Ordinary Differential Equations (noc 24 ma 78)

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Lecture 32

Welcome students to this video on Stumbleable Problems. This is one of the most important aspects of the study of ODEs. So, what is this topic about? We are essentially interested in the equation like this, a second order equation given by P naught of x y double prime plus P1 of x y prime plus P2 of x y equals to r of x. Let us say, let us call that 1. So this is your second order equation.

Is it okay? Now this is defined on some alpha beta. Defined let us say in alpha beta. alpha, beta. And here we are assuming P0, P1, P2 and R are to be smooth.

So, you know, maybe sufficiently, you know, C1, C2, whatever is that you can choose is not a big problem. And we also have the boundary data right what is the boundary data so first of all we'll call it a l1 of y let's just call it like this which is given by a naught of y at the point alpha plus a1 y prime at the point alpha plus b naught y at the point beta plus b1 y prime at the point beta let's just call that to be a And there is another condition L2 of y which is again the similar sort of condition but with C0 and D0. C0 y at the point alpha plus C1 y prime at the point alpha plus D0 y at the point beta plus D1 y prime at the point beta.

This is B. this is okay so l1 and l2 if those two are uh these two sort of conditions are there so basically one along with these two conditions here a i b i c i d i a and b are constants okay are constants now we want to wanted to study this problem now you see this problem if you remember we have seen that for Some values, it will depend on the value of the initial data, depend on L1 and L2. But the thing is, this problem may have solution, may have unique solution, may have no solution or may have infinitely many solutions, right? We have already seen that part.

Now, the thing is this, you see, a very important aspect in mathematical physics, okay, is to determine, see, if let us say, if we put lambda over here, let us say, lambda over here.

Okay, let us, so essentially, let me write it this way. Let us say, now, if we have a boundary value problem, which will look like this, p y prime whole prime, okay, plus q y plus lambda r y, okay? Equals to 0. Let us just say if we have a problem like this.

Is it okay? Right. So see R is here I wrote R like this. So basically think of it as R to be 0 and the thing is P2 looks like this. Okay.

Q plus R Y R lambda R. Sorry. Yes. And this sort of equation, basically in mathematical physics, what happens is there are many different situations where you have to determine the value of this parameter lambda. Okay. So the question generally arises as determine lambda.

Determine lambda. Okay. So the value of lambda theta. Here you see a lambda is in a complex number C. That's what we generally assume. Okay.

For which non-trivial solutions exist. Non-trivial solutions exist. Exist. Okay. See it is not that this is only from mathematical physics.

Even we doing you know solving some pd's and all. We actually are going to need this sort of problem. So this is a very very important class of problems. These are called Sturm-Liouville boundary value problems. Okay.

So we are going to study this problem. So basically what is the equation? The equation is given by this. Let's just call this 2. Yes.

And so 2 along with uh maybe i can write it like this right maybe i can write this as okay let me I should introduce the problem properly. So essentially, okay. So let me do it properly, okay?

So you got the idea. Essentially, this is the equation. And for this equation, you have to find, so you have to find lambdas for which, you know, non-trivial solutions exist. You do realize that for all lambda, it may happen that for some lambda, there are one solution, unique solution, right? For some lambda, there is no solution.

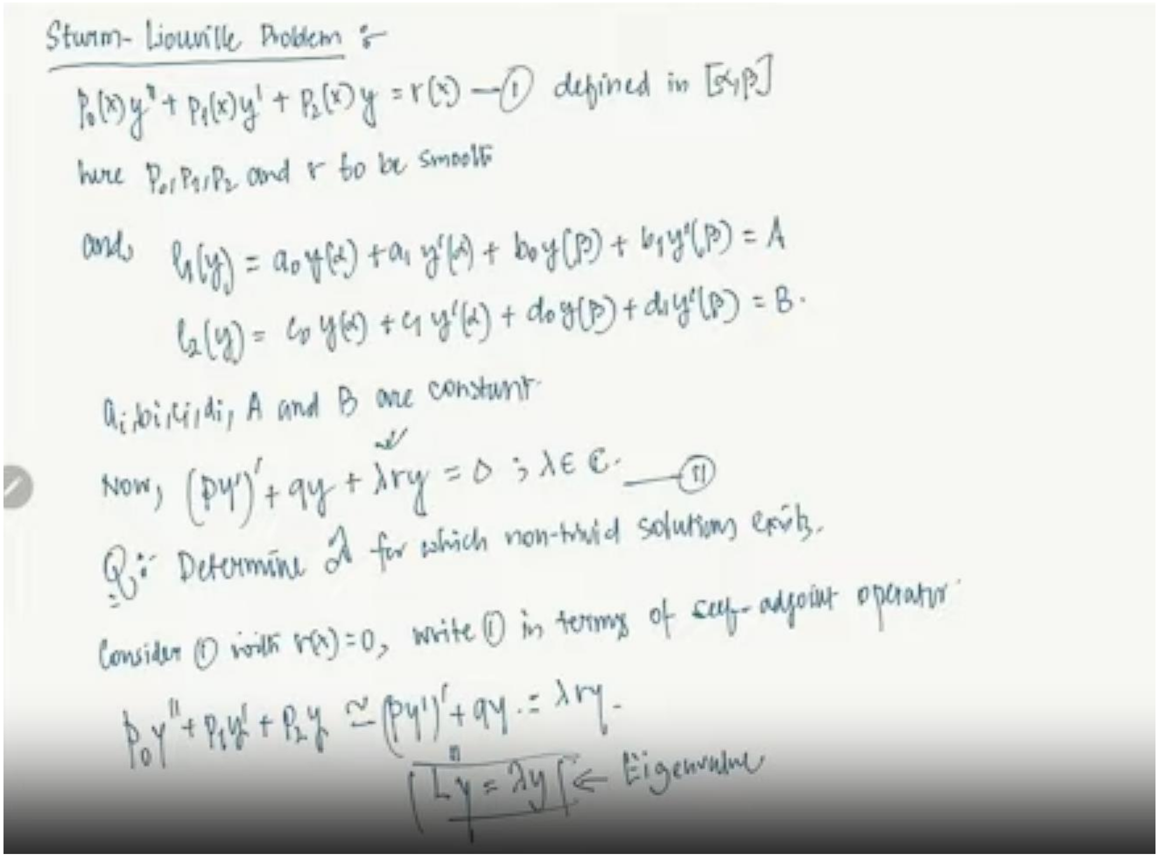
For some lambda, there may be many solutions, okay? So we have to keep track of such lambdas for which there are non-trivial solutions. And what is the boundary data? That we have to determine. So what I am going to do is maybe introduce the problem in the next page properly.

Let me do it this way. See, the thing is, first of all, in 1, let me introduce it in this way. See, consider 1 with rx equals to 0. So basically it is a homogenous problem. rx equals to 0 yeah now you see for the homogeneous problem what happens is this if you are i mean this problem is in the general form right now if you multiply it properly with the integrating factor and all we know we can actually uh write it in terms of a self-adjoint operator okay so write one write one in terms of self-adjoint operator terms of self-adjoint operator

adjoint operator operator okay and then what happens is if we can write it in terms of self adjoint operator then it will look like this py prime plus qy right this whole form see then this p naught y double prime plus p1 y prime plus p2 y may be written as okay can you we have seen right we can write it as p y prime whole prime plus qy Now the question is this that let us say that if this is equal to let us say lambda times some r y. Okay. Lambda can be positive, negative, doesn't matter, but lambda, it can be compressed also. So let's say we want to find out for which values of this, this is equals to lambda times ry.

Okay. Then these sort of problems are called sum-label boundary value problems. If you write it properly, you see what is happening is, let's say this is the operator l of y. and let us say r is 1 for now, then it will look like lambda times y. You see, these sort of equations are the eigenvalue problems, right? So, basically, we are trying to find out the eigenvalues of L and the eigenfunctions, of course, okay?

So, these sort of equations are called eigenvalue problems. So, basically, some of the problems are called eigenvalue problems because of this reason, okay? Now, so let me now formally introduce the problem, and then we are going to look at some properties of this, okay? So, consider the Consider the differential equation.



What is the differential equation? It is given by p y prime whole prime, okay? See, first of all, again, anyways, this term-level boundary-valve problem, we are always considering, going to consider a self-actual differential equation, okay? So, plus q y plus lambda r y, okay, equals to 0. That is the equation given to us.

Yes. And of course, P and Q. Let me write it down. So P and Q are continuously differentiable. Continuously differentiable. P and Q are continuously differentiable.

This is given to us. What is lambda? Lambda is a parameter. For the starters, what we are going to do is we are going to assume lambda is some number is a parameter which is varying at, you know, in C. So basically lambda will be from C. That's what we are going to assume. OK.

OK. So this is there. And very important property. So these properties are very, very important in order for this problem to work. OK.

P and r are positive. Please remember this thing. Very, very important, okay? Without these properties, none of the things which I am going to do is going to work, okay? So, first of all, P is this, yeah?

The first coefficient. r is this, the coefficient corresponding to lambda. Those two problems. So, okay? We are going to assume the problem, you see.

First of all, we are going to write it like this. ly plus lambda ry equals to 0. Here, this lambda corresponding to, this r corresponding to lambda will be called a weight function. This is called a weight function. Is this okay?

We will call it like this. And we will write this operator like this, ly plus lambda ry equals to 0. Is it okay? Yeah, this is the form which we are going to write it. Again, if we are writing like this, please, please remember p has to be positive.

The first coefficient p corresponding to py prime whole prime. Okay. This p has to be positive and along with that lambda, the r which is along with lambda, the weight function r, that has to be positive. These two conditions is vital for this sort of thing to work. Is it okay?

Otherwise, you know, the problem is this. Without this, if you want to do this problem, so what happens is, first of all, you know that if P is 0 or something like that, then, I mean, we can have that this is Lothar regular problem, boundary value problem, okay? So, this will be, in this case, a singular boundary value problem. And, of course, we are also assuming all of these in a closed interval alpha beta. That is also there.

Closed interval alpha beta. Why? Because again, we are basically looking at regular boundary value problem. You see, of course, what are the boundary data along with the boundary data? Along with the boundary data, what is the boundary data?

We are going to assume this boundary data. a0, y at the point alpha, plus a1 y prime at the point alpha is 0, okay? And d0 y at the point beta plus d1 y prime at the point beta is 0. This is okay.

These are the two boundary conditions which you are going to assume along with this equation. So, let us for now, let us call this equation as star, okay? The whole thing, yes, the equation along with the boundary. So, maybe I can write it here, star. This is the star problem.

This is the problem which we want to study for next few one or two, I mean, you know, videos. Whenever we are talking about Sturm-Liouville boundary value problem, we are going to look at this particular problem. We will call it a Sturm-Liouville boundary value problem. boundary value problem. Is it OK?

Strum level boundary value problem, OK? And generally speaking, we also go, I mean, like to call it as a eigenvalue problem. So that also is generally done, OK? And please remember, this is the data which we have considered, p, q, the interval i, the alpha, beta. So this is, of course, a regular boundary value problem.

So it is generally there. It is also called a regular strum level boundary value problem. Is it OK? Because this is a regular data, right? Regular boundary data.

This we already know. So, it looks like this, okay? See, the thing is the lambdas which we are going to find. So, basically a small note here. Now, what is the problem?

The problem is to find lambda, right? Those will be called as eigenvalues, right? eigenvalues. Clear? And the corresponding non-trivial solutions.

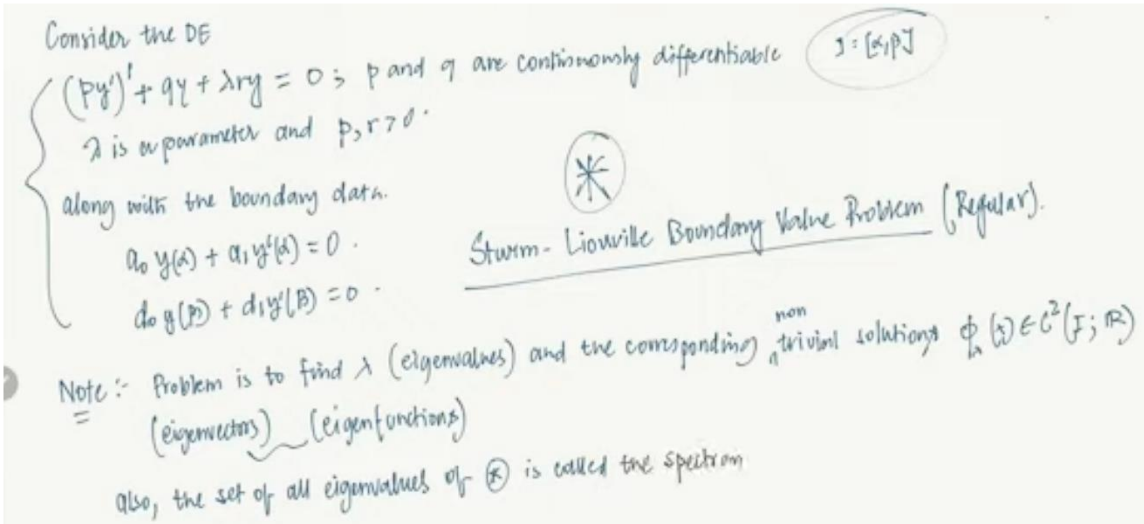
Is it okay? This is the problem. Corresponding non-trivial solutions you have to find. And we will call it a phi lambda of x, right.

So, any solutions of this for a given lambda, any solution will look like phi x lambda, right, which is again a function. So, it is sufficient regularity, yes. So, in C2 of i, i is i. So, this is defined in i, right. All of this is in i, which is alpha beta, let us say. If you remember, and this is finite, those boundaries are I to R, that sort of function, okay?

Non-trivial function. This will be called eigenvectors. If you remember, this is called eigenvectors, right? Or eigenfunctions also we will call it. It is not a problem.

Synonymously, it is also called eigenfunctions. So, I am going to call it eigenvectors or eigenfunctions, whatever, I mean, everything is fine. It is not a problem, okay? Now, If you put all the set of all eigenvalues, okay?

Now, again, also, the set of all eigenvalues, all eigenvalues. So, if you take the set of all eigenvalues together, right, of a regular problem of star, okay? That set is called the spectrum. okay what is spectrum of what spectrum of some operator right so the operator is given by l which is p y prime whole prime plus q y that's the operator okay so i mean that is as you understood is it okay right so that is the whole problem that's the introduction of the problem okay so now let's work out one you know example and then we go to the theory part so first of all how do we compute such eigen value and eigen function let's just look at that So you see, first of all, consider this problem.



Consider y double prime plus lambda y equals to 0. Is it okay? And then y0 equals to y at the point pi equals to 0. Yes? If you remember, see here, if you assume p is, of course, 1.

Here, here, P of x is identically equals to 1. Okay. What is Q of x? Q of x is essentially 0.

And what is R of x? R of x is essentially 1. Okay. So, you see here for the problem P and R has to be positive. Q can be anything.

I do not care. You can be negative, positive, whatever. As long as Q is continuous, it is fine. Okay. And the data is given on 0 and pi.

0 and pi, what sort of, I mean... do i put it what sort of interval it is it is a closed and bounded interval right so what sort of problem is it can you think of it what is it called this sort of problem so this sort of problem is called regular stumble will boundary value problem okay so we call this sort of problem is what is it regular so we'll i will write it like this huh regular term leave will boundary value problem is this okay we'll call it this right okay now how do you solve this thing so first of all see what we are going to do is i'm just going to concentrate on lambda in r okay so we are going to we are going to concentrate concentrate On lambda in R. So what I mean by this C. Generally speaking lambda can be in C right. But the thing is we are not really interested in C right now.

Why not? That will be covered in the theory part okay. So you do not have to look for lambda in C. So essentially if there are lambdas. okay for which non-trivial solutions exist those lambdas has to be real for this sort of problem okay that's come that comes from theory so you don't have to worry about it all you have to worry about it is the real lambdas okay so first of all so what can be the cases first of first case is either lambda is zero or lambda can be non-zero or a positive or negative right okay so first of all if lambda is zero if lambda is zero Okay, then what happens to this equation y double prime equals to 0 and the general solution y of x will look like c1 plus c2 of x. Is it okay?

And c1 c2 is in R. c1 c2 is in R. Okay, and you see it satisfies these two equation and you see 0 equals to y at the point 0. That is nothing but c1 plus c2 times 0. Okay, that will give you c1 is 0. Now if c1 is 0, if you are putting it like this, then y is c2 of x, right? Now, therefore, what we have is y of x is nothing but c2 of x. Now it also has to satisfy the other condition.

So what is the other condition? 0 equals to y at the point pi, right? So which is nothing but c2 times pi. So c2 times pi is 0. That will imply c2 has to be 0, right?

Hence, c1 and c2 is 0. So, yx identically equals to 0. Is it okay? That is the only solution available. Now, you see what is the question?

The question is this. You have to find, please remember, you have to find lambdas for which non-trivial solution exists. See, this is the homogeneous problem, right? So, for any lambda, trivial solution exists, right? See, let me put it as a small remark.

See, these regular sum-level boundary value problems, they are homogeneous problems. So for any lambda, for any lambda, trivial solution is always a solution, right? For any lambda, trivial solution exists. Solution exists. exists.

Is it okay? Now, the thing is, this trivial solution is fine, but what about the other non-trivial solutions? Okay. Now, you have to find, if you have to find something like that, then you have to work out. So, basically, you have to find out lambdas for which non-trivial solution exists.

So, here you see, if lambda is 0, you will see that lambda ayx equals to 0 is the only solution. Is it okay? So, the thing is, since this is the case, therefore, lambda equals to 0 is not an eigenvalue. It is not an eigenvalue.

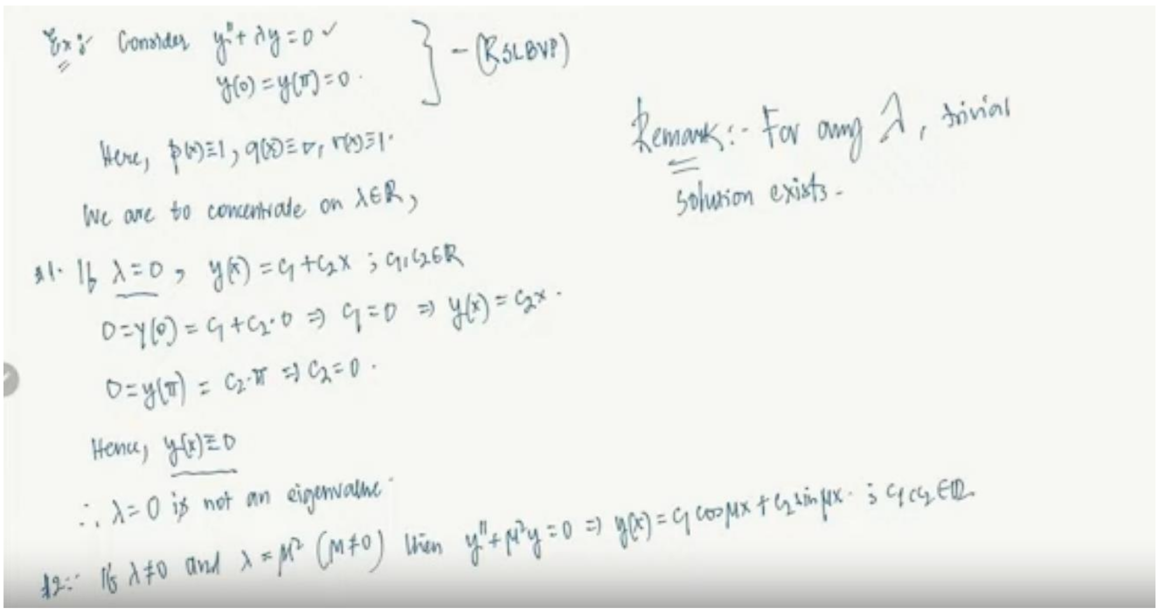
Is it okay? It is not an eigenvalue. Okay. Right. So, this is case 1, let us say.

Case 2. Let us say lambda is not 0. So, if lambda is non-zero and lambda can be positive negative. So, lambda is let us say mu square. Okay.

Mu not 0. Clear? So, it is non-zero basically. So, in that case, then the equation which you have is y double prime plus mu square y equals to 0. Is it okay?

And that will imply yx equals to c1 times cosine mu x plus, one second, this is plus minus m square, what is the solution? m square plus mu square. So, c1 plus c2 times sine mu x. Sine mu x, yes, right? That is what the solution is.

Yes, I hope this is, I am correct. Yes, fine. Now, c1 and c2 is of course in R. c1 and c2 is in R. Just give it. Now, see, if this is the case, if this is the case, now you have to satisfy that equation. So, what is it?



First of all, the first equation is this, 0 equals to y at the point 0. And what is this y at the point 0? c1 times sine mu 0 plus c2 times, I have wrote it as sine cosine, right, or cosine sine. So, I should write it properly. So, cosine mu x plus sine mu x. So, I should write it like cosine mu 0 plus c2 times sine mu 0.

Is it okay? Yes, now you see sin 0 is 0 essentially. So, c1 times cosine 0 is 1. So, that will imply c1 is going to be 0. Is it okay?

So, therefore, now the second condition is y equals to 0 equals to y at the point pi, which is equals to c2 times sin mu pi. That is what you have, right? Now you see, is it okay? Because c1 is 0, right? So the only thing which is remaining is c2 times sin pi x. So basically now y equals to c2 times y pi x and we satisfy y at the point pi is 0.

So basically c2 times sin pi mu pi is going to be 0. Is it okay? Now, of course mu is non-zero, yes? So sin mu pi is not going to be 0, right? Generally speaking, c2, let us say c2.

So here what we have is c2 times this sin is 0. okay now if c2 is 0 see if c2 is 0 c1 is 0 c2 is 0 then we have what we have is a trivial solution in that case okay we do not want previous solution because that's not a interesting yes right okay we want non-trivial solution so if so we have trivial solution if c2 is 0 we have trivial solution which we don't want trivial solution okay so If c2 is not 0, what happens? If c2 is not 0, then you have non-trivial solutions, okay? Then y of x, since sine of mu pi is 0, okay?

If c2 is non-0, then sine of mu pi has to be 0. So now, what does that mean? Sine of mu pi has to be equal to sine of n pi. That's the only case when they are going to be 0, anything, z. Is it okay?

Otherwise, there is no other option, right? Okay. So, if this is the case, what does that mean? What is pi? Pi is nothing but n. n is in z. Is it okay?

yeah so what it means is to see if we are looking for non-previous solutions okay positive for positive eigenvalues lambda positive you see this is lambda equals to mu square so positive what positive eigenvalues exist if mu equals to so basically what is the corresponding eigen functions mu equals to n okay so what is mu mu mu is lambda root lambda so lambda is mu square right so therefore what are the eigenvalues eigenvalues are positive, eigenvalues, okay, are given by, are given by, see, lambda, eigenvalues are lambda, which is mu square, and mu square is n square, so eigenvalues are given by n square, okay, n is in n, let us say, z also does not matter, right, it is n square, basically, huh, And what are the corresponding eigenfunctions and eigenfunctions? What are the eigenfunctions? What are the solutions of this?

See, if lambda is n, the solution is given by, see here, after all of this, what is y of x? y of x is nothing but c2 times sine mu x, right? That was the solution. And then That solution satisfied this y at the point pi equals to 0.

Okay. So now you see if mu equals to n, if mu equals to n, then yx. So basically for those, what is your function which satisfies? Sine of nx, right? Constant time sine of nx but constant can be 1 also.

Any constant multiple is fine. So basically it is sine nx. And eigenfunctions are given by r sine nx. And n is e. For now let us just say n. It is not a problem. n can be z also, but it does not matter.

So, it is basically the same thing. So, you see, these are the things. So, essentially what we can say is the eigen pair. What is an eigen pair? Eigen pair is the eigen function with its corresponding eigen vector or function.

So, eigen value with its eigen vector. So, eigen pair is rr. Let us just write it as r n square sine of nx okay n is in n let's say or z you can put it as z i think it will be much appropriate okay these are the eigen pairs is it okay so let let us take here to be z n is in z is it okay so these are the eigen pairs eigen pair is eigen vector the eigen value the first thing is eigen value and the second part is the eigen corresponding eigen function okay right Now, so we have these positive eigenvalues.

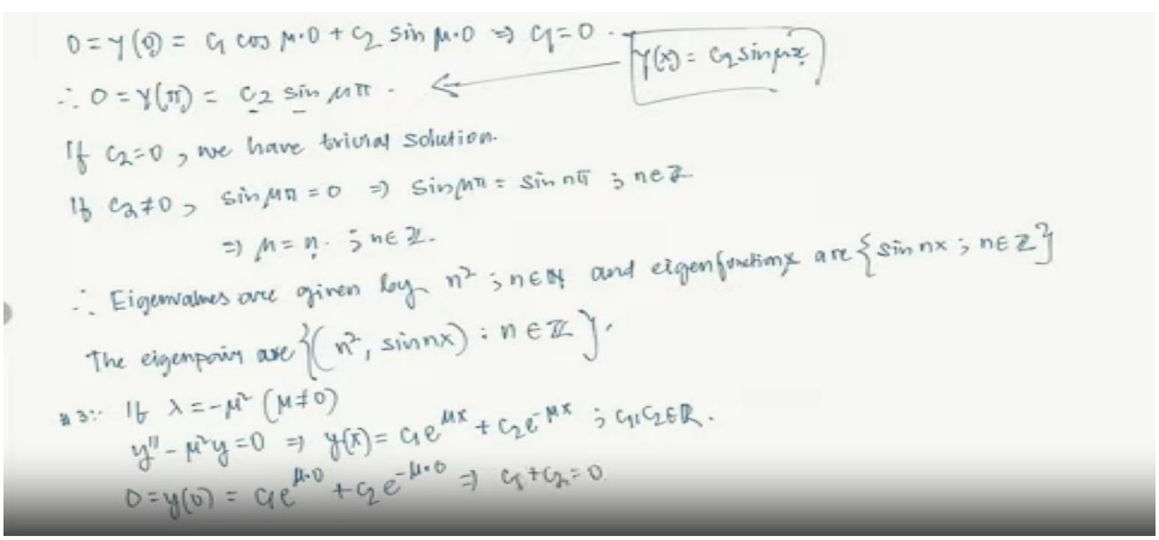
What about negative ones? So, now case 3. If lambda is, how do I put it? Let us say negative. So, minus mu square.

Mu non-zero. Let us just take it like this. Is it okay? Then what happens? Let us just check that.

What happens? See now, y, the equation turns out to be y prime minus mu squared times y equals to 0. Okay? We can solve this equation, right? And if you solve it, the solution which you are going to get yx equals to c1 times cosine mu x plus c2 times sine mu x.

sin mu x, right? Yes. Oh, sorry, sorry, sorry. This shouldn't be the case. I am writing this mistake e power mu x plus c2 times e power minus mu x, okay?

c1, c2 is in R here. Now, it has to satisfy those two equations again. Again, 0 equals to y at the point 0, which is equals to c1 times e to the power mu times 0, plus c2 times e to the power minus mu times 0. Okay?



So, that will give us c1 plus c2 equals to 0. Is it okay? So, that is your first equation. Let us just follow and put it. Okay? And another one, 0 equals to y pi mu equals to 0. That is there, right? So, the second one is 0 equals to y at the point pi, which is c1 times e to the power mu pi plus c2 times e to the power minus mu pi. okay so this is two now it is quite clear that from one and two from one and two okay what you have c1 equals to c2 equals to zero there is no other way okay so please check this part because the determinant is basically non-zero so tb acceleration is only going to with the solution also only solution so basically please check this part okay so c1 equals to zero therefore yx is identically equal to 0.

It is the only solution. It is the only solution. Only solution. What does that say? That says that there are no negative eigenvalues.

Is it okay? So, that will imply there does not exist any negative eigenvalue. Is it okay? Yeah. So, what did we have?

We have positive eigenvalues. Okay. Along with eigen functions which is given by sine of nx. Here. Yes.

That is what we have. Okay. And the thing is we do not have any zero eigenvalue or we do not have any negative eigenvalue. That is what. So, this is how you solve these problems.

Now, a very important part here is this. See, we have assumed. Where is it? We have assumed lambda is in R. We did not look at the complex formula. What happens if lambda is complex?

And those things will be taken care of now. So, what we are going to do is this. We are going to define some new theory. And with the help of that theory only, I will justify why we do not have to consider the complex things. So, first of all, let us write down a definition.

definition what is definition the definition is the set of functions the set of functions okay set of functions phi n of x phi n of x uh maybe i can write it this way n equals to 0, 1, 2, each of which are piecewise continuous. For now, we are just starting with piecewise continuous. See, the definition, this definition which I am writing has nothing to do with the theory. Is it okay?

So, that is why I am just giving it for the most general case. Okay. But the thing is we are not going to need so much general cases in our, this theory. Okay. So, this is just for definition in terms of definition.

So, the most general way of putting it. Which are piecewise continuous in an infinite or finite set. Okay. Infinite set. or finite or finite interval alpha beta interval alpha beta okay now is said to be is said to be orthogonal orthogonal if oh sorry with respect to rx

with respect to a non-negative negative function non-negative function rx if phi m phi n The inner product of those two. We will call it as inner product of phi m and phi n. Given two functions, the inner product will depend like this. Alpha to beta, phi m of x, phi n of x, r of x, dx. This has to be 0.

Is it okay? For all, m not equals to n. Is it okay? And, And alpha 2 beta r of x phi n square of x dx is non-zero for all n. Yes. See, the thing is, if those two conditions hold, okay, this r, of course, is called, as I told you, this is called the weight function.

Weight function. Okay. So, if these two conditions hold, then we call, you know, the set, the phi n set is a orthogonal set. Is it okay? Orthogonal set.

Of course, if this non-zero, it is the integral rx phi n square is non-zero, right? If it is the 1, then we call it an orthonormal set. Okay. So, remark. Remark.

If alpha to beta here, right? phi n square x dx, if that is equals to 1 for all n, then we call it a, then the set phi n of x n in n is called orthonormals. It's called orthogonal. Is this okay? Right?

That's just the definition. Now, you see one small detail which we need to understand here is this. See, for something like this to happen, the integral from alpha beta rx times pn square, let's say, it is summable, right? Yes. For something like this to happen, we will assume that rx only has a finite number of zeros.

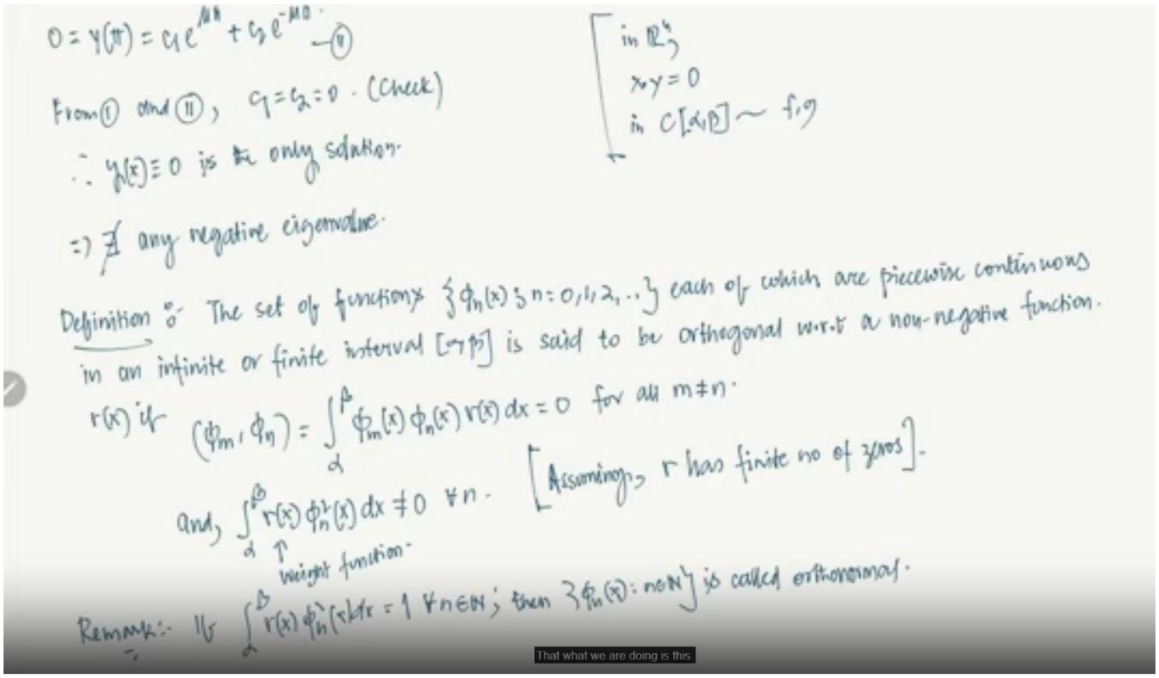
So, this is assumed. Assuming, let me put it this way. Assuming, See if you remember, maybe we have assumed that r is non-negative function, right? But we did not assume.

Maybe I can, so okay. Assuming r has finite number of zeros. Finite number of zeros. Is it okay? Now, what is the idea?

See, basically what is happening is this. Let us say in Rn, I will give you some motivation. In Rn, what happens is this, two functions, two points, vectors x and y, they are, I mean, orthogonal to each other, x dot y, basically they are perpendicular to each other if x dot y is zero. So in Rn what is the inner product? Inner product is given by x dot y. That's the inner product.

Basically the dot product. Now let's say in a space like this. Let's say c of alpha beta. This sort of set. You have two functions f and g.

And we say they are perpendicular to each other. But how are you going to define that perpendicular to each other path? That what we are doing is this. Basically, we are defining it as the inner product of f and g is given by integral fg dx. Is it okay?



And then if that is 0, then we call it an orthogonal. So basically this is just a generalization. In a vector space, finite dimensional vector space Rn, if you have to define the inner product, you define it like a dot product x dot y, which is your usual thing. But in a vector space here, we can define, we can think of it as an inner product space. And what is the inner product?

It is given by integral f g t x. There can be others also, but this is the inner product which we are basically going to assume. Is it okay? Great. So, orthogonal set, orthonormal set. Clear?

Right. Now, the thing is this. See, question is, question. Does there exist orthonormal set? I mean, orthonormal, orthogonal is okay.

So, but does there exist one? Can you show there is one? So, you see, the thing is, let us look at this thing. The thing, sine of nx. n in n n in n or n z it's not a big issue but n in n for now let's just assume that huh so this set let's say call it the set a okay you see a is we are we will you know prove that this is the orthogonal set is it okay

And what is this orthogonal set? It is orthogonal somewhere. See, we say that two vectors, let us say 1, 0 and 0, 1, they are orthogonal to each other. Those two vectors are in some space, in some vector space, that is in R2. Here, they are orthogonal set, let us say, in C infinity also you can put if you want, but let us just put it as C1.

C1 of, let us say, R2. minus pi to pi or 0 to pi also 0 to pi let's just say 0 to 5 so basically in this state they are orthogonal okay right so how do you prove something like this so in this case what you have to do alpha to beta okay R in this case is 1. So, it is orthogonal with respect to the weight function R, which is identically equals to 1. See, R has to be non-negative.

That is the only condition and it has to have finite number of zeros. R is 1. If R is 1, all of those conditions are satisfied. So, this is the orthogonal set with respect to the weight function 1. So, how do you prove it?

Sine nx times sine mx dx. Okay, now, of course, you can do this integration. So please check that this is going to be 0 if m not equals to m. Okay, so this you have to check it yourself. It is basically integration, very, very easy stuff. Okay, and you can actually show that, I mean, the integral from n moreover, and integral alpha to beta sine square nx dx.

nx dx sorry i should write 0 to pi and not alpha to beta i should write 0 to pi 0 to pi and again 0 to pi sine square nx dx this is going to be most probably pi okay so please check this point okay you have to check it for all n for all any it is So, what does it give you? It gives you that this is a is orthogonal set. It is an orthogonal set with respect to the weight function r. So, in c1 of 0, 1 with respect to the weight function, weight function 1, function wx identically equals to 1. Is it okay?

That is just, I mean, one of the, you know, examples of orthogonal Okay. Now, the important part is this. There are some theorems which we need to prove. And these are very, very important theorems.

Okay. So, the first theorem. These are the properties of regular term-label boundary value problem. Okay. The first theorem.

So, what is the first theorem? The eigenvalues. The eigenvalues of a Regular. Stumbly wheel.

Boundary value problem. I will write it like this. So please understand that. This is just a shortcut. I don't want to write the whole thing down all the time.

And what is the problem? Star. Yes. R simple. R simple.

What does it mean? Let's just understand what does it mean. It means that if. That is. If lambda is an eigenvalue.

Is an eigenvalue. Okay. And phi 1 of x and phi 2 of x. Okay. Are the corresponding eigenfunctions. Are corresponding eigenfunctions.

So, you do realize that eigenfunctions for one particular eigenvalue is never going to be unique, right? Because if phi is an eigenfunction constant and phi is also going to be an eigenfunction. Is it okay? So, phi 1 and phi 2, let us say there are two corresponding eigenfunctions for a given eigenvalue lambda. What you can show is then phi 1 and phi 2 are linearly independent.

So they cannot be linearly dependent. Linearly. Sorry they are linearly dependent. So they cannot be linearly independent. So they are linearly independent.

So basically dependent. So basically what it means is phi 1 has to be a constant time phi 2. There cannot be two extremely different functions which are like linearly independent. So what is the proof? Proof.

Now you see that phi 1 and phi 2 are both solutions, right? So phi 1 and phi 2 are both solutions. Solutions of what are these solutions of the solutions of star with lambda with lambda. right so you know this equation if you write it with lambda where is the equation sorry uh where is the equation i wrote somewhere no star if we write lambda let's say lambda is an eigenvalue then phi 1 and phi 2 both satisfies this equation is it okay so and with this boundary data okay so let us let me write it down that will make life much easier So, therefore, p phi 1 prime whole prime plus p cube phi 1 plus lambda r phi 1 equals to 0.

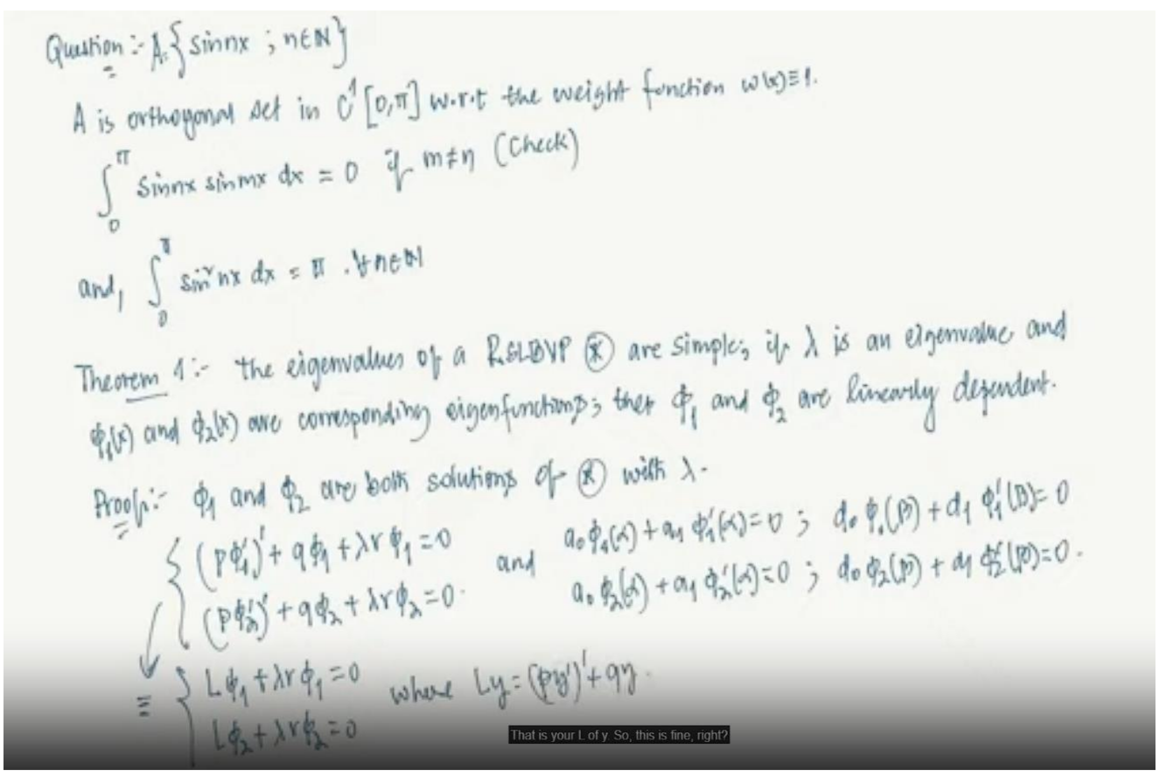
And also this equation is there, p phi 2 whole prime plus q phi 2 plus lambda r phi 2, this is 0. Is it okay? We have also the boundary data, right? And moreover, and you have the boundary data, a naught of phi 1 at the point alpha plus a 1 of phi prime, phi 1 prime at the point alpha is 0.

Yes? corresponding to phi 1 right and the same thing also holds a naught and i hope i wrote b let me just check that part so the other one is a 1 and a a naught and a 1 d naught and d 1 i wrote up so let's write down d naught and d 1 and this thing d naught phi 2 sorry phi 1 at the point beta phi 1 at the point beta plus d 1 phi1 prime at the point beta, this is 0. And similarly here, phi2 also satisfies this equation. So, a0 times phi2 at the point alpha plus a1 times phi2 prime at the point alpha, this is 0.

And correspondingly, d0 phi2 at the point beta plus d1 phi2 prime at the point beta, this is going to be 0. Yes, so phi 1 and phi 2, they are both the eigen functions corresponding to the eigen value lambda, then all these conditions work. Is it okay? Now you see, what does this condition say? It says that the Wronskian of phi 1 and phi 2 at the point alpha, this is 0.

And similarly, for beta also, we can say the exact same thing, okay? But there is, what it is saying is this, there exists a point alpha, yeah? In that interval alpha beta, there exists an alpha where the Wronskian of two functions are 0. Essentially, that will imply, the Abel theorem, if you remember, that will imply phi 1 and phi 2, phi 1 and phi 2, one second, oh, sorry, sorry, sorry, what am I saying? sorry i should do it in this way okay so first of all i will come to this so essentially first of all let's just look at this see first of all these two equations are there yes what is this equation if you write it down it is l phi 1 plus lambda r phi 1 equals to 0.

This is equivalent to this. This part is equivalent to this. And this is L phi 2 plus lambda R phi 2 equals to 0, where L of y equals to P y prime whole prime plus Q y, right? That is your L of y. So, this is fine, right? Now, you remember?



If this is the case, you can use Lagrange identity. You remember I told you that we are going to use Lagrange identity later on. So, we can use Lagrange here. So, using Lagrange identity, one has that this equation, from these two equations, use the Lagrange identity, you are going to get phi x times the Wronskian of phi 1 and phi 2 at the point x is a constant c. c is a constant.

You remember we did this. So please go back to this adjoint equation and exact equation part that video. You are going to get this thing. So this is why Lagrange identity or Green's identity also you want to call it. It does not matter but basically it is the same thing.

You integrate one part. So basically we are going to get it. Now you see this C we have to find what this C is. Is it okay? We have to find this C. C, the thing is, that is what I am trying to say.

From here, from these two equations, what do we have? We have that this is nothing but the Wronskian of phi 1, phi 2 at the point alpha. This is equal to 0. Is it okay? This is what it is saying, right?

Or even this is also saying the same thing. The Wronskian of phi 1, phi 2 at the point beta is 0. This is what it is saying. Now, the thing is, if you remember... Now, P is non-zero, right?

P is non-zero given to you, okay? So, we have to find the value of C here. What happens to the value of C? Okay. Now, since the wrong square of phi 1, phi 2 at the point alpha is going to be 0, then what is C?

So, that will imply that C has to be equals to 0. There is no other way, right? Yes? I hope this is clear. See, other theorem, phi 1 and phi 2 are both solutions of this equation, right?

So, if the Wronskian is 0 at one point, it has to be 0 everywhere, right? That is just the Apple theorem. So, basically, if that is the case, then in this case, Wronskian of phi 1, phi 2 at any point x is going to be essentially 0. So, in that case, c has to be equal to 0 because p is assumed to be positive. You remember?

p is assumed to be positive. We have always assumed. Since p is greater than 0. So, c has to be 0. Now, if c is 0, therefore, what does it imply?

It implies that phi 1 and phi 2 are linearly dependent. I hope this is clear to you. So, what do we have for one eigenvalue if you have two different eigenfunctions? This thing. phi 1 and phi 2, then phi 1 equals to c times phi 2.

That is what it is saying. So, basically, the other way of saying it is that the eigenvalues are simple. So, now let us look at the second important theorem, theorem 2. What does it say? It says that you look at the set of eigenvalues lambda n, n equals to 1, 2.

So, n is in n essentially, be the eigenvalues Eigenvalues of a regular Sturm-Liouville boundary value problem. Is it okay? These are the eigenvalues of a regular Sturm-Liouville boundary value problem. Okay.

And phi n. Is it okay? phi n, n equals to 1 to this beta corresponding eigenfunctions. I am writing it in terms of set, but you do realize what I am trying to say. Understanding? Eigen functions.

Is it okay? Then, the set phi n of x, n equals to 1 to this set, is orthogonal. is orthogonal okay in alpha beta in alpha beta with respect to with respect to the weight function rx equals to zero weight function rx equals to zero is it okay so basically what it is saying is this sorry rx equals to one rx equals to one Okay, so what exactly is he saying is this, you see, if you have, so if you have, let us say lambda 1, lambda 2, it goes on like, lambda n, it goes on like this, okay. Now, corresponding to lambda 1, you have phi 1.

Corresponding to lambda 2, you have phi 2. Corresponding to lambda n, you have phi n. Now, the relation, what is the question is this? What is the relation between phi 1, phi 2, phi n? The relation is this. They are saying they are going to be mutually orthogonal to each other.

So, phi n times phi n, if you indicate it with respect to rx equals to 1, of course, it is going to be orthogonal. okay maybe rx is 1 not required so with respect to the weight function rx is fine yes don't have to work rx equals 1 also will work but the thing is generally here there is rxy we don't want it it's fine so basically what i'm trying to say is this they are orthogonal so basically it means that integral phi m x phi nx times rx dx, this is going to be 0. For m0 equals to n, of course, for m0 equals to n, this is going to be 0. This is what he is going to say.

Now, before we prove this thing, let me tell you why this is very important. See, essentially, this equation is important in the sense that it actually gives you a set you actually see if you have a regular sum level boundary value problem and if you can somehow find some set of eigenvalues and eigen functions for that then that say eigen function set will act as a set of orthogonal functions right Yes. So, if you have to somehow, let us say for some problem purposes, you have to somehow produce an orthogonal set of functions.

Yes. What do you do? You just find a regular sum level, you just construct a regular sum level boundary level problem, solve it and you will be done. Is it okay? Yes.

So, here in this theorem, this actually guarantees that you are going to get something like that. Okay. So, let us look at the proof. Proof. So let us say that let lambda k and lambda l, lambda l, k not equals to l be eigenvalues and phi k, phi kx and phi lx, phi lx are the corresponding eigenfunctions.

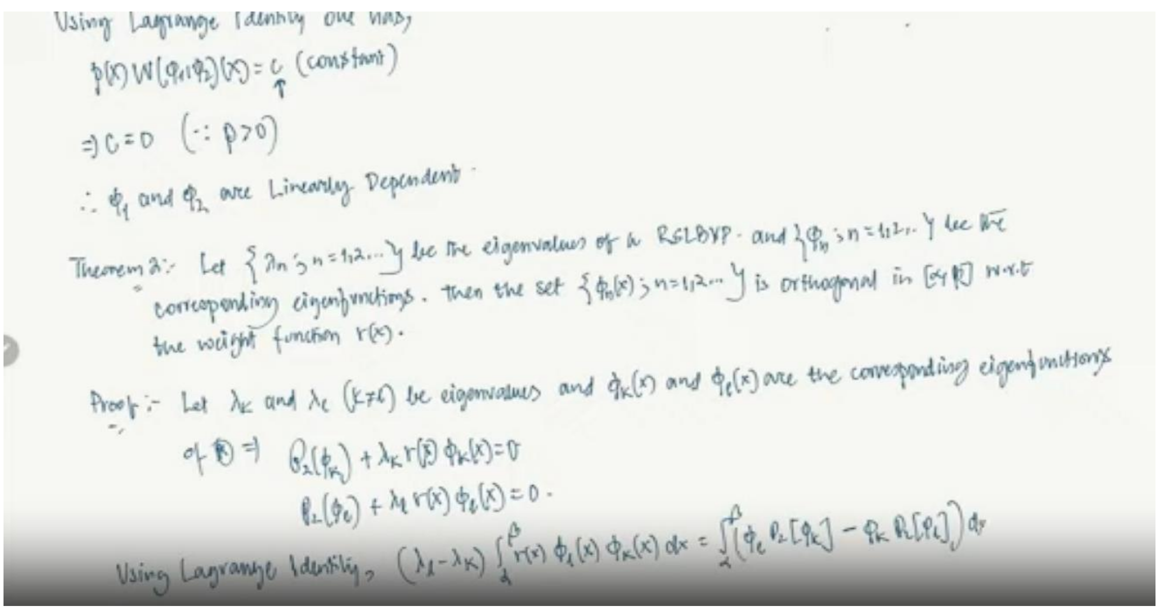
Are the corresponding. Eigen functions. Eigen functions. Of the of course the of star. So that is given of star.

Clear. Now since phi k and phi l are solutions. We have this equation right. Phi 2 of that implies. Phi 2 of.

Phi 2 of phi k sorry. Plus lambda k. R of x. phi k of x equals to 0. And p2 of phi l plus lambda l r of x phi l of x, this is going to be 0. Is it okay?

Now, what we are going to use to is, now again same, Lagrange identity or Green's identity, which you want to call it, using Lagrange identity. I will just call it Lagrange identity. If you want, you can call it Green's identity. No issues there.

So, using Lagrange identity, you remember, you multiply this thing with phi L, you multiply that thing with phi K. The first with phi K, phi L and the second with phi K and then, you know, you indicate it. You are going to get this Lagrange identity. So, what you are going to get is phi L, sorry, lambda L minus lambda K. Integral alpha to beta, r of x, phi l of x, phi k of x dx. This is nothing but alpha to beta, clear?phi l of p2 phi k, yes, minus



Using Lagrange Identity one has,

$p(x) W(\phi_1,\phi_2)(x) = c$ (constant)

$\Rightarrow c = 0$ $(\because p > 0)$

$\therefore$ ϕ_1 and ϕ_2 are Linearly Dependent.

Theorem 2:- Let $\{\lambda_n ; n = 1,2,\ldots\}$ be the eigenvalues of a RSLBVP and $\{\phi_n ; n = 1,2,\ldots\}$ be the corresponding eigenfunctions. Then the set $\{\phi_n(x) ; n = 1,2,\ldots\}$ is orthogonal in $[\alpha,\beta]$ w.r.t the weight function $r(x)$.

Proof:- Let λ_k and λ_l $(k \neq l)$ be eigenvalues and $\phi_k(x)$ and $\phi_l(x)$ are the corresponding eigenfunctions of (*) $\Rightarrow$ $P_2(\phi_k) + \lambda_k r(x) \phi_k(x) = 0$

$P_2(\phi_l) + \lambda_l r(x) \phi_l(x) = 0$.

Using Lagrange Identity, $(\lambda_l - \lambda_k) \int_\alpha^\beta r(x) \phi_l(x) \phi_k(x)\, dx = \int_\alpha^\beta (\phi_l P_2[\phi_k] - \phi_k P_2[\phi_l])\, dx$

phi k p2 of phi l. d x. Yes. Now, and what is it? See, this is again alpha to beta. You remember that Wronskian part is there, right?

So, this is nothing but phi l x. Sorry, p x is there, right? Let me write it here. p 2 of phi k Maybe it is okay. Let me write it this way.

So, you see, this is why greens, right? Direct calculation of greens. Now, if you write it properly, it will be P times the raw scale. You know, we also talked about it here. P times the raw scale.

So, basically this. So, then what you have? Therefore, lambda L minus lambda K integral alpha to beta. rx, phi kx, phi lx, dx. This is nothing but P of x, phi l of x, phi k prime of x, minus phi k prime of x, phi k of x, sorry, phi l prime of x, phi l prime of x, phi k of x.

Px times Wronskian of P L P k. See, similar sort of thing we got it here also. So, exactly the same thing which I wrote here. Now, this evaluated at the point alpha beta. So, I have to integrate this part, right? See, integration of alpha to beta this.

Inside this is the Wronskian part, P times Wronskian. So, I have to integrate it between alpha and beta. Okay. So, now, you see, we have to evaluate what happens to this. Yes.

Now, let us understand this. See, since P k And VL satisfies the boundary data, right? VL satisfies the boundary data. The boundary data.

Okay? So, what does that mean? It means that A0 times VK at the point alpha plus A1 times VK prime at the point alpha is equal to 0. And similarly, A0 times VL at the point alpha plus a1 times phi l prime at the point alpha, this is 0.

And similarly, d0 times phi k at the point beta plus d1 times phi k prime at the point beta is 0. And here also similar sort of thing. d1 times phi, I think, l prime at the point beta is 0. So, those two conditions are satisfied. Now, you see from these two conditions, we have that phi k alpha phi k prime of alpha minus phi k prime of alpha phi l of alpha.

This is equals to phi k of beta phi l prime of beta minus phi k prime of beta. phi l of beta this is zero right this is what we are going to get why we are going to get this can you can you understand what is happening here see the boundary is satisfied right now think about it this way this for this problem to work okay see the thing is a naught a1 d naught d1 this is like an arbitrary thing it is non-zero also so basically what happens is this uh these two system of equation will have solutions other than the trivial solution okay so the wrong scale has to be the corresponding coefficient matrix that has to be uh singular so basically uh the wrong this thing the determinant has can be zero so should be zero so basically that is what i wrote it here Now, once this is there, so therefore, if you put this condition here, if you put this condition here, see what is happening is this. Basically, it means that this integral between beta to alpha, if you calculate it from here and here, we are going to get this part is 0 because p is given to be positive.

And therefore, what you have is, that will imply lambda L minus lambda k is Integral alpha to beta r of x phi k of x phi l of x dx. This has to be equals to 0. k not equals to l. Fine. I hope this, you know, the logic behind this is fine. See, if you remember again, what is the condition here for this thing to work?

a0 square plus a1 square is non-zero. And the condition here is d0 square plus d1 square is not zero. What does that mean? It means that this system has a non-trivial solution. Trivial solution is of course there, but a non-trivial solution.

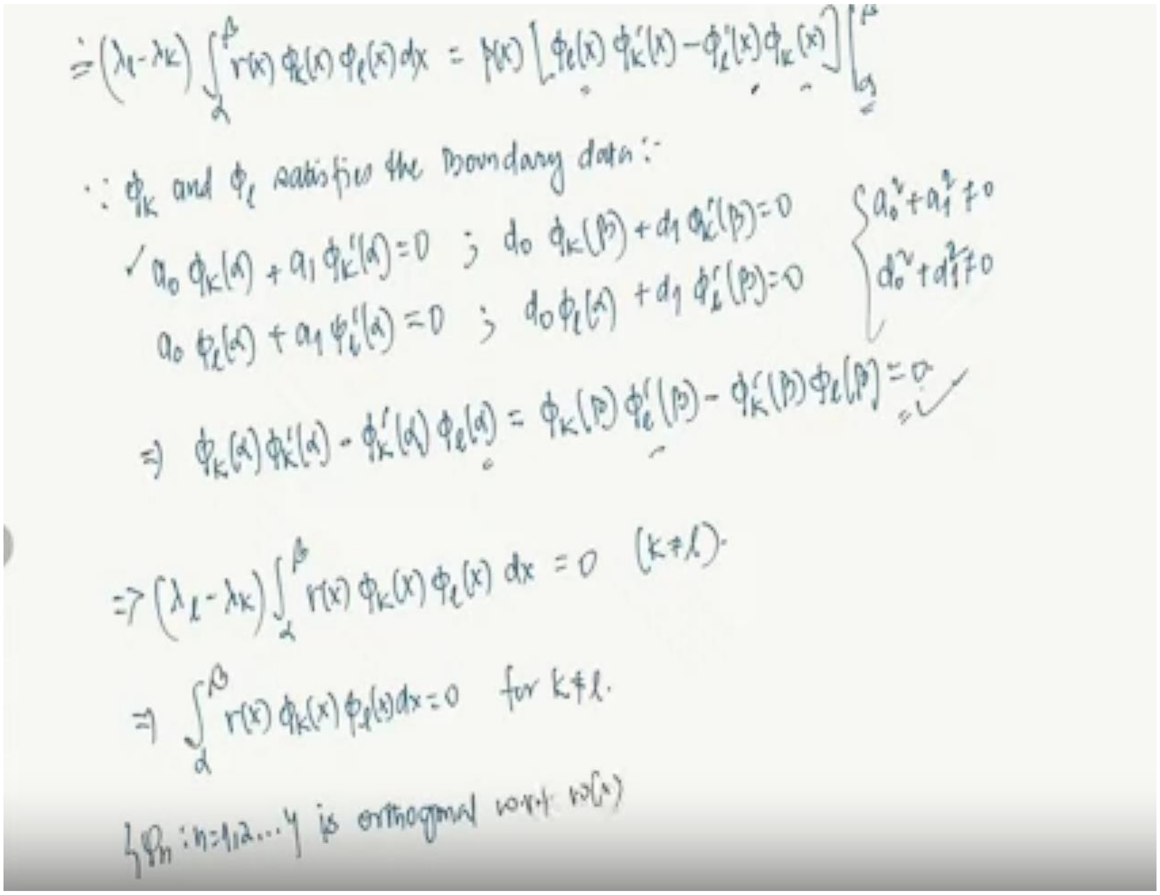
This system has a non-trivial solution. That is what this means. So either one of the a0, a1 or d0, d1 and d0, d1 is non-zero. So if that has to be the case, the corresponding, you know, the coefficient matrix has to be singular. So we are getting this.

Now this is basically nothing but this particular thing. Evaluated at beta minus evaluated at alpha, which is basically 0. So, that will give us this. Is it okay? So, what does this give you?

See, lambda n and lambda k are not same. This is what we assume. Lambda k and lambda l are basically not same. They are distinct eigenvalues, right? So, that will imply alpha to beta, r of x, phi k of x, phi l of x, dx.

This is going to be 0. Is it okay? This is going to be 0 for k not equals to m. So, what does it mean? It means that the set phi n, n equals to 1 to this set, this is orthogonal with respect to

The weight function w of x. Is it okay? So, this is proved. Yeah. So, what did we prove? First of all, how do you produce an orthogonal set of functions?



You just look at the eigenvalues and corresponding eigenfunctions. The corresponding eigenfunctions of a regular stumbling wheel boundary value problems are going to be orthogonal. And also, what did we prove? We also showed that the eigenvalues corresponding to a regular stumbling wheel boundary value problems are simple. Is it okay?

Now, in the next distinct video, we are going to look at what happens when... if they are i mean why not complex okay so basically what we are going to show is we are going to show that the eigenvalues of a regular sum wheel boundary value problem are always going to be real okay that is for the next video so thank you so much