Ordinary Differential Equations (noc 24 ma 78)

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Week-05

Lecture 33

So, welcome students to this second part of Stumbleville problem and in this part we are going to continue our study of Stumbleville problem. So, let us write down the problem which we have in our hand p x y prime whole prime plus q x y plus lambda rxy equals to 0 right that's the problem which you have and here what is the assumption here you see p q and r are smooth okay smooth that is differentiable continuously differentiable with with p and r positive this is what our assumption is in the interval i which is alpha beta Now, lambda is given with the parameter, right? That is just the given quantity.

You do not have to worry about it. And the question is, okay, the initial data is there, right? Sorry, the boundary data. And the boundary data, okay, what is the boundary data? The boundary data is a naught y at the point alpha plus a1 y prime at the point alpha equals to a, let us say, and...

uh i have assumed d naught right y at the point beta plus d1 i think yes this is what we used actually from the last lecture beta y prime at the point beta this is because this is let's say maybe i can take a equals to yeah i think i can take that it is better to do that let's take 0 0 i think it is better to do 0 0 huh let's do it for this so boundary data it is this is given to us right that's the boundary data which we are working with right now okay so the problem is called so again let's just call the whole thing as star and this is called a regular strongly with boundary value problem we will denote it like this okay Now, what did we, we learned two different, you know, properties of the eigenvalues, okay, and eigenvectors. And the third property, now we will start here. So theorem 3.

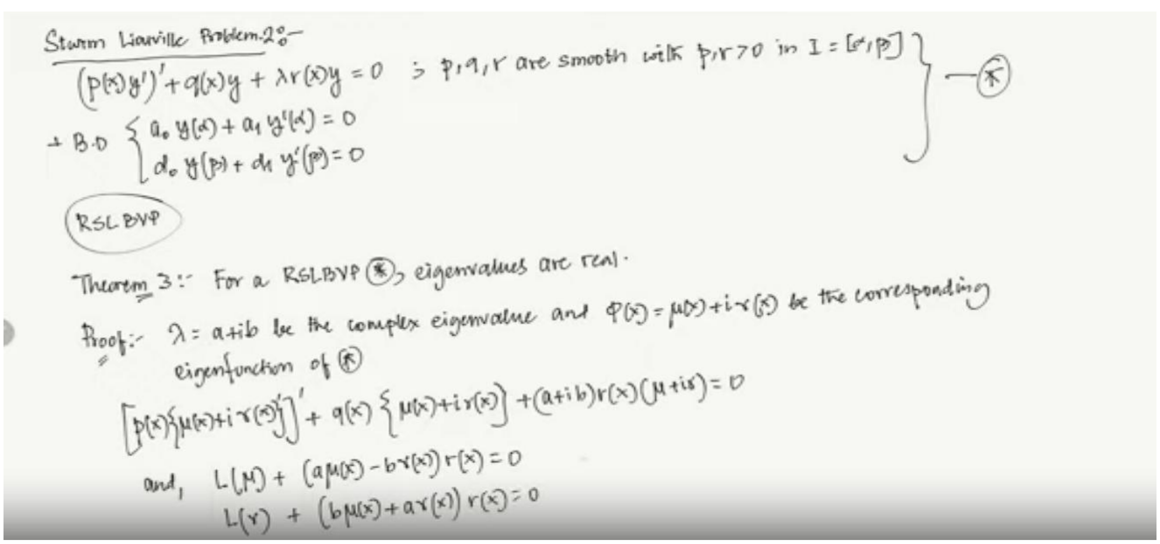
Theorem 3. So what is the third property? Third property says that if for a regular storm level boundary value problem, for a regular strom-leuville boundary value problem star, let us say, star, eigenvalues are real. Yes?

Is it okay? So basically what it is saying is this. See if you have you remember the first in the last video we talked about one example and in that example we definitely do not require you know we did not calculate what happens when eigenvalues are complex. Of course we could have done but we did not. So the thing is we said that they are real.

So, you do not have to worry about complex. But why are they real? This is the theorem which actually shows that they are going to be real. So, proof. See, if they are not real, let us start with complex eigenvalues lambda equals to a plus ib.

Let us say, be the complex eigenvalue, be the complex eigenvalue of star, eigenvalue of and what is the corresponding eigen function let's just call it p of x and what is it it is some function of mu x plus i uh gamma x let's just say gamma x okay be the corresponding eigen function of star of course corresponding eigen function function of star is it okay yes okay so then what do you have so this lambda and that phi is going to satisfy the equation right okay so what does it mean it means px times mu x plus i i let me maybe let me put it in a second bracket i mu x derivative of that right I think I am missing another derivative here, derivative of this, okay, plus qx, qx, mu x plus i gamma x, okay, plus a plus ib lambda rx, okay, and then mu plus i gamma, this is equal to 0, is it okay? Now if you have something like this you see you can we can actually break it up into real and imaginary parts right and we can write and what we will have is this the linear this L of mu plus

a mu of x minus b gamma of x. This I want you guys to check yourself. Please do that. L and what is l mu and l gamma, right? l gamma plus b mu x mu x minus plus a, sorry, gamma x here, a gamma x times r of x. cost is low and where what is l l as we have talked about in the last video also l of y is p y prime whole prime plus q y that's your l yeah okay so the these two is uh possible so please check this part out.



Sturm Liouville Problem 2:-

$(p(x)y')' + q(x)y + \lambda r(x)y = 0$; p, q, r are smooth with $p, r > 0$ in $I = [\alpha, \beta]$ — (*)

+ B.C $\begin{cases} a_0 y(\alpha) + a_1 y'(\alpha) = 0 \\ d_0 y(\beta) + d_1 y'(\beta) = 0 \end{cases}$

RSL BVP

Theorem 3:- For a RSLBVP (*), eigenvalues are real.

Proof:- $\lambda = a+ib$ be the complex eigenvalue and $\varphi(x) = \mu(x) + i\gamma(x)$ be the corresponding eigenfunction of (*)

$$\left[p(x)\{\mu(x)+i\gamma(x)\}'\right]' + q(x)\{\mu(x)+i\gamma(x)\} + (a+ib)r(x)(\mu+i\gamma) = 0$$

and, $L(\mu) + (a\mu(x) - b\gamma(x))r(x) = 0$

$L(\gamma) + (b\mu(x) + a\gamma(x))r(x) = 0$

OK, now what happens is once this is true, there is a boundary data also and the boundary data. Let's look at the boundary data. The boundary data is a naught times again that also you can, you know, break it up. So it is mu of alpha plus a1 mu prime of alpha is 0. Yes.

And again A naught gamma of alpha plus A1 gamma prime of alpha is 0. So, breaking it into real and imaginary parts that is. And simultaneously it is D naught mu at the point beta plus D1 mu prime at the point beta is 0. And here you have D naught sorry, gamma at the point beta plus d1 gamma prime at the point beta is going to be 0.

Is it okay? So, those two conditions are fine. Now, you see, you remember we did that Lagrange identity, right? Exactly the same Lagrange identity, we are going to use it here. So, what we are going to do is this.

Now, note that if you have mu times L of, sorry, gamma times L of mu minus mu times L of gamma. Yes, you remember we We computed something like this, right? This is the Lagrange identity part there, the exact same.

If you do write it, then it will become alpha to beta. You see, l of mu, you know what is it. See, l of mu is minus a times mu plus b times gamma times r, right? And similarly, a of gamma is also like this. So, we can just put it here.

So, what we are going to do is, I am going to write it like this, minus a mu i'm not going to write as a function of x so that is always given right minus b gamma times gamma r of x okay uh forget about writing x let's not write x plus you have b mu plus a gamma okay and then mu times r This is what you have. Okay. I'm just calculating.

I'm just putting L of L of mu and L of gamma from here. Yes. I'm just putting it here. That's all. I'm not doing anything.

Now, if we just compute it, it is alpha to beta. You see, basically, mu and gamma, that part is gone. Right. So basically, we are left out with B times gamma square plus mu square. Okay.

R dx. This is what we have. Now, if you just calculate it, it is basically nothing but P of x gamma mu prime minus mu gamma prime from alpha to beta. Is it okay?

And that is going to be 0. Yes. Why is it true? Can you guess why it is true? See what is happening is this.

This from here to here is fine, right? It is just a calculation. Where am I getting from here to here? How am I getting it? I know by Lagrange identity this is equals to p times the Wronskian right this we talked about in the last video also exact same thing or you can go back to the week 9 second video most probably then also you can see the Lagrange identity you get it there so basically this is equals to this from Lagrange okay so once that is true then what we can say is and the thing is

now if you calculate this thing uh mu gamma mu prime minus mu gamma prime from alpha to beta because of this initial data okay sorry the boundary data if we you can just check the boundary data if we just put it here why this is true because in the boundary data since a naught square plus a 1 square is not 0 and D naught squared plus D1 squared not equals to 0. So the basically, you know, the corresponding the coefficient matrix that is going to be singular. And from there, you can actually calculate this part that it is going to be 0. This is exactly the same thing we did in the earlier video.

So I'm not going to explain it. That's fine. Okay. So that is there. Now, if this is 0, then what does it say?

See, it says that this particular thing is 0, right? That's what it is saying. see, I should have write it like this. Here, I should have write it as P of x gamma mu prime minus mu gamma prime, you evaluated between alpha and beta is this, right? And then, so basically, if you do that, then you see, this is nothing but 0.

So, basically, that means, b times integral a to b gamma square plus mu square r times dx is going to be 0. What does that imply? That will of course imply that b is 0. Why?

Because, see, gamma square plus mu square, gamma and mu are the real and imaginary part of phi, right? Yes? See, since gamma and mu are real and imaginary part of phi, imaginary part of phi of phi and what is phi phi is a non-trivial solution and phi is non-trivial non-trivial okay so you see you gamma and mu are not simultaneously zero it cannot be both zero at the same time so basically mu square plus gamma square cannot be zero Yes, at least something has to be there, positive.

And then what happens is if you are taking r, r is always given to be positive. So between alpha and beta, that particular term is always positive. So beta times a positive term is 0, that implies beta, b is 0, right? If b is 0, therefore lambda has to be equal to a, which is in r. Is it okay? So if it has, so basically what it says is this, for a regular sum level boundary value problem, the eigenvalues are always going to be real.

And, $a_0\mu(\alpha) + a_1\mu'(\alpha) = 0$; $d_0\mu(\beta) + d_1\mu'(\beta) = 0$ $\quad a_0^2 + a_1^2 \neq 0$

$a_0\nu(\alpha) + a_1\nu'(\alpha) = 0$; $d_0\nu(\beta) + d_1\nu'(\beta) = 0$. $\quad$ and, $d_0^2 + d_1^2 \neq 0$

Note,

$$\int_\alpha^\beta [\nu L(\mu) - \mu L(\nu)]\,dx = \int_\alpha^\beta [-(a\mu - b\nu)\nu r + (b\mu + a\nu)\mu r]\,dx$$

$$0 = p(x)(\nu\mu' - \mu\nu')\Big|_\alpha^\beta = b\int_\alpha^\beta (\nu^2 + \mu^2)\, r\, dx$$

$\Rightarrow b = 0$ ($\because$ ν and μ are real and img part of ϕ and ϕ is non-trivial)

$\therefore \lambda = a \in \mathbb{R}$.

Is it okay? Now, let us look at our theorem. And this is very, very important theorem. The thing is this. We are not going to prove.

I am not going to prove this theorem for this course. The proof is little complicated. Okay. The thing is this. Why?

And it is, I mean, it is very lengthy actually. Yes. If you are interested in looking at the proof, please. So, for the proof, let me put it this way. For the proof.

Of the following theorem, one may look at the book, the Coddington-Levingson book, sorry, Coddington book, Coddington ODE book. Okay. It is there. It is very explicitly written. It is very well written.

Okay. So you can go through that if you want. But I mean for this course, I am not going to do it. I am going to explain the theorem. Okay.

Theorem. And this is very, very important theorem. Yes. Okay. So what is it?

So let me put a star here. Double star. Very important. Okay. What does the theorem say?

It basically says, see, the thing is this. Let us say, if I give you a problem, you can solve the problem, let us say, and then you can find what are the eigenvalues and eigenvectors.

What is the guarantee? That given any regular something will be a problem, there exists, yes, there exists a, you know, eigenvalue. How do you know there are eigenvalues?

And after that also, if there are eigenvalues, how many eigenvalues are there? How are we going to guarantee that? So, what we have is this. For a regular termly wheel boundary value problem. Is it okay?

Which one? Star. The problem which we are talking about. There exists an infinite number of eigenvalues lambda n. of eigenvalues lambda n. Is it okay?

And this is n equals to 1, 2, it goes on. Clear? So, basically what it is saying is this. I really do not we really do not care what p q r is yes you can always get a infinite sequence of eigenvalues lambda n and the beautiful part is this see these lambdas although they are real is not like a continuous spectrum right is always a discrete spectrum lambda s lambda 1 lambda 2 basically a countable number of you know values okay lambda n these eigenvalues these eigenvalues these eigenvalues can be arranged.

So basically, we can have a monotone sequence of eigenvalues. And as a monotonically increasing sequence, as a monotonically increasing sequence, increasing sequence. how is it lambda 1 that's the lambda 2 and it goes on like this right such that such that so basically where is it going is it converging somewhere the sequence yes it converges it goes to infinity basically converges in in that sense it diverges so in this to infinity is it okay so And now another moreover, a very important part. So basically, first of all, what is this saying?

You have an infinite sequence of eigenvalues. Yes. And of course, if you have lambda s corresponding to lambda s, you also have, you know, phi n's. That's there. Now, the thing is, what are the properties of lambda n?

You know, you can go on doing this lambda n. So, basically, you can actually arrange them in such a way that lambda n is a monotonically increasing sequence, right? And it goes to infinity as it enters to infinity, okay? Moreover, you can say something about the zeros of lambda n. So, what you can say is this, moreover, eigenfunction eigen function phi n of x okay which is corresponding to corresponding to to lambda n yes so basically let's say phi 1 lambda 1 is there so corresponding to that phi 1 is there right so now the question is how many zeros will there be huh in an interval alpha beta open interval alpha beta so corresponding to lambda n's okay has exactly has exactly n minus 1 zeros in alpha beta.

It is a brilliant theorem, right? So, you do realize what I am trying to say. See, what I am saying is this. Let us say lambda 1 is there. Corresponding to lambda 1, you have a eigenfunction phi 1.

Now, how many zeros does phi 1 have? In this case, n is 1. n 1 minus 1, 0, 0. So, there is no 0. So, phi 1 either positive or it is negative.

It does not have any 0 anywhere in alpha beta. So, it is not sign changing. phi 1 is not sign changing. What about phi 2? How many 0s does phi 2 have?

So, corresponding to lambda 2, you have phi 2 and phi 2 will have exactly one 0. So, there exists some x naught between alpha beta such that phi 2 of x naught is going to be 0. Is it okay? So, brilliant this thing, theorem, right?

And it is a very, very important theorem, yes, because it actually guarantees the existence of eigenvalues, okay? So, again, I am not proving the theorem, yes? And please, I mean, if you want, you can go through it using the Coddington-Levin, sorry, Coddington-Bochmann-Knot-Levin subject, yes? Now, let us look at some examples of singular strum level boundary value problem. So, regular strum level boundary value problem, we have some good ideas.

Now, let us look at the singular strum level boundary value problem. So, singular strum level boundary value problem. By the way, the theorem which I just talked about, this last theorem, we are going to use this theorem. It's not like I just gave you for fun. This is very important in the sense that this will actually help us to get through the next part of this video.

But not this part. This is basically about singular boundary value problem. I just want to put some problem. This is basically a remark kind of thing you can think of. It's basically a remark.

So, this will actually tell you that if there is some difference here. So, first of all, we will just look at some examples. Now, let us say that you, let us look at the problem y double prime plus lambda y equals to 0. Is it okay? That problem is there.

And the data now is given to be y 0 equals to 0. yeah and we do not have any data on the other part but we know that mod y x is less than equal m is it okay i don't have any other data we just have mod y is less than equal m which is less than infinity so basically it is finite for all x in zero infinity okay yes So, what you can do is this. So, you have to do it yourself. Now, you can check this part.

Very important. You can check this part. That for each lambda between 0 and infinity. Okay. Sine root lambda x is a solution.

Solution of the above problem. Okay, now what do you think that implies? You see what this is implying is this. This problem is definitely not a regular sum-label boundary value problem, right? Okay, now what did we just looked at this theorem?

It says that for a regular boundary value problem, sum-label boundary value problem, you will have a sequence of eigenvalues, right? Lambda 1, lambda 2, lambda 3, lambda n, which are increasing, going to infinity. Yeah, but the thing is this, the spectrum spectrum is the you know the set of eigenvalues that is the spectrum is discrete for a regular sum level boundary problem the spectrum is discrete is it okay now for a singular sum level boundary value problem this theorem does not work okay so the spectrum is not exactly discrete how do i know it because you see here you can actually show that for each lambda between zero and infinity this is a continuous uh this is a interval right zero to infinity So, you can take any lambda you want and you can get this function sine of root x lambda.

Sorry, root lambda x. And this is a solution. Yes. That is an eigen function. Yes. So, eigen values are any value between 0 and infinity.

And corresponding to that eigen functions are given by sine of root lambda x. Yes. So, the spectrum here. So, spectrum may be. Yes, you see, it is not like a guarantee that it will be continuous for a singular problem. It is may be continuous.

Continuous. But you see, in regular stem-label Barnaby problem, it has, it cannot be continuous. It has to be, how do I put it, discrete. Is it okay? Yes.

Now, the question, let me put in another question here. Yes. Yeah. So, can we, so question, question. So, I want you to do, think about this question.

Yes, I am not doing this part. So, let us say that you look at this problem. Look at, okay, let me write it this way. Consider, consider 1 minus x square, y double prime minus 2xy prime plus lambda y equals to 0.

Is it okay? And the conditions which are given is on the solutions are limit x tends to minus 1 y of x is finite. Yes. And limit x tends to 1. You see.

I have limit at two points, minus 1 and 1. So, basically, yx is finite. Now, the thing is, I want you to find the eigenvalues of this problem. Find the eigenvalues. Or maybe, let me give it.

The eigenvalues are given by lambda equals to n, n minus 1. So, lambda n equals to n, n minus 1. n is in n. Okay, and the eigen functions phi n let us say, phi n of x will be given by phi n minus 1 of x. These are the Legendre polynomial. Okay. Now, can you do this?

Can you show this thing? So, can you please check this part? Check. Okay. So, please try to do this part.

Is it okay? So, now the question is. So, I hope this is clear. So, basically what I am trying to say is this. Whatever we have proved the theorems and properties of this.

These are all for regular sum level boundary value problem. Okay. So, what I am going to do is, in the next video, we are going to talk about a different kind of sum-level boundary value problem, which is called a periodic sum-level boundary value problem. So, with this, I am going to end this video.

For the proof of the following theorem one may look at the Coddington. ODE Book

Theorem :- For a RSLBVP ($*$), $\exists$ an infinite number of eigenvalues λ_n ; $n = 1, 2, \ldots\ldots$ These eigenvalues can be arrange as a monotonic increasing sequence $\lambda_1 < \lambda_2 < \cdots$

such that $\lambda_n \to \infty$ as $n \to \infty$.

Moreover, Eigenfunction $\phi_n(x)$ corresponding to λ_n has exactly $(n-1)$ zeroes in (α, β).

Remark: Singular Sturm - Liouville Boundary Problem :-

Ex:- $y'' + \lambda y = 0$; $y(0) = 0$, $|y(x)| \leq M < \infty$ $\forall x \in (0, \infty)$

(Check) $\to$ Each $\lambda \in (0, \infty)$, $\sin \sqrt{\lambda} x$ is a solution of the above problem.

Spectrum $\boxed{\text{may}}$ be continuous.

Q:- Consider $(1-x^2)y'' - 2xy' + \lambda y = 0$; $\lim_{x \to -1} y(x) < \infty$, $\lim_{x \to 1} y(x) < \infty$ (Check)

Eigenvalues :- $\lambda_n = n(n-1)$; $n \in \mathbb{N}$ and $\varphi_n(x) = P_{n-1}(x)$ (Legendre Polynomial)