Ordinary Differential Equations (noc 24 ma 78)

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Week-05

Lecture 35

Hello students, in this video we are going to talk about object function expansion. So, let me start with some preliminaries and then we will see what exactly I will try to do here. So, first of all see thing is in Rn let us say that let let E1 into en okay so this be the basis of r right basis of r is it okay and then if that is this is the case then every let's say X in Rn.

Okay. For every X in Rn. So there exists. There is a unique choice. There is an unique choice.

Unique choice. Of constants. Of constants. Okay. What is the constants?

Alpha 1. Alpha 2. Alpha n. For which. For which. x equals to summation i equals to 1 to n alpha i e i. Is it okay?

I can write it like this. Now, you see, let us say that if such a representation is already given, or let us say if we have given any, you know, eigenvalues, given a basis, and if we want to find out what the a i's, alpha i's are, okay, how do you do it? See here what we are using is this. Now, since e i's, are orthonormal for all i between 1 and n. What you can do is this.

You can actually see what is the action of x on ei. is it okay and what is that that will be nothing but the summation alpha i e i acting at e i okay i goes to 1 to n and that we can use the linearity property of the drop product and we can write it like this right so this is nothing but summation of alpha i e i acting at Maybe I can write this thing as E j that will be better. So basically I want to find alpha j that is. So E okay this is fine and then E j is okay.

now if we do that so i equals to 1 to n okay so i'm just taking any j between 1 to n i'm just writing it like this okay so if i do that then you see e i e j essentially what is happening is this i should write i here e i e j you see for i equals to j it is 1 otherwise this is going to be 0 so that will actually give you for me the coefficient alpha j And this holds for every j between 1 and n. So, hence the unique representation given orthonormal basis. Hence the unique representation given an orthonormal basis. orthonormal basis is x can be written as summation. You see the coefficients is nothing but x acting at ej and times ei, i equals to 1 to n.

So, that is your unique representation and that is how you write any element x in terms of its normal orthonormal basis that is. Now, you see we want to generalize this idea. Is it okay? So, basically there is a given orthogonal set of functions in some interval alpha beta. Yes.

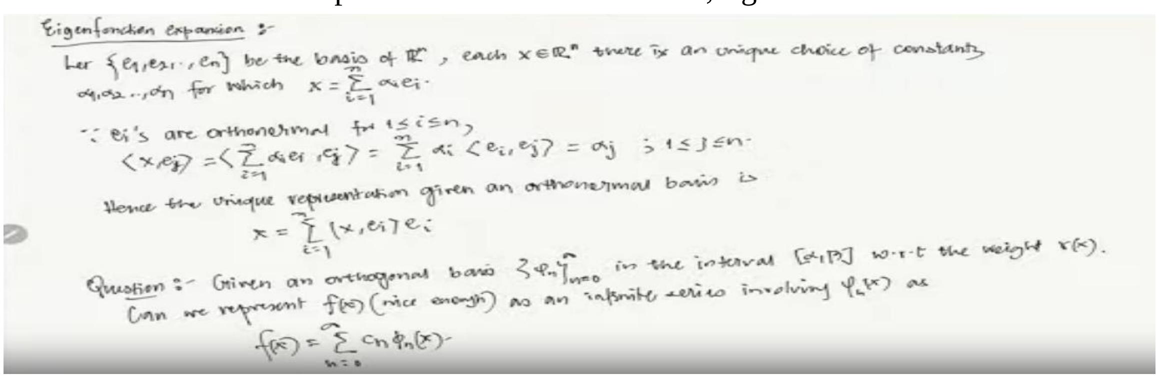
With respect to some weight let us say rx. Yes. Can we represent an arbitrary function? So, the question now is this question. Question is this.

given okay and ortho so here it is orthonormal basis right so let's just start with the orthogonal basis of course we can make it normal so let's start with the orthogonal basis now okay so orthogonal basis basis and what is it phi n okay phi n and any n equals to 0 to infinity. So, the orthogonal basis given like this, orthogonal basis, yes, in the interval alpha beta, in the interval alpha beta, is it okay, with respect to the weight, with respect to some weight, okay, with respect to the weight rx, weight rx, is it okay. Now, other thing is this, can we represent Can we represent another function?

Let's say f of x. Nice enough. Of course, you can't expect that you can represent any function like that. But nice enough, let's say. Yes, we don't know what nice enough means right now. But let's just, I mean, you know, it's a nice function, exponential sine cosine, that sort of function.

Can we do that? That's the question. As an infinite series. as an infinite series, infinite series involving phi n's, involving phi n's, phi n of x. Is it okay? So, essentially f of x should look like this.

You see, summation n equals to 0 to infinity cn phi n of x. Is it okay? See, if you remember, why are we writing this as infinite series? Because the orthodontic sets which we are getting from the Sturm-Liouville problems. We talked about it, right?



Eigenfunction expansion :-

Let $\{e_1, e_2, \ldots, e_n\}$ be the basis of $\mathbb{R}^n$, each $x \in \mathbb{R}^n$ there is an unique choice of constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ for which $x = \sum_{i=1}^{n} \alpha_i e_i$.

$\because$ e_i's are orthonormal for $1 \le i \le n$,

$$\langle x, e_j \rangle = \langle \sum_{i=1}^{n} \alpha_i e_i, e_j \rangle = \sum_{i=1}^{n} \alpha_i \langle e_i, e_j \rangle = \alpha_j \; ; 1 \le j \le n.$$

Hence the unique representation given an orthonormal basis is

$$x = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$$

Question :- Given an orthogonal basis $\{\varphi_n\}_{n=0}^{\infty}$ in the interval $[\alpha, \beta]$ w.r.t the weight $r(x)$. Can we represent $f(x)$ (nice enough) as an infinite series involving $\varphi_n(x)$ as

$$f(x) = \sum_{n=0}^{\infty} c_n \varphi_n(x).$$

So, if you have a regular Sturm-Liouville boundary value problem, yes, you are going to get, let us say, that... For a distinct eigen values right. So you are going to get no eigen functions and those eigen functions are orthonormal to each other right. Yeah that we know yes and the thing is this is like an infinite series right. So basically you see there is a way of actually producing orthonormal basis.

Now the thing is the question is this can we use that idea to represent any function. Is it okay? So, let us understand this what happens without you know going into I mean proper mathematics let us just quickly understand that how should we formally proceed. So, how to formally proceed to formally proceed without convergence without the question of convergence. question of conversions what does it mean see the in the earlier case okay when we were in rn it was like a finite number of

points you see this this series this series which we are talking about okay summation alpha e i this is like from i to one i from one to n right so it's a finite series in case of finite series you really don't have to worry about convergence because it is like a finite sum right but once we have like infinite kind of cases then we have to talk about convergences We do not know whether this actually converges or not. So what we are going to do is basically without the equation of convergence, let us just assume that all of these converges, they are nice, everything is fine. And can we find the coefficient cn? That is just the question.

So what we do is this. See, basically such that fx must look like summation n equals to 0 to infinity cn phi nx. Can we do that? Yes, let us just understand that. So what we do is let us just call this as 1, 1.

So, we multiply, we multiply 1 with R of x times Pm of x. Is it okay? We just multiply it and integrate between alpha to beta and integrate between alpha to beta. to beta. Is it okay? Yes.

What do we get? You see, if you multiply between, integrate between alpha to beta, r of x, phi m of x, f of x, right, dx. That is what we are going to get. I am multiplying it by r of x, phi m of x, clear? So, that will be written by alpha to beta.

Summation, n equals to 0 to infinity, cn, r of x, phi n of x, phi m of x dx. Is it okay? See, similar sort of thing what we are doing here. Here we are multiplying by ej, right?

That is my basis kind, the orthonormal basis. Now, here I am multiplying with my orthonormal basis element phi n, but there is a weight involved, right? r of x. So, basically we are incorporating that r of x also here, right? right so once we do that you see the thing is here there is a problem if i want to take this integral inside this summation there is a problem but the thing is we are just assuming that we can do that let's just assume that yes so once you do that then what happens is this see from here what can you write this is summation n equals to 0 to infinity yes and this integral is from alpha to beta here r of x phi m of x, okay, phi n of x dx, yes, and that is nothing but alpha to beta, okay, r of x, phi m of x, f of x dx, is it okay?

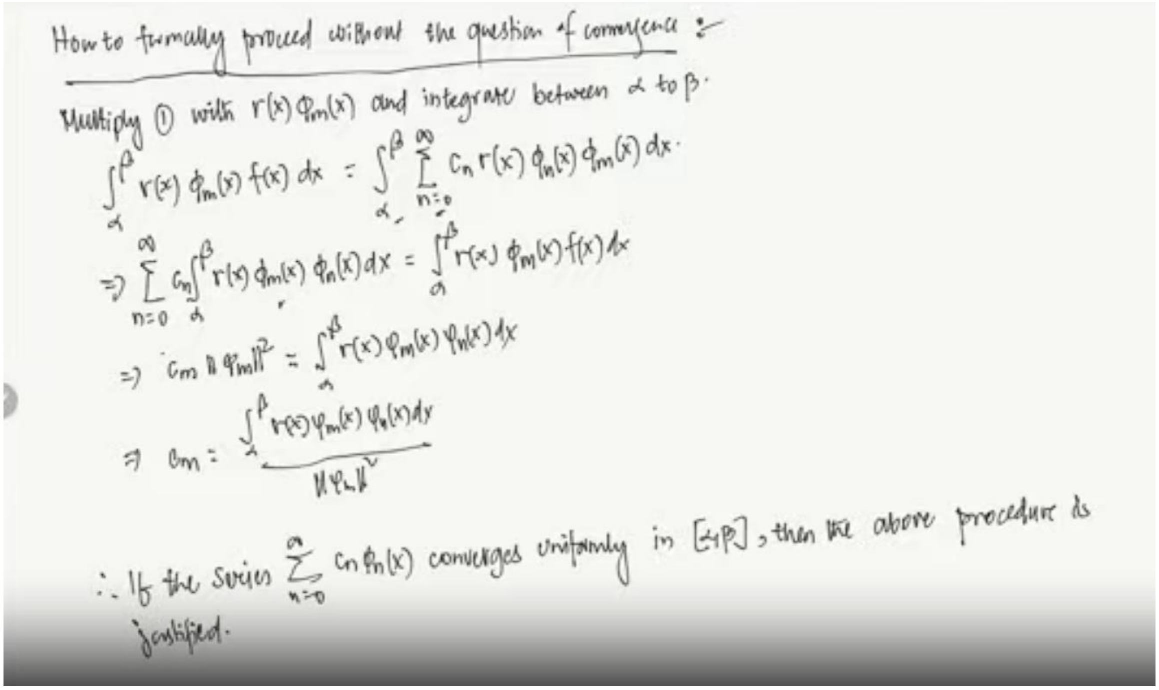
yes and by the way what about d this is nothing but you see this is a let's say this is the orthonormal basis right phi n's are not so normal so basically for i n not equals to m we are have basically we have the integral alpha to beta it is orthonormal with respect to the weight r right so basically r phi m phi n the integral of that is going to be 0 for all n not equals to m yes so the only part which is remaining is m equals to m so in that case it is just the r of x norm of phi m square, yes, between alpha to, okay, one second, sorry, sorry, sorry, it is not r, sorry, I made a small mistake somewhere, right, that cm is, I did not write again, I am really sorry about it, let me just write it, this cm should be there, no, cm, cm, let us just write it, cm, Okay. Now, you see, the thing is, this is the only thing which is remaining is cm times norm phi m square. Is it okay?

And then this part, what we have is, this is alpha 2 beta r of x phi m of x phi n of x dx. Is it okay? So, therefore, what we have Cm is nothing but alpha 2 beta r of x phi m of x phi n of x dx. by norm of phi m square, okay. Phi m is orthonormal, sorry, orthogonal, right.

So, phi m square is of course not 0, yes. And if it is orthogonal, so basically that will be 1 in that case, yeah, if we start with that, okay. So, in that case phi m will be just a to b, alpha to beta or phi m phi m x, okay. So, see the thing is therefore, See, if this operation, the first operation, the integral and the sum, okay?

If we can, you know, interchange that. So, basically, when can we do it? So, basically, if the series, the series, what is the series? Summation n equals to 0 to infinity cn phi n of x, okay? If it converges uniformly, if it converges uniformly, okay?

In alpha beta, then the above procedure is justified. Procedure is justified. I hope it is clear to you, very, very simple, very, very simple, okay. Now, the thing is this, see, the coefficient Cn which you are going to get, this Cm which you are getting, those coefficients, if, see, if, let us say, all of this is possible, so that this convergence, convergence uniformly and everything is fine, then what we are going to do is we are going to call this thing as the Fourier coefficient, Fourier coefficient of The function f of f of x. Is it okay?



So, now let us look at a problem and then we will go from there. So, first start doing a problem. So, exercise example 1. So, you see, we know that the set, this set 1 cosine n pi x by l sine n pi x by l And L positive n is greater than equal to 1.

This is orthogonal. This is orthogonal with respect to what is the weight? rx equals to 1. rx equals to 1 okay uh okay first of all how do you show this orthogonal of course you can take the integral and show that that's going to be 0 for n not equals to m or it is going to be not 0 for n equals to m okay that of course you can do the other way of doing it is basically you see y double prime equals to y you remember that problem we did y at the point with a periodic solution this thing with a periodic data yes and we have seen that these are the eigenvalues right yes sorry this is the eigen functions and we know from that theorem which we did in the earlier class also that those eigen functions going to be orthogonal so you have this set which is orthogonal you can do that yeah so this is orthogonal in minus l to l so we did it for minus pi to pi but for minus l to l this is what you are going to deal with okay now um

Of course, then what happens is minus L to L, minus L to L that is cosine square n pi x by L dx. This is nothing but 2L and L. This is for n equals to 0 and this is for n greater than equal to 1. And minus L to L sine square L. n pi x by l dx. This is l for n greater than equal to 1.

What about n equals to 0? It is basically 0. So, you do not worry about it. So, in that case, you see, by going by this earlier idea, what happens is this f of x. So, given any f. So, see, we define something like this. So, the Fourier series, the Fourier series,

Fourier series of f of x, f of x is defined to be, is defined to be f of x, okay, it will look like this, a naught by 2 plus summation n equals to 1 to infinity a n cosine n pi x by l plus bn sine n pi x by l. Is it okay? We can write it like this. Where, let me just write it. What is an or an?

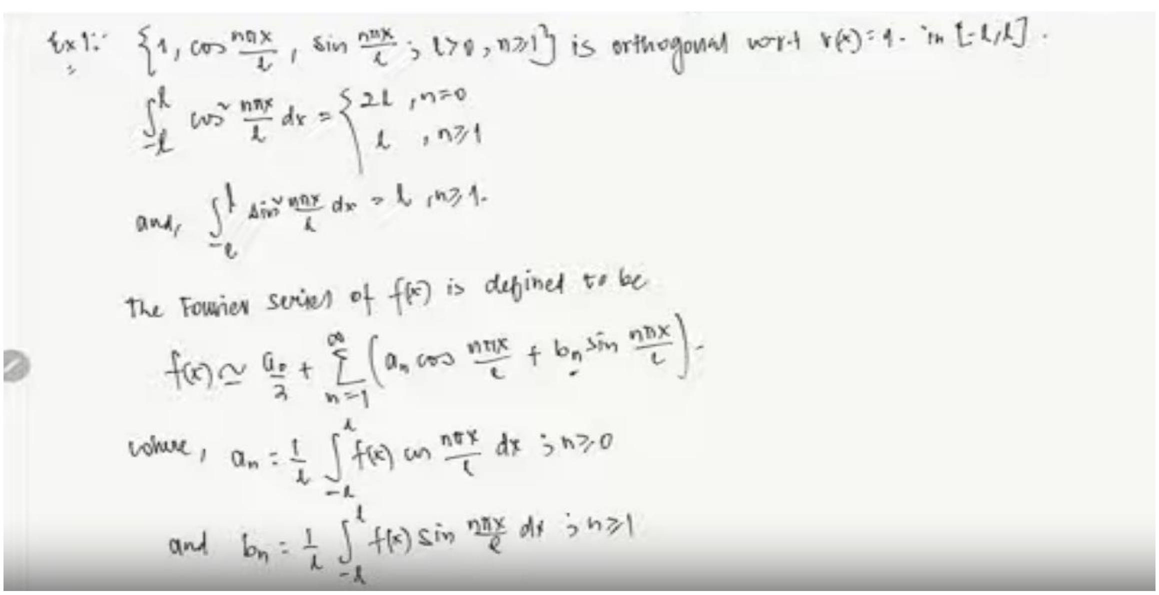
an is 1 by L. Why 1 by L? Because you remember this part is there, phi m square. What is phi m here? They are like sines and cosines, right?

So, a n is basically that cosine part. So, basically that 1 by L thing is there. It is phi m square. So, phi m square cosine square. So, it is, let me write it, it is 1 by L f x

cosine n pi x by l dx, okay, n greater than equal 0. Is it okay? Let me just once check if everything is fine what I wrote. a n is 1 by l. Yes, this is final. Minus l to l. And let us write b n. So, if you can see, if we do the exact same procedure, b n, this b n will be in respect to, you know, sine.

So, in that case, it will be again 1 by l sine. And this is minus l to l. And it will be just a multiplication. So, let me write it. So it is sine of n pi x by l dx. So that is there.

Now the thing is this. See the thing is what we will do is this. So this is fine. So if you do it like this you can just write down the Fourier series of f of x. So by the way this Fourier series in particular is called a trigonometric Fourier series. Now, the thing is this.


Ex 1: $\{1, \cos\frac{n\pi x}{l}, \sin\frac{n\pi x}{l}; l>0, n\geq 1\}$ is orthogonal w.r.t $r(x)=1$ in $[-l,l]$.

$$\int_{-l}^{l}\cos^2\frac{n\pi x}{l}\,dx = \begin{cases}2l, & n=0\\ l, & n\geq 1\end{cases}$$

and, $\int_{-l}^{l}\sin^2\frac{n\pi x}{l}\,dx = l, n\geq 1.$

The Fourier series of $f(x)$ is defined to be

$$f(x)\simeq \frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right).$$

where, $a_n=\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx;\ n\geq 0$

and $b_n=\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx;\ n\geq 1$

You see, as I told you already that all of this is valid provided the Fourier series converges. So, how are you going to show that this Fourier series actually converges? And so, for this, you know, seeing this theory to work what we are going to do is we are going to assume so the convergence of Fourier series let us just write it like this we are talking going to talk about the convergence of the Fourier series Fourier series okay so see the thing is first of all we are going to start with some phi n's so let phi n N equals to 0, 1, 2 and f of x. B are piecewise continuous.

Piecewise continuous in alpha beta. I know, I hope you guys all know what piecewise continuous in alpha beta means. It basically means that there are finite number of points. Okay. But they may not be continuous except that everywhere they are going to be continuous.

So, that is what we call all the piecewise continuous. Okay. Okay. Now, you see, let and we will call define S capital N of x. This we are going to define it as the sum of summation n equals to 0 to capital N. The first n plus 1 term of Cn phi n of x. Is it okay? Okay.

yes, and we donate it by capital SN of x, yeah. And now, you see, we consider, we consider the difference, we consider the difference SN of x minus f of x, okay. This difference for large values of n, large values of capital N. Is it okay? Now, you see, if, if given epsilon greater than 0, given epsilon greater than 0, there exists an integer n epsilon, which is positive, okay, such that mod Sn of x minus f of x

can be made less than epsilon, then the Fourier series, the Fourier series converges uniformly, right, uniformly, uniformly to f of x. for all x in alpha beta. Is it okay? So, the idea is this. See, essentially what we are saying is this.

We are approximating f. Yes. And how are we doing it? We are going to use f of vx. Okay. In such a way that we are basically taking the finite sums.

And after that, we are actually saying that for large n of n, what happens is this. The difference between sn and f can be made arbitrarily small. Right. And if that holds for all x, that is important, then we say it converges uniformly. Is it okay?

And of course, if n depends on, so let me make a small remark here. So, you guys already know this, but I still wrote it in case of, since there should not be any confusion here about the definition. So, what is the remark? See, n, the capital N. Okay, here depends only on epsilon, depends only on epsilon, only on epsilon and not on x. Yeah, and not on x, not on x. So, if it depends on x also, then we basically say this is a point wise converter.

Okay, right. So, now let us look at a definition. Let me just write down the definition and then I will try and explain what I am trying to do here. So, let us write down a definition. Okay.

So, we are going to start with some function phi n of x. Okay. So, let phi n of x n greater than equal to 0. Of course, natural. Yes. And phi of x are piecewise continuous.

Piecewise continuous. in alpha beta okay we say we say the sequence phi n of x okay this converges in mean this is very important okay converges in mean this is the See there are different kind of convergences. This convergent is convergent in mean 2 phi of x. 2 phi of x with respect to the weight function r of x. The weight function r of x. See here I am incorporating r of x. If you want you can just take r of x to be 1.

It is not a problem. In alpha beta. This is the meaning. See, what we are saying is this. The norm of phi n minus phi is that if you take the square of that norm and then you take the limit n tends to infinity.

That is, let me just write down what is the definition. That is even by limit n tends to infinity. What is this norm? This is nothing but the integral alpha to beta r of x. phi n of x minus phi of x square dx.

This is going to be 0. Yes, we talked about this sort of norms and all right. If you remember in the first lecture itself, we talked about it. So, we are using that norm. So, this norm is defined by integral r phi square.

So, if this goes to 0, then we say that the, you know, the Fourier series converges to f of x essentially. So, therefore, The Fourier series, the Fourier series converges in mean, in mean to f of x, f of x provided, provided limit n tends to infinity and alpha to beta r of x. Sn of x minus, you do not have to put a mod here, it is basically the same, square is there, minus f of x square dx, square dx, this should go to, then we call this convergence and the convergence in mean. So, now, see, Convergence of Fourier Series :-

Let ϕ_n ; $n=0,1,2,\ldots$ and $f(x)$ are piecewise continuous in $[\alpha,\beta]$.

Define, $S_N(x) := \sum_{n=0}^{N} c_n \phi_n(x)$.

we consider the difference, $|S_N(x)-f(x)|$ for large values of N., if given $\epsilon>0$, $\exists\, N(\epsilon)>0$ s.t. $|S_N(x)-f(x)|<\epsilon$, then the Fourier series converges (uniformly) to $f(x)$ for all $x\in[\alpha,\beta]$.

Remark :- N depends only on ϵ and not on x.

Definition :- Let $\{\varphi_n(x)\}_{n\geq 0}$ and $\varphi(x)$ be piecewise continuous in $[\alpha,\beta]$. We say the sequence $\{\varphi_n(x)\}$ converges in mean to $\varphi(x)$ (w.r.t the weight function $r(x)$) in $[\alpha,\beta]$ if

$$\lim_{n\to\infty}\|\varphi_n-\varphi\|^2 = \lim_{n\to\infty}\int_\alpha^\beta r(x)\left[\varphi_n(x)-\varphi(x)\right]^2 dx = 0$$

$\therefore$ The Fourier series converges in mean to $f(x)$ provided $\lim_{N\to\infty}\int_\alpha^\beta r(x)\left[S_N(x)-f(x)\right]^2 dx = 0$

Is it okay? Now, you see, the thing is, we will see a small thing that what happens, okay, if there is a possibility of representing f in terms of a different series. Is it okay? So, basically, let f of x, okay, f of x is represented, is represented by summation n equals to 0 to infinity dn phi n of x, okay, where phi n are not necessarily, are not necessarily Fourier coefficients, okay, Fourier coefficients

of fx. Is it okay? You understand what I am trying to say, right? See, it may happen that there is a different representation which is basically not a Fourier representation and we can write it like this, okay? So, now, and we want to see what happens if we do something like this, okay?

So, now, you know, define. Let us see what happens. another finite sum, dn. What is the sum? It depends on x, of course, and then d0, d1, dn, okay?

Just to specify this particular thing, that it depends, the dn is a finite sum with respect to the coefficients d1, d2, dn, right? And that is given by summation n equals to 0 to infinity dn times phi n of x, okay? Now, and let us say we define capital E, sorry, E capital N, okay, that quantity is nothing but the norm of Tn minus F, okay, we want to see what is the difference, Tn minus F, the norm, if we calculate it, let us just call that Tn, I want to calculate what that is, okay, now you see, since phi n's are orthogonal, phi n's are orthogonal, are orthogonal, this way is given to us, right, orthogonal, Then, what is E n square?

Let us just calculate what is E n square. E n square is nothing but summation dA capital N minus f square. Is it okay? And that is given by, by definition, what is it? It is alpha to beta r of x and then summation dN phi n of x, n equals to 0 to infinity minus f of x square dx.

Is it okay? Yes, we can write it like this. Now, you see, we can just break it up, yes? Let us just break it up and write it. So, this is nothing but summation n equals to 0 to infinity, okay?

dn square, dn square, alpha to beta, r of x, phi n square, phi n square, and dx. The first term, right? Whatever the last term, let us just write down the last term now, plus... alpha to beta, r of x, f square of x dx. The last term and then the middle term, two every term, minus 2 times summation n equals to 0 to infinity dn integral alpha to beta, r of x, phi n of x, f of x dx.

Let me just check once if everything is fine or not. Yeah, this is okay, right? This seems everything is fine. Okay, now you see what exactly is this term? This is nothing but phi n square, right?

So, this is nothing but summation. I am not writing 0 to infinity. It is all over. Sorry, sorry, sorry, sorry. What am I doing?

Sorry, sorry. This is not 0 to infinity, right? This is 0 to n. Sorry, I am doing it with e n. So, this is 0 to n. That is why I can change the limit. Sorry, yeah. This is my mistake.

Where is it? Sorry about it. 0 to n here. So 0 to n I am not writing. So that is given.

Because it is a finite time, I can take the integral inside. So it is not a problem. So this is dn square. dn square and phi n square. That is just this thing, definition.

Minus 2 times summation dn. dn cn norm of e n square okay plus f square is it okay so i will just write down everything in terms of norm so that is given by and of course the integral is from zero to n okay let me just write zero to n zero to n okay so if this is true you see i can write this as Summation 0 to n, this is nothing but norm of phi n square dn minus cn square. dn minus cn square minus sum n equals 0 to n phi n square cn square. plus non-phi n F square.

I hope everything is fine. Let me just once check. Yes, this is fine. Yes, 2 times this and plus F square. I think it is fine.

So, you see, if we can write it like this, now the thing is this. You can see that the quantity E n, E capital N, when is this least if dn equals to cn. So, therefore, capital en this quantity is least so basically the norm of capital en is least this if cn equals to dn for any 1 to 0 1 2 right yeah and therefore therefore see the thing is what we proved is the following let me just write it there so we prove this theorem theorem We proved the theorem.

Let me write down the theorem first. So, for a given non-negative integer n, so given n in n, the best approximation, the best approximation in mean very important in mean, to a function f of x, f, okay, x, by an expression, what sort of expression? 1, by an expression of the form summation dn phi nx, okay, is obtained is obtained when the coefficients when the coefficients dn are the Fourier series are the Fourier series of f of x.

f of x. I hope everything is clear here. See, what we did in this part is this. fx, we wrote it as summation dn fn. dn is not necessarily the eigen, sorry, not necessarily the Fourier coefficient of f of x. And what we did is this, we calculated en square, which is nothing but the finite sum with respect to those coefficients dn minus f square. So, in the mean, they will converge in the mean.

So, you see, we have to talk about this particular term. That is what we are doing here. Now, you see, e n square is nothing but summation d n minus f square, right, which we can write it like this. And now, you see, this is a positive term, yeah. So, when is e n least?

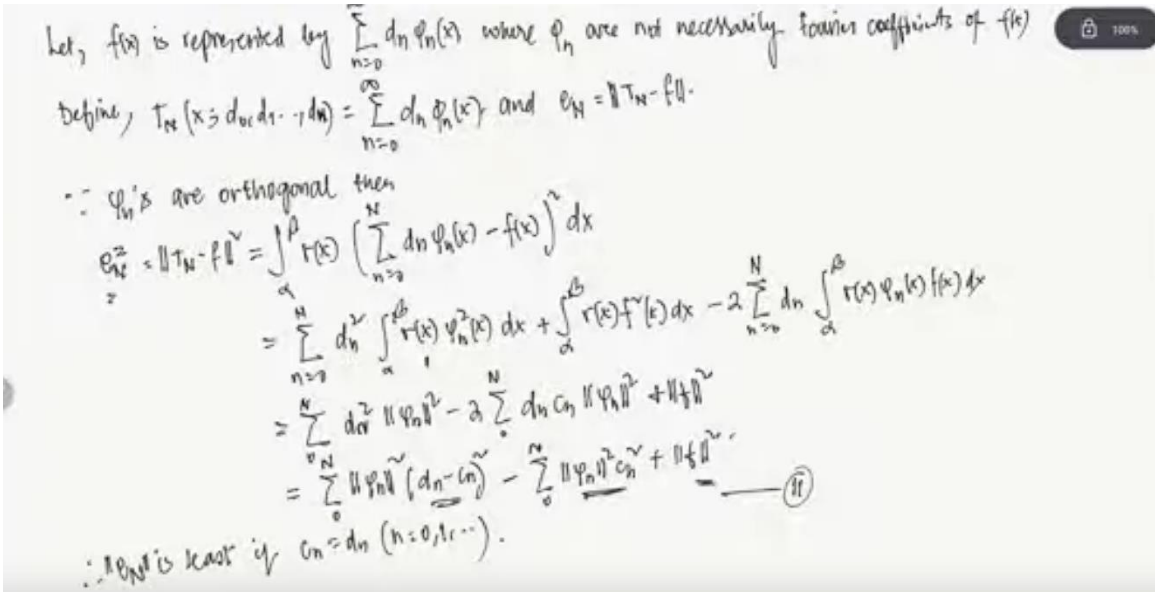
e n is least if d n is equal to c n essentially, okay. So, that is what. So, what we proved is this, that basically if you want to write f of x in terms of a Fourier series, looks like this, n equals to 0 to infinity d n f e n x, then the best, you know, the best Candidate for the coefficients has to be the Fourier coefficients. Is it okay?

Right. Now, you see, the thing is, let us say that here, let us just call this thing as 2. Let us just call this thing as 2. Now, into, into, if dn equals to cn, okay? It is a Fourier coefficient.

They are Fourier coefficients. So, n equals to 0, 1, capital N. to obtain, to obtain, okay, one second, sorry, this, I should write it as a finite, we did it for 0 to n, capital N, right, for any capital N, we can read, okay, so into, if dn equals to cn, what do we obtain, we obtain that summation sn minus f square, okay, this is nothing but norm of f square minus summation n equals to 0 to infinity, phi n square c n square, right? That is what we are going to get c. This t will be s in that case because the coefficient d is basically e i's, okay?

Sorry, d and c are basically same. d n's and c n's are basically same. So, this t n is basically s n, right? Okay. Now, so what is the norm of s n minus f square?

So, that is nothing but this one and this one. And this is gone because d n equals to c n. Is it okay? That is what I wrote it here. So, now, therefore, what we can say is this norm of Pn minus F square. That is nothing but summation 0 to capital N norm of Pn square Dn minus Cn square.



That is what we actually showed. and then S capital N minus F square. We can write it like this, yes. The last two terms is in terms of S N minus F, we can write it like this, okay. Now, if we do that, see, since this is a positive term, all of these are positive, okay.

So, we can say 0 will be dominated by norm of S N minus F, which is again dominated by norm of T N minus F. right. This is a positive term, we can throw it away. So, S n minus f square will be dominated by T n minus f square and hence this inequality holds, okay. Now, you see, if therefore, if summation n equals to 0 to infinity, now it is infinity, yes, d n phi n x, if this converges uniformly, converges, sorry, not uniformly, in the mean, in the mean to f of x, in the mean,

to f of x, okay, then the Fourier series converges, right, then the Fourier series, because this is with respect to the Fourier series, right, the Fourier series converges in mean, converges in mean to f of x, to f of x, okay, okay. But then let us just call this thing as 3. But then 1, 2, 3 implies that limit n tends to infinity. Summation 0 to capital N phi n square dn minus cn square. This equals to 0, right?

Yes? See, the Fourier series converges to f of x in mean, right? So, basically this is 0, yes? And of course, this also is given to be, it converges. So, that is 0.

So, basically, we have limit entrance to infinity. This particular term is going to 0. Is it okay? And this is, and when is this possible? See, this is a positive term.

This is a positive term. I am taking a finite sum. And then I am taking the limit. I am telling that that is 0. It only happens if dn equals to cn.

Therefore, the above is only possible. The above is possible. possible only if dn equals to cn okay for n equals to 0 1 2 it goes on like this is it okay yes and what we proved now is essentially this thing so let me write it here what we proved is this theorem theorem 2 what is this theorem so if a series of the form series of the form summation 0 to infinity dn phi n of x converges in mean, converges in mean to f of x, is it ok? If it converges in mean to f of x, then the coefficients, then the coefficients

Theorem 8: For $N \in \mathbb{N}$, the best approximation in mean to a function $f(x)$ by an expression of the form $\sum_{n=0}^{N} d_n \varphi_n(x)$ is obtained when the coefficients d_n are the Fourier coeffs of $f(x)$.

in (11) if $d_n = c_n$ $(n = 0, 1, \dots N)$ to obtain

$$\|S_N - f\|^2 = \|f\|^2 - \sum_{n=0}^{N} \|\varphi_n\|^2 c_n^2 .$$

$$\therefore \|T_N - f\|^2 = \sum_{0}^{N} \|\varphi_n\|^2 (d_n - c_n)^2 + \|S_N - f\|^2 \quad \text{(III)}$$

$$\Rightarrow 0 \le \|S_N - f\| \le \|T_N - f\| .$$

$\therefore$ If $\sum_{n=0}^{\infty} d_n \varphi_n(x)$ converges in the mean to $f(x)$, then the Fourier series converges in mean to $f(x)$

but then (III) implies $\lim_{n \to \infty} \sum_{0}^{N} \|\varphi_n\|^2 (d_n - c_n)^2 = 0$

$\therefore$ the above is possible only if $d_n = c_n$; $n = 0, 1, 2, \dots$

coefficients dn must be the Fourier coefficients coefficients of f of x. Okay. See, we just saw, right, that if, I mean, if you have something like this, that dn converges to f of x in mean, then it has to be the Fourier coefficient, dn has to be cn. So, basically, that is what we showed, that if the series converges to f of x in mean, then it has to be the Fourier coefficient. So, the series converges, it is basically the Fourier series, that is what, okay, right. So, now, also, let us see, also,

Yeah, so this is proved, this is proved. Now, let us look at a different thing here. See, the thing is from this inequality, from this inequality, I do not know, I wrote it, yes, here. Let us just call it star. Also, from star, from star, what do we have?

We have 0 is Sn plus 1 minus f is dominated by Sn minus f, right? See? See, star. So, basically, if you are taking n plus 1, there is another term. So, I, we are basically have a decreasing sequence, right?

Non-increasing sequence that is, okay? So, thus, thus, you know, this sequence norm of Sn minus f, okay? And n equals to 0, 1, this, this sequence is non-increasing and bounded, is non-increasing, okay? and bounded okay is non-increasing and bounded okay below by 0 of course below by 0 by 0 okay what does it mean it means that it must converge right and if it converts to 0 then of course the Fourier series fx converges to the Fourier series of f converges to f of x in mean right See, of course, right, if s n minus f, this sequence, they are basically convergent.

If it converges to 0, so basically in that case, we just say that the Fourier series converges in mean. That's there. Okay. So, also, also, we will have this. See, summation n equals to 0 to infinity phi n square c n square is dominated by norm of f square.

Is it okay? Yeah, that is true. Now, because of this, right? See, because of this, this is always dominated by, this always dominates 0. So, basically phi n square minus c n square is dominated by f of f square, right?

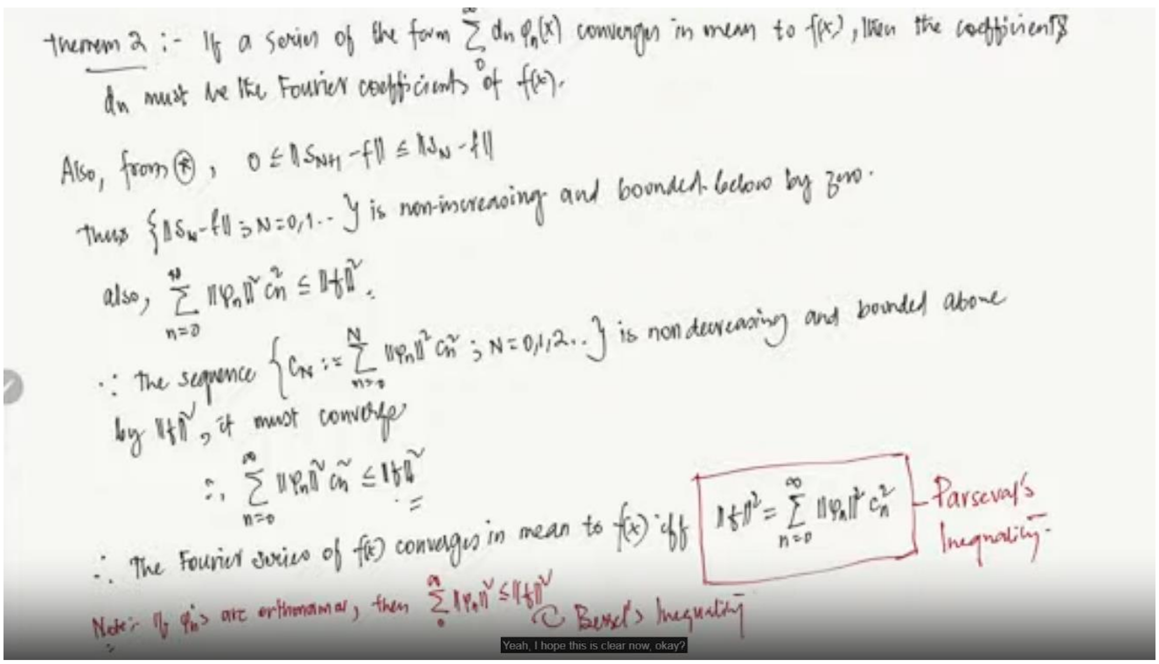
Okay, now you see the sequence, if I, now since the sequence, the sequence is Cn, what is Cn? Cn we will define it to be summation n equals to 0 to capital N, okay, free n square, Cn square, okay, this sequence and n equals to 0, 1, 2, yes, is non-decreasing and bounded, is non-decreasing and bounded. above by bounded above by norm f square it must converge it must converge here and therefore, what we have is summation once again I did a mistake again here this should be up till n because all of those earlier calculations are up till n so then it must follow that zero to infinity norm of phi n square cn square must be dominated by normal square yeah this is non-decreasing at this hold for all n so basically i can write it as the the update the the sum is from 0 to infinity right we can do that now you see

Therefore, what we can say is therefore, the Fourier series of f of x converges in mean to f of x. If and only if this is true, right? If and only if this and this is true, right? So, summation F square is equals to summation n equals to 0 to infinity phi n square C n square. I hope this is clear.

This convergence if and only if norm F square has to be equals to this. So, Now, this inequality what we wrote here, this is very important. This inequality is called the Parseval's inequality. So, this is called, how do I put it?

Maybe I can put it in a red. This inequality is called Parseval's inequality. inequality is it okay yeah and this particular inequality this inequality you see if phi n's are basically you know orthonormal so let me put it this way small note if phi n's are orthonormal orthonormal then this thing implies the summation phi n square 0 to infinity is always dominated by f square, okay? And this is called Bessel's inequality.

Bessel's inequality. Yeah, I hope this is clear now, okay? So, let us put everything in summary and I will put it in red so that it becomes much clearer what we did up till now, okay? So, let me put it as this theorem. theorem so this is about fourier uh this thing series huh so let we we start with phi n of x okay phi n of x and this is for n equals to 0 1 this thing okay b r be an orthonormal set or to normal set orthonormal set is it okay and let c n c n be the fourier coefficient



Fourier coefficient of f of x, of f of x given by f of x is summation cn phi n of x. Is it okay? So, this is between n equals to 0 to infinity. Yes. Okay. Then the following holds.

Then the following holds. following poles what is that so it is a the series this is summation 0 to infinity cn square okay converges converges this is what we showed right you see it is dominated by norm f square so basically uh this series converges cn square okay and what does it mean it means that limit n tends to infinity cn equals to limit n tends to infinity integral alpha to beta r of x phi n of x f of x dx. This is going to be 0. See, the thing is here that norm of phi n square is not there because phi n is taken to be orthonormal.

Now, b, the second thing is the basis inequality holds. Basis inequality, what we wrote this last page, this inequality holds. Basis inequality holds, holds. And C, what is the C which you have? You have that the Fourier, the Fourier series of f of x, of f of x converges in mean, converges in mean.

Mean 2 f of x. Is it okay? If and only if the passable inequality holds. If and only if passable inequality holds. Is it okay? Yes.

So, this is the idea of Fourier series. So, basically what we have is this. Let me conclude what we had and why, where is it coming from. See, we know that if you have a regular or a periodic sum level boundary value problem, then basically corresponding to the distinct eigenvalues, we are going to get a, you know, series. So, basically the infinite series of ortho

functions, right, which we are basically eigen, we call it an eigen function. And the thing is, the thing is, since they are orthogonal, okay, that set we can start with and we can actually formulate a Fourier series out of it, yes, and then we can represent any f with respect to those Fourier series according to whatever theory we just developed. So, that is the idea. So, basically what happens is if you want to produce, you know, different Fourier series, where are you going to get the supply of f of x? Those supplies are actually, we can get it using the sum level boundary value problems.

So, with this I am going to end the video.

So, if you have a regular Sturm-Liouville boundary value problem, yes, you are going to get, let us say, that... So, if you have a regular Sturm-Liouville boundary value problem, yes, you are going to get, let us say, that.

Theorem: Let $\{\varphi_n(x); n = 0, 1, \ldots\}$ be an orthonormal set and let c_n be the Fourier coefficient of $f(x)$ given by $f(x) = \sum_{n=0}^{\infty} c_n \varphi_n(x)$. Then the following holds:-

(a) $\sum_{0}^{\infty} c_n^2$ converges and $\lim_{n\to\infty} c_n = \lim_{n\to\infty} \int_a^b r(x)\varphi_n(x) f(x)\,dx = 0$.

(b) Bessel's Inequality holds

(c) The Fourier series of $f(x)$ converges in mean to $f(x)$ iff Parseval's Inequality holds.