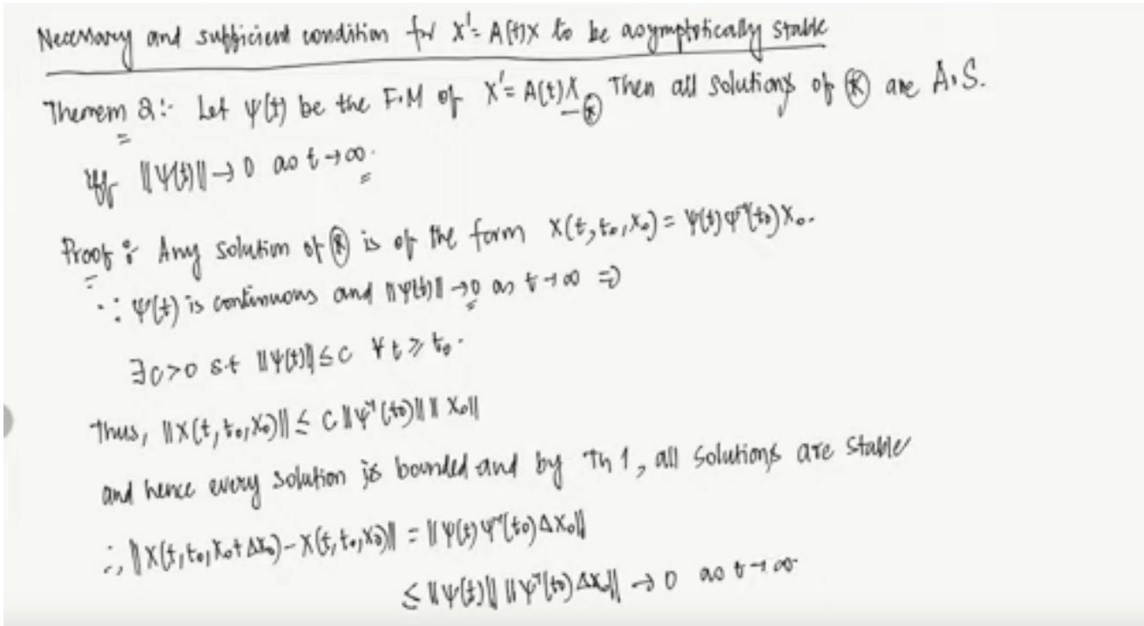
So, phi is definitely continuous. What does it imply? Then that implies since it is continuous and it goes to 0, you see, it goes to 0. And norm of phi t goes to 0 as t tends to infinity. So, that will imply that there exists c positive such that norm of phi t can be made less than equal c for all t greater than equal let us say t0.

Is it okay? See, the thing is, let's say that from t0 to some number t1, it is a closed bounded set. So in that closed bounded set, since v is continuous, it is bounded, right? So you can make it less than equal some bound. And after t1, so since this goes to 0, so after some sufficiently large t1, norm of pt can be made less than epsilon.

So, basically you take the maximum of epsilon and the first bound, then we can say that there is just t such that this happens. I hope this is clear. Now, thus, what do we have? Thus, norm of xt t naught and x naught this is can be written as dominated by c times norm of phi inverse t naught norm of x naught this we can do yes and then and hence every solution

solution is bounded is bounded right see uh once the fundamental matrix is fixed at the point t naught t naught is some fixed point right so inverse t naught that is a fixed matrix so the norm is fixed we know that what the norm is okay that's the constant and norm of x naught is also fixed so therefore what happens is every solution has to be bounded okay now the thing is this since every solution is bounded of this thing we can use theorem one and that will imply an by theorem one zero one all solutions are stable but still okay now what happens now so therefore We have to now again we have to do show that they goes to 0 right the difference. So you see norm of x of t t naught x naught plus delta x naught minus x of t t naught x naught.

This difference is nothing but phi of t phi inverse t naught times delta x naught. right now this is again bounded by phi of t phi inverse of t naught delta x naught here and you see this goes to zero as t turns to infinity is it okay So therefore, every solution of star is asymptotically stable. Again, I am not writing it properly, but A s is asymptotically stable.



Necessary and sufficient condition for $X' = A(t)X$ to be asymptotically stable

Theorem 2:- Let $\psi(t)$ be the F.M of $X' = A(t)X$ —(*) Then all solutions of (*) are A.S. iff $\|\psi(t)\| \to 0$ as $t \to \infty$.

Proof: Any solution of (*) is of the form $x(t, t_0, x_0) = \psi(t)\psi^{-1}(t_0)x_0$.

$\because$ $\psi(t)$ is continuous and $\|\psi(t)\| \to 0$ as $t \to \infty$ $\Rightarrow$

$\exists\, c > 0$ s.t $\|\psi(t)\| \le c$ $\forall\, t \ge t_0$.

Thus, $\|x(t, t_0, x_0)\| \le c\|\psi^{-1}(t_0)\|\,\|x_0\|$

and hence every solution is bounded and by Th 1, all solutions are stable

$\therefore$ $\|x(t, t_0, x_0 + \Delta x_0) - x(t, t_0, x_0)\| = \|\psi(t)\psi^{-1}(t_0)\Delta x_0\|$

$\le \|\psi(t)\|\,\|\psi^{-1}(t_0)\Delta x_0\| \to 0$ as $t \to \infty$.

I hope this is clear. So, you understood this particular term, right? See, phi of t, it is given, it goes to 0 as t tends to infinity, right? So, and this is bounded. So, basically, this is a bounded term and this term goes to 0.

So, the whole term goes to 0. I hope this is clear, yes? Now, what about the converse? Converse. Conversely, let's say that if all solutions, if all solutions are asymptotically stable, are asymptotically stable, so I am writing it as A s, right?

Then the trivial solution will also be asymptotically stable. Then the trivial solution, the trivial solution Okay, what is it? It is x of t, t0, 0. Because the trivial situation will be corresponding to the 0 boundary data, right?

So, that equals to 0 is asymptotically stable. All solutions are asymptotically stable. So, the trivial situation has to be, right? Therefore, what is norm of x of t, t0, delta x0? What is it?

You see, this is nothing but this. This, in that case, goes to 0 as t tends to infinity. Okay? Therefore, so, you see, if this goes to 0 as t tends to infinity, okay, see, the trivial solution is asymptotically stable. So, basically, in the trivial solution, I mean, the difference then must go to 0 as t tends to infinity.

That's just the definition. That will imply this. I should write it like this. So, this is the definition. Now, if this is the case, therefore, what is this?

This is nothing but x of phi of t. So, this is nothing but phi of t, phi inverse of t0 and delta x0. This goes to 0. That will imply this goes to 0 as t tends to infinity. If this is the case, therefore, norm of phi t has to go to 0 as t tends to infinity.

This is no other way. Since this, the whole term, this term, it goes to 0 as t tends to infinity, you can just put it as epsilon delta. So you can just do that. Please check this part. And you can show that norm of Pt can be less than epsilon for a particular choice.

Since this goes to 0, norm of PT can be made less than some particular choice of epsilon. You can do that and you can show that this goes to 0. Since this goes to 0. This is okay. So, hence it is proved.

So, essentially what we will show is the asymptotic stability depends on the norm of the fundamental matrix. If it goes to 0, we have asymptotic stability, otherwise we do not. Now, you do realize that for a constant coefficient equation, if the eigenvalues are basically negative, then what happens is you essentially have asymptotic stability. okay so i'll put it in the assignment also you can check it there uh i'm not writing this as a part of corollary but it is clear right okay so for as a constant coefficient please check when we have asymptotic stability right now what we have is this we will talk about another time of stability and then we are going to end this lecture so the stability is called another new definition definition so we have First of all, stability.

Yes, stability. Stability. Solution stability. Next, we have something called the asymptotically stable. Yes.

Now, we have something called a uniformly stable. So, a solution, by the name of it, you do realize what I mean by this, right? A solution x of t, which we write it as x of t, t naught t. x naught is said to be uniformly stable see this stability as you can understand this is basically a you know we are taking the definition of continuity and we are just morphing it into our needs okay so here what we are going to do is basically we are going to change the definition of continuity to uniform continuity so the solution x t equals to x of t t naught x naught is said to be to be uniformly to be uniformly stable if for each epsilon positive there exists a delta yes please remember this delta depends only on epsilon positive such that such that any solution any solution

I am writing it in a little generalized way. So what I am doing is this. I am writing x1 of t. What is x1 of t? It is given by x evaluated at the point t with the initial point at the point t0 and the initial data is x1. of the problem so please remember the stability exam definition which i am giving i am giving it for a general non-linear equation not only for linear equation of the problem x prime equals to f of tx x at the point t naught is x1 okay uh i should write it like a solution x of t of this problem x prime equals to f of

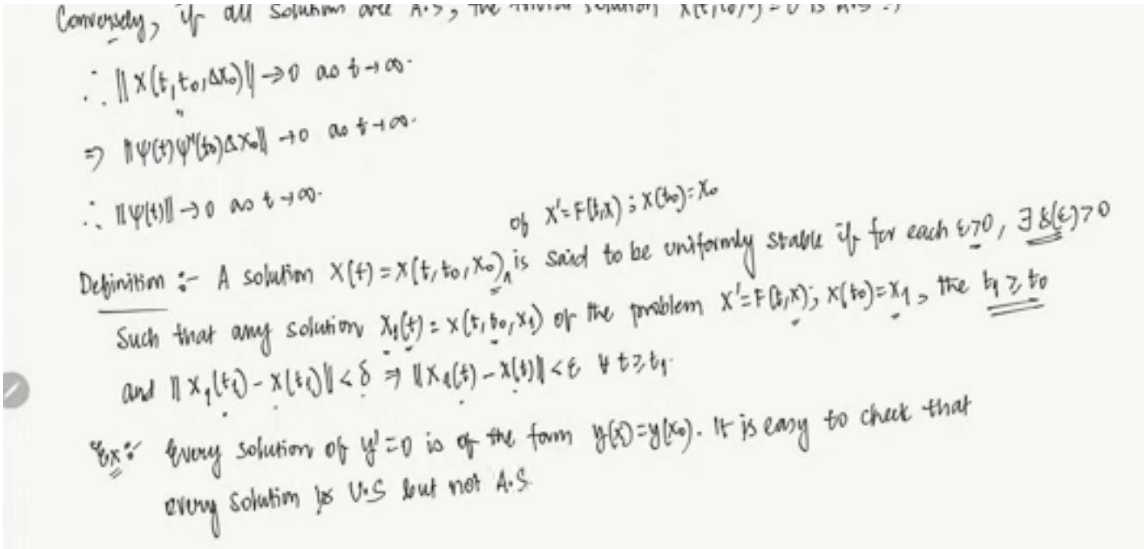
tx, x at the point t0 is x0. It is said to be uniformly stable if with this initial data the inequality the inequality t1 greater than equal to t0 and and if x1 sorry e1 this we called it as x1 we called it as x1 right okay so let's just write it x1 at the point t yeah which is given by this right minus x at the point t this difference is made there's an epsilon for all t greater than equal t1 i hope this is clear Okay, so what we are basically saying is see this is this is just definition of uniformly stable. What we are saying is this, if you have this solution, okay, your initial solution, and then you have the problem, this problem with the initial data as x1.

Then, this solution, the resulting solution with the initial data x1 and the original solution x of t, that can be made less than epsilon. That's all. And this holds for every epsilon. You can actually find a delta such that this holds. I forgot to write one delta area once again.

one second i should write it sorry about it that is why okay so and norm of x1 x1 i'm sorry x1 of t1 minus x of t1 less than delta implies this. See, if T1 is confusing you, what you do is you choose T1 to be T. T1 to be T0. No problem here. So, what we are doing is this.

I am saying that you can change your initial point also. Not only initial data, you can change the initial point also. But for now, let us just assume that if T1 equals to T0. In that case, the difference of those two, if it is less than delta, then the difference of the solution can be made less than epsilon. okay for all t greater than equal t naught why it is uniform because the delta which you are going to get that will be independent dependent only on epsilon and nothing else okay so let's look at an example that will make life easier for us right so every solution every solution

of y prime equals to 0 okay is of the form is of the form y x is given by y at the point x naught right if it passes through the point x naught it has to be y x equals to y l so do you think it is uniformly stable exactly this is actually uniformly stable it is very easy to see right so it is easy to check so please check this part up easy to check that every solution is uniformly stable is uniformly stable uniformly stable yes please just write down the definition and see that it is uniformly stable this is very easy to do i am not doing that but we have already seen that this is not asymptotically stable but not asymptotically stable Is it okay? So, this is an example where the thing is, of course, if it is uniformly stable, it has to be stable. That is there.



But the thing is, it may not be asymptotically stable. Is it okay? Yeah. So, let us look at, this is the last theorem which I am going to do before we finish this particular lecture, the theorem 3. Okay.

What is that thing? So, let phi of t be the fundamental matrix, fundamental matrix, I will write it as f of m, fm, fundamental matrix of The system x prime equals to a t times x. So, for linear systems, let us just understand when we have a sufficient and condensatory condition for some e system to be, for all solutions of the system to be uniformly stable. So, let phi t is a fundamental matrix. Then, all solutions...

of x prime equals to 80 times x right all solutions are uniformly stable are uniformly stable if and only if okay you need a different condition now so the condition which you are going to need is the following so you remember in the last thing not last or maybe some two, three lectures back, we talked about a particular condition, okay. So, the condition will look like this. See, if, you know, leave norm of phi t, phi inverse of t, sorry, sorry, sorry, you know, phi of P, phi inverse of S, this is less than equal to C for P.

greater than equal t naught, less than equal, or maybe I should write it as s, s less than equal t, and which is less than infinity. You remember this thing we talked about when we talked about boundedness, right? You remember? Also, if it's bounded, that is the condition, right? When we talked about asymptotic stability.

So, this is that same condition. So, when is it uniformly stable? If and only if this particular condition holds. Okay. So this is the hero.

Right. Now the thing is what I am going to do is this one part I will leave it for yourself. So basically the thing is check this part. This you have to do it yourself. Check.

That if all solutions are uniformly stable. If all solutions. Solutions are uniformly stable. Uniformly stable. Okay.

Then. Then the trivial solution. Previous solution. Is. Uniformly stable right.

Yeah. So use this. Use this. To show. That.

This. This condition. P of T. P inverse of S. T S. P of T. P inverse of S. This. Is bounded. So, please do this part.

I want you to do this part yourself. I will show you just this part that what happens if this condition is holds. Then I will show that they are uniformly stable. So, conversely this part I will do it. Conversely let us say that let x of t which is given by x of t t naught t.

and x naught solves x prime equals to 80 times x. Is it okay? Then for, you see, t1 greater than equal to t naught, we have to show that t1, t naught, you see, for t1 greater than equal to t naught and this particular delta condition, we have to show some, that is, x1 minus x has to be less than epsilon, right? So, for t greater than t naught, you have x of t i can write it as phi of t phi inverse of t1 and then x of t1 right i can do it like this see this is our new initial data at the point t1 right now if x1 I wrote it x1 like this or I wrote it like this.

Let us write it like this. x1 of t which is in our case you see x1 of t satisfies this thing this problem with initial data at the point x1. So if x1 of t satisfies this thing phi of t inverse of t1 and then x1 at the point t1. Is it okay?

So, is any other solution, is any other solution and let us call this as double star condition and double star holds. What can we say? What we can say is this, see, x1 of t minus x of t, this is nothing but, this can always be dominated by phi of t, phi inverse of t1. This is common in both, right? And then I have x at the point t1 minus x1 at the point t1.

right i can say this yeah now you see this can be made less than equal c times because this is less than c see this is given less than c times x at the point t 1 minus x 1 at the point t it is okay i hope yes now see that this holds for all T0 dominated by T1 which is dominated by T which is finite. Thus if epsilon greater than 0 that is given right epsilon greater than 0 then T1 greater than equal to T0 and the initial data x1 at the point t1 minus x at the point t1 less than epsilon by c which we will choose it as delta positive implies norm of x1 of t minus x of t is less than epsilon, right?

See, this is what we need to see. x1 of t1 minus x of t1 less than delta, you see? We chose it like this. So, delta we choose it like epsilon by c. If we do that, then of course, if this is epsilon by c, c and c gets cancelled out. So, it will be less than epsilon here.

And hence, what happens is, then the solution, therefore, the solution is uniformly, solution is uniformly Uniformly stable. Is it okay? Yes. So, it is not very difficult to actually prove something is uniformly stable.

This is how you do it. Yes. You just use the definition. That is all. It is not very difficult.

So, with this I am going to end this video.