Ordinary Differential Equations (noc 24 ma 78)

Dr Kaushik Bal

Department of Mathematics and Statistics

Indian Institute of Technology, Kanpur

Week-05

Lecture 34

So welcome everyone and again in this video we are going to talk about periodic sum level boundary value problem okay so in the last video we talked about regular sum level boundary value problem right this is also a similar sort of thing a little different okay and let me define the what i mean by this problem so basically again we have the same equation so consider consider the problem Problem. What is the problem? It is P Y prime whole prime plus Q Y plus lambda R Y equals to 0. Yeah.

So, this is the problem which we have to remember. We are assuming P Q R to be C 1. You don't need it to be, but let's just do it. It is not a problem. And and

P and R, we will assume it to be positive in whatever the domain is, alpha, beta. Let us say in the interval which we are talking about, alpha, beta. Yes? And the boundary data. And in this case, you see, what we are going to do is a little different.

Since we are talking about periodic stimulable boundary value problem, okay, what we will do is we take P of A equals to P of B. This is the condition on P. Along with P and R positive, P is sufficiently smooth. P has to be equal to, will be equal at, sorry, AB means, this is AB. Let me just put it in A, B, AB. Is it okay?

And And what is the boundary condition? The boundary condition is given by this y at the point A equals to y at the point B. y prime at the point A equals to y prime at the point B. So, you see, it is a beautiful condition, right? Basically saying that y and y prime at the point A is basically the same with y and y prime at the point B. So, basically y A equals to y B, y prime A equals to y prime B. Yes, the position and the speed of the particle, let us say at the point A,

yes is same as the position and the speed of the particle at the point b that's what it is saying so basically it is coming back to this original configuration somehow so that is uh this type sort of problem is called periodic sum level boundary value problem okay so what i'm going to do is uh start by uh working out an problem And then we look at some properties. Yes. OK. So the first problem what we have, which we have is C. Let's say this equation is there.

So for the question is this for lambda in R. OK. Solve this equation. Solve y double prime. Plus lambda y equals to 0. Yes.

And periodic conditions, right? So, see, p equals to 1 here. p at the point a and p at the point b is same, basically. So, y at the point 0 equals to y at the point pi. And y prime at the point 0 is y prime at the point pi.

Right. That's your periodic condition. Yes. Now, we want to solve this problem. Yes.

What do you do? So, first of all, case 1. Again, why I chose lambda to be R? Here the question is lambda to be R. You do not have to worry about C, right? Now, of course, you can think of C and all, but the thing is, again, we will prove our theorem, which will actually guarantee, like, you know, the last video we talked about, right?

There also we have shown that the real eigenvalues are going to be real. Here also the same sort of thing will happen. So, you really do not have to worry about that. So what is the case 1? Case 1 is let lambda is 0.

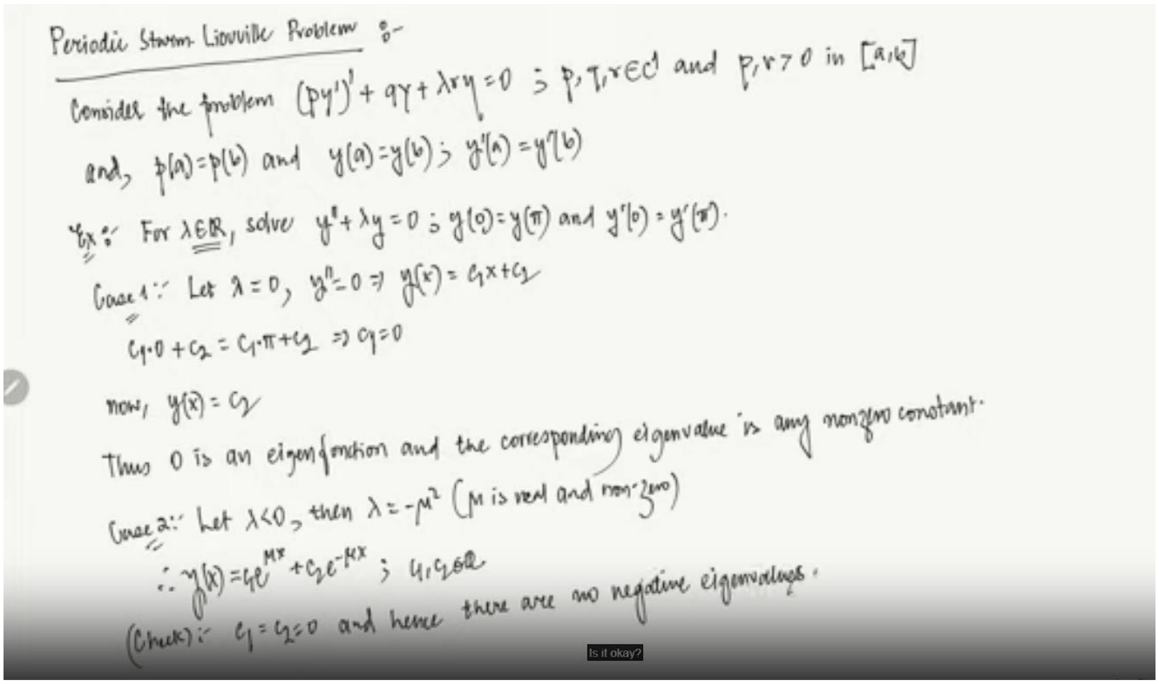
Lambda 0 case. So in that case, y double prime equals to 0. And the solution that will imply what is y of x? The general solution will look like c1 times x plus c2. Now you see y0 equals to y pi.

So basically c1 times 0 plus c2 equals to c1 times y. pi plus c 2 right so what does it give you c 2 is c 1 pi plus c 2 so that will give you c 1 has to be 0 there is no other option okay so that will give you c 1 is 0. now if c 1 is 0 y prime at the point a equals to y prime at the b what is y prime y prime now y of x, let us say, will look like c1 of, sorry, c2 of c2. y prime should be only c2, right? c2.

Now, that will satisfy y prime of 0 equals to y prime at the point pi. Because it is a constant, y prime is 0. So, basically, 0 function at the point 0, 0 function at the point pi is basically 0. Right? So, it satisfies.

So, what are the eigenvalues in this case? So, basically, in that case, it is basically any arbitrary eigenvalue. So, it is 1, eigenfunction. Okay? So, we can think of it as 1.

You can take any other number. It is not a non-zero constant, basically. You can take. So, thus, we can say that 0... is an eigen function eigen function okay and what is the eigen value and the corresponding eigen value is basically any non-zero constant and the corresponding eigen value eigen value.



is any non-zero constant zero constant is it okay yes so simple right okay let's look at case two yes what is the case two case two is let lambda is negative yes so if lambda is negative then then lambda equals to minus mu square we can write it right and mu is mu is real and non-zero real and non-zero. Non-zero. Is it okay? Now, if that is the case, then what is the general solution?

Therefore, the general solution is given by yx equals to okay so y x in this case is looks like e power c 1 times e power mu x plus c 2 times e to the power minus mu x right and c 1 c 2 is in r yeah now given those boundary data so please check this part this is easy so i am not doing this so the boundary data if you incorporate it that will imply c 1 equals to c 2 equals to 0. Is it okay? And hence, what do you have? Yes.

So, the only solution is to be a solution in this case here. And hence, what happens is there are no negative eigenvalues. There are no negative eigenvalues. Is it okay? Right.

Okay. So, now let us look at this. What happens if there are positive eigenvalues? So, case 3. Case 3.

So, let lambda positive. Okay. Then what do you have? Lambda is mu squared. Okay.

Mu is real. Mu is real and non-zero. Is it okay? And what is the general solution? General solution is given by yx equals to some constant c1 times cosine mu x plus c2 times sine of mu x. Is it okay?

Now, y satisfies the boundary data. So, y satisfies the boundary data. The boundary data. Data. If and only when is it satisfies y0 equals to y pi.

So, y0 equals to y pi will give us a sine mu pi plus b 1 minus cosine mu pi is equals to 0. And we will also get a y prime 0 equals to y prime of pi. The row should be reversed. So, a 1 minus cosine mu pi. Okay, minus b sine mu pi is 0.

These are the two which we are going to get, right? Now, this system, let us call this system as 1. Now, the system 1, the system 1, okay, admits a non-trivial solution. I want it to admit non-trivial solution. Otherwise, you know, there are no non-trivial solutions basically, okay, sorry.

What am I writing? What is A, B? This is C1, C2, C2, C1. So, the system admits non-trivial solution. Non-trivial solution.

If the determinant, what is the determinant? Sine of mu pi 1 minus cosine mu pi 2. And 1 minus cosine mu pi minus sine of mu pi. This is going to be 0. That will imply that cosine of mu pi has to be equal to 1.

That will imply that mu has to be equal to plus minus 2n. Is it okay? So, n is in n. Yes. So, what is lambda? Therefore, lambda has to be equal to 4n square.

Yes. n is in n. Yes. So, what are the eigenvalues? The positive eigenvalues. Therefore, the positive eigenvalues are 4n square.

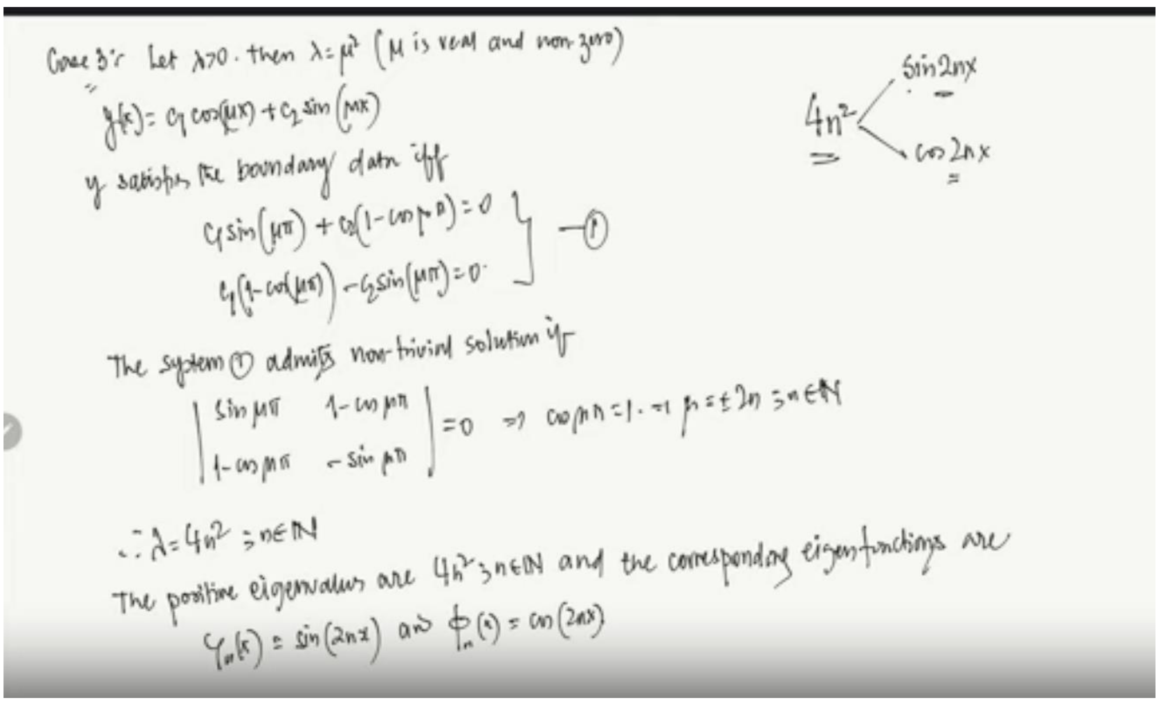
Okay. n is in n and the corresponding eigenfunctions and the Corresponding eigenfunctions are, let us say, phi n of x is sine of 2nx and phi n of x is cosine of 2nx. Is it okay? Yes?

see the thing is this has a non-trivial solution okay so there is a non-trivial set c1 c2 both are not simultaneously zero that is okay such that i mean this equation has i mean you know there is a non-trivial solution so basically In that case, what is the solution? Either it is sine and cosine. So basically, if you put c1 to be 0, c2 to be 1, you have sine. And then for c1 to be 1, c2 to be 0, you have cosine.

So basically, you have two linearly independent eigenfunctions for each eigenvalue. You see, the motivation behind giving you this particular, you know, how do I put it? example is the following. In this example, what we have seen is this, for let us say 4 n square is an eigenvalue for some n, right, n whatever it is, 5, 10, whatever. Now, you get two eigenfunctions, sine of 2nx and cosine of 2nx.

And we know that sine of 2nx and cosine of 2nx, these two are linearly independent. Yes. So, for each eigenfunction, you are going to get two linearly independent eigenvalues. That actually says that this eigenvalue, sorry, eigenfunction. So, that actually says that this basically that eigenfunction 4n square is not simple.

Yes, but you know for a regular stumbly wheel boundary value problem we have seen that these sort of eigenvalues are always real. Sorry, simple. Yes, simple. But in this case it is not going to be simple. Yeah, so this is one major difference between regular and periodic stumbly wheel boundary value problem.



Let me make a small remark here. Yes, remark. Remark. The eigenvalues. of a periodic Stumleyville boundary value problem are not real.

Sorry, what am I saying? They are not real and are not simple. They are real, but they are not simple. Now, the thing is, why they are real? That is also we need to understand.

So, basically what I am going to do is this. I am going to write down a theorem with two parts. Okay. So, the thing is, the eigenvalues, the eigenvalues, values of star, this equation star, the original equation, what is the equation?

Yeah. So, this is, let us say, let us call this star. Okay. The eigenvalue star, if any, if any, are real okay so basically they are not simple but they are going to be real yes what is the proof you use the exact same sort of proof we did for regular sound level boundary level problem only you remember you just use that the boundary data is basically different use that fact the boundary data is different right use that boundary data and you show that they are going to be really exactly the same sort of you know

Proof works. Just a little tweak is required. I hope you can do it yourself. So, this whole theorem is for you to check. Okay.

Right. And the thing is I proved another theorem. Right. Okay. The second part.

The second theorem. So, what about that, you know, orthogonal thing. Yes. So, that also holds here. So, what is the theorem?

The theorem says the eigen function eigenfunctions, okay, of a periodic stumbleable boundary value problem, periodic stumbleable boundary value problem, okay, corresponding to distinct, corresponding to distinct eigenvalues, eigenvalues, are orthogonal orthogonal with respect to r in a b is it okay so basically what i'm trying to say is this see there are basic three theorems which we prove the properties yeah other than the existence three theorems we prove we prove for the regular sum we will boundary develop problem the first one is the eigenvalues are going to be real if there exists they are real so that holds for both regular and periodic yes Okay. What about the orthogonality of distinct?

So, basically, let us say you have two distinct eigenvalues corresponding to that two distinct eigenfunctions. Yes. You can actually show that they are going to be orthogonal. You can show it for the regular or the periodic strum-double boundary value problem. Okay.

Now, what about the... the other property whether they are simple or not this is with this example we have shown that this is not true for periodic some limit boundary value problem is it okay right now uh another uh this thing theorem which we want to do so this is about that theorem existence theorem very very important again this is again we are not going to prove this theorem if you want you can just look it up in the corinthian book okay so um what does the theorem say so basically as regular, regular, sorry, not regular, periodic, periodic, periodic Sturm-Liouville boundary value problem, okay, has an, has an infinite sequence of eigenvalues. Okay, you see, you remember for regular sum level boundary level problem also we showed that there are like an infinite sequence of eigenvalues. Okay, here also the same thing lambda n, lambda n, n is in n. Okay.

satisfying this particular property. So, basically, you know that monotonic property is also satisfied here. So, basically, minus infinity can actually show that this goes on doing this thing. Lambda 1 is less than lambda 2. Yes.

So, there are at least two there. And then they may repeat themselves. So, less than equal lambda 3, less than equal, less than equal lambda n, less than equal, this goes on. And now the thing is the first eigenvalue. The first eigenvalue, eigenvalue lambda 1 is simple.

Simple. Okay. This is always true. The first eigenvalue lambda 1 is going to be simple. Okay.

And the number of linear and this, the number, number of linearly independent eigenvalues linearly independent Li, eigenfunctions, eigenfunctions corresponding, corresponding to any eigenvalue mu, to any eigenvalue, eigenvalue mu, is equals to, is equals to the number of times mu is repeated in the above listing. See, this is different here. You remember, if you think about it, in the, for the regular sum level boundary value problem, this like distinct, lambda 1 less than lambda, sorry, less than lambda 2, which is less than lambda 3, which is less than lambda 4, it goes on like this, such that lambda s goes to infinity.

That is what it was given there, in the regular sum level boundary value problem, okay. In this case, however, What happens is this and what happens is lambda goes to infinity also. You remember lambda goes to infinity. For the periodic sum level boundary problem you do not say something like that.

What you can say is of course there is a lambda 1 which is of course greater than minus infinity. and which is strictly less than lambda 2 that is there that we know but after lambda 2 it can be all the same you know everything is lambda 2 lambda 2 lambda 2 it will go on like this yeah and if it happens so basically you see they may you know I mean, they may recur itself. So, lambda 2 can be 2, twice, thrice, 5 times. That can happen.

Now, the thing is, what you can say is, let us say mu is there. Mu is any one of them, lambda i, something, mu. And corresponding to that mu, let us say mu is actually repeated 10 times in this scheme. So mu is 10 times repeated. Then what happens is the number of eigenvalues, linearly independent eigenvalues corresponding to that mu will be 10.

Is it okay? So basically they are going to be same. How many times it is repeated? Yes. So here also you see what is the eigenvalue.

You see here the eigenvalue is 4n square. This is a periodic sum level boundary level problem, right? In this problem, the eigenvalue is 4n square, yeah? And this is repeated twice. So basically there are two, sorry, the eigenvalues are basically two.

So basically 4n square, right? Okay. So the thing is how many linearly independent eigenvectors should be there corresponding to this 4n square eigenvalue. So it will be sin nx cosine nx. So basically it is 2.

You see, since it is, you know, this actually in the scheme, if you think about it, it is coming twice. So that is why there are like two linearly independent eigenvalues. It is okay. okay so that is there now the thing is i will finish this particular video with the problem which i want you guys to do it yourself okay so let's look at an exercise this is a this is for you to do it y double prime plus lambda y equals to zero okay y at the point pi equals to y at the point minus pi And y prime at the point pi equals to y prime at the point minus pi.

Okay. So, this what you want to do is find the eigenvalues and eigenvectors. Eigenvalues and eigenvectors. Okay. Right.

So, with this I am going to end this video.