Ordinary Differential Equations (noc 24 ma 78)

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Week-05

Lecture 31 : Maximum Principle

Welcome students to this video and in this video we are going to talk about Maxwell Principles. Now what are Maxwell Principles? See these are sort of theory is a particular type of you know theory which we actually so we call this sort of problems as a a priori estimate. What is a priori estimate? A priori estimate is essentially

you know given an equation yes we do not know that whether uh that equation let's say has a solution or not okay any silly equation doesn't really matter let's say y equals to zero okay you do not have to know whether there is a solution or not what we will do is we will assume that let's say that z let's say solves one Solves one. Okay. Now, can you say something about z? What is the behavior of the z in the interior of points?

How does it behave in the boundary? Let us say you are given that z satisfies this ly equals to that equation, right? Now, can you say that any solution of Z has a maxima somewhere or a minima somewhere, right? That sort of questions, maximum principles, you know, deals with. Essentially, these are called a priori space because without knowing the existence and uniqueness of solution, we derive some properties of the solution.

A priori means beforehand, okay? Now, you see, the thing is and where is it important, okay? Now, To be very frank with you, I really cannot tell you, emphasize enough how important is maximum principle is. You can, so uses, let us just put a small thing like uses where we can use this thing.

Although we are not going to do this in this course specifically, but you can use this to show existence and uniqueness. Or maybe I can show you, it is not a problem. Let me see if we can do that. Existence and uniqueness of solution. okay and you can also uh i mean use it to you know look at the stability properties of the solution stability properties okay properties okay so this is very like a very very important theorem and to be very frank with you uh this is probably the most important you know uh

How do I put it? Principle, most important theory which one can learn. So basically given an operator, whatever the equation is, there is a corresponding operator, right? So given an operator, if you know what, if there is a maximal principle, your life becomes much easier, okay? So what is the idea of the maximal principle?

Basically that we are basically looking for a maxima or the minima, that sort of thing of solutions, okay? So without much ado, let me start with the theorem and by explaining the theorem, I can better explain what exactly are we trying to achieve here. So let us say if y is in C2 alpha beta, what is C2 alpha beta? It is twice continuously differentiable functions defined on close bounded interval alpha beta. Okay, so you are looking at a function y which is twice continuously differentiable in alpha beta such that the function, the double derivative of the function of x, okay, this is always greater than or equal to 0.

What does it mean? It means that the function is convex, okay, in open alpha beta. Double derivative is greater than or equal to 0 means it is a convex function, right? And yx, yx attends its maxima, attends its maximum. At an interior point.

At an interior point. Interior point. What is the interior point? Some interior point of alpha beta. Of alpha beta.

Then an amazing thing happens. Yx is identically constant. Identically constant. constant in alpha beta okay so beautiful theorem right so basically what it is saying is this see we are not really now basically you see initially for this theorem we are not actually working with some equation or anything we are just saying that you take any function right doesn't really matter what is so equation it solves it's just some function y which is twice continuous differentiable such that it has to be convex y double prime x is rather than equal to convex What he is saying is if it attains a maxima at the interior point of alpha beta, then it has to be identically constant in alpha beta.

There is no other way. Is this okay? So let us look at the proof of this here. Proof. See that first of all, we will assume that let us say that y double prime of x is not greater than equals to 0, but strictly positive in alpha beta for now.

Let us just assume that. So, suppose y double prime of x is strictly positive in alpha beta. You will see why we can deduce something like this. Now you see, let us say, If yx attains its maxima in an interior point of alpha, beta.

of alpha beta, then what can you say? Let us say, so basically what I am trying to say is at an interior point of alpha beta, let us say x naught and the point is x naught. Then what can you say? y prime at the point x naught has to be 0, right? Okay.

And what can you say about double prime? y double prime at the point x naught is less than equal 0. That is just the definition. The second derivative test you know, right, that if it is a maxima at an interior point, it has to be critical point. So, y prime x naught is 0 and y double prime x naught has to be less than equal 0.

Now, see this. So, basically you have a point x naught where it is less than equal 0, but we have assumed that y double prime x is positive. So, that is a contradiction, that is a contradiction. Hence, what can we say that if thus, what can we say, if y double prime of x is positive in alpha beta, alpha beta, then the function then the function cannot attain, cannot attain its maxima, its maxima at an interior point of alpha beta, at an interior point of alpha beta.

I hope this is clear to you that the thing is you it has to be i mean it has to be constant in that case there is no other way so uh y double prime x if it is greater than zero then it has to be constant it cannot have a maximum at an interior point okay it has to be everywhere so basically same okay now let now what happens if there are some point where it is zero so would the derivative double derivative is zero so now suppose Suppose y double prime at the point x is greater than equal 0 in alpha beta. In alpha beta and we will also assume that yx attains its maxima. Attains its maxima at x1.

which is in alpha beta open alpha beta right some some point now if y at the point x1 yeah is m so what is the m m is the maxima then what can we say then y of x should always be less than equal m okay this is for all x in alpha beta okay now you see what we will do is this let let there exist a x2 okay which is between alpha beta such that yx2 is less than m here so if x2 greater than x1 okay so you see If yx1 is equals to m, it means that the maxima is attained at the x1 point. So, basically let us just assume that is m. So, for any point x in alpha beta, y of x has to be bounded by m, right? That is what we wrote.

Now, let us just assume that there is a point where yx2 is less than m. Of course, there has to be such a point. If there is no such point, then y will be a constant, right? It is identically equals to m and it is a constant. So we have already proved it. It is nothing to go on.

So we will assume that y of x2 is less than m. Now once that is true. So see x1 is here. And now y of x2 is less than m. y of x2 can be here. x2 can be here. x2 can be here.

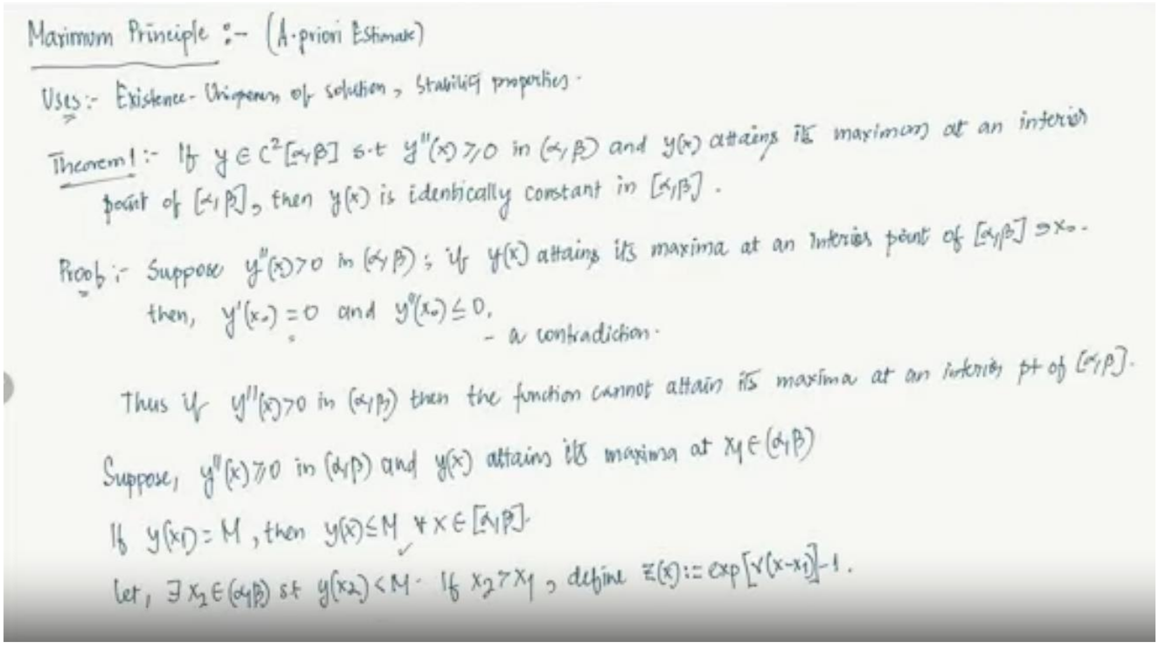
Let us say this is alpha. This is beta. Now, x2, we do not know where x2 is, but we know that there is a x2 such that y of x2 is less than m. Clear? Then, what we will do is this. So, let us say x2 is greater than x1.

So, let us say x2 is here, not here. Here, this case. Then, we define a function. Define z of x to be exponential gamma x minus x1 minus 1.

What is the need of this function? See, basically what we are trying to do is this. We know that if y double prime is strictly positive for all x, right, then the maximum cannot be added in the interior. We want to use this fact for the next part. So,

for next part let us say we know that there are some points where it can be zeros y double prime can be zeros right so what we are doing is we are building a new function like this so okay and what happens is this if you see if we build z like this okay what we have is this see clearly z of x is negative for x in alpha x1. It has to be negative. x2 is less than x1 right and at the point x2 y of x2 is less than m so basically if you define this you see uh z of x if you look at between alpha and x1 x minus x1 right x minus x1 so in between this particular thing this is basically negative right it is exponential gamma some negative term minus 1 so basically it is less than 1 so this has to be negative right now what is z of x1 of course that is 0 see z of x1 if i am putting it is exponential 0 so is 1 minus 1 so basically 0 and if x is greater than x1 okay so basically in that case this is exponential of a positive minus 1 so basically you have this positive term right so you have and

Z of x is positive for x in x1 beta. Is this okay? Also, you see, this is important. Z double prime, if you calculate of x, you see, this is exponential gamma type things, right? So, it is basically gamma square exponential gamma x minus x1.



So, minus 1 is gone, right? With the derivative. now this is positive exponential is a positive term gamma squared is positive okay this is positive for all x in closed alpha beta okay so now what we will do is this see this is a small trick which you are going to use just to you know i mean get our result okay so what we are doing is see we got a z such that z double prime is positive yeah now what we will do is we define a new function define w of x which is y of x plus epsilon z of x. Where this epsilon, we will specify here, you will see why we need this epsilon. So epsilon is positive and which is less than m minus yx2 by zx2.

Is this okay? I mean, we will define this new and this epsilon is such that something like this happens. See, if this thing happens, what is the thing? So, it becomes at the point of w of x2. If we define epsilon like this, what happens to w of x2?

It is y of x2 plus epsilon times z of x2, right? z of x2 is, so this is strictly less than m minus y of x2 by z of x2, okay? times z of x2. So, this is gone. So, this will be strictly less than m. You see, we are constructing this w in such a way that w at the point x2 should be always less than m. Why we are doing it, I will explain to you right now.

So, this is the reason why we are choosing a epsilon like this. Now, since you see y of x2 is less than m, as I have explained, and z of x2 is positive. See, z of x2, x2 lies between x1 and beta, right? So, z of x2 is positive, and y of x2 is less than m, because y, I mean, we have assumed, this is our assumption, you see?

This is our assumption, okay? So, therefore, this sort of axis, hence, epsilon exists. Clear? Okay. And we also have that w of x is less than y of x. Okay?

And which is less than equal m? Clear? for x in alpha x1 plus c between alpha x1, where is it? Here, between alpha x1, z is negative. So, basically w will be dominated by y of x, clear?

And w of x2, we have just calculated it, right? w of x2 is strictly less than m, yes? And what is w of x1? w of x1 is y of x1 plus epsilon times z of x1. z of x1 is 0.

So, it is y of x1, okay, which is n. See, y of x1 is, what is it? y of x1 is n. The maximum is at n there, okay. So, this is just a trick, because I want to use this particular thing, that a function, if it is strictly positive, then, you know, it has to be constant. This is what we need to use, okay. Now, see, since

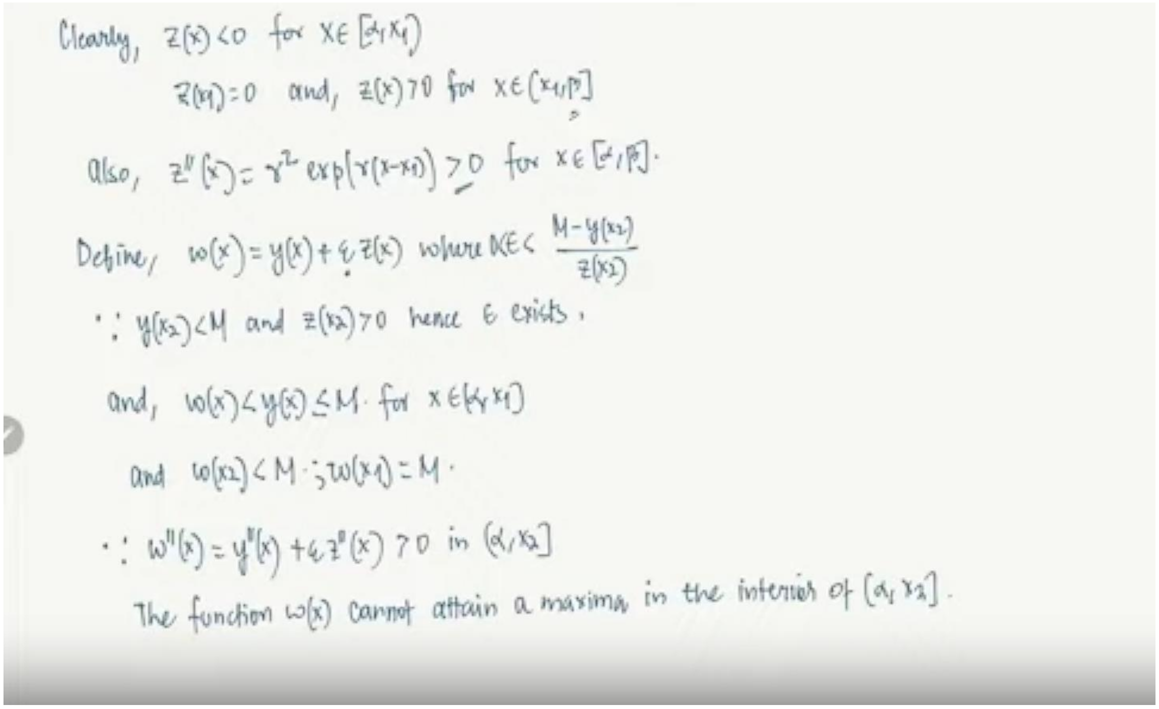
W double prime of x, what is it? It is y double prime of x plus epsilon z double prime of x. Okay, this is strictly positive in alpha x2 closed. Is it okay? Of course, this is right. See, alpha x2, so it starts from alpha until x2 closed.

y double prime is given to be positive, we are assuming, you see, y double prime, we have assumed that y double prime x is strictly greater than 0. And what about this z? z is such that z double prime is strictly positive, you see, between alpha, beta. Here it is given, z double prime is strictly positive. So, y double prime x plus epsilon times z double prime x is strictly positive.

Is this okay? And so, in between alpha x2, what is happening is this, w double prime of x is strictly positive. Okay. Now, since, one second.

Yeah. So, we have actually found out a function which is strictly positive in alpha x2. Okay. So, therefore, the function, the function, if you restrict it to alpha x2, w of x cannot attain, cannot attain Okay, a maxima in the interior of alpha x2.

Is it okay? Okay, it cannot have, this is, where are we getting it? We are getting from the first part. Okay, now let us look at it. The thing is between alpha x2, we have it.



Okay. Now you see, since what happens now, let us look at what happens at W at the point alpha. W at the point alpha is less than m. Right. This is what we calculated here. You remember?

W at the point alpha is less than m. Right. See, W at the point alpha is less than y at the point alpha, which is less than m. Right. Okay. right so w at the point alpha is less than m w at the point x2 again is less than m and w at the point x x1 equals to m is it okay see w is such that think about it this way at the point alpha this is x1 this is x2 x2 is here this is what we assume that x2 is bigger than s1 this is beta W is less than M. So let us say this is your M, height is M. This is the M line.

At the point alpha, W is less than M. At the point x2, W is less than M. So somewhere here let us say. Here at the point alpha, here at the point x2. And at the point x1, it is equal to m. So, basically, it does this, right? Minimum, that is the minimum it has to do. So, you see, then if it does, then wx must attain a maxima.

Greater than equals to m. Is it okay? At some point it is taking m. So basically the maxima will be at least m or it will be greater than equal to m. It has to be in between alpha x2. Yes. I hope this is fine. At an interior point of alpha x2.

See, between alpha x2, there is a point where it is taking at least n. At least. So, basically the maxima is greater than equal n. Yes. Okay. So, this is a contradiction, right? Because we just showed that Ws cannot attain any maxima in alpha x2.

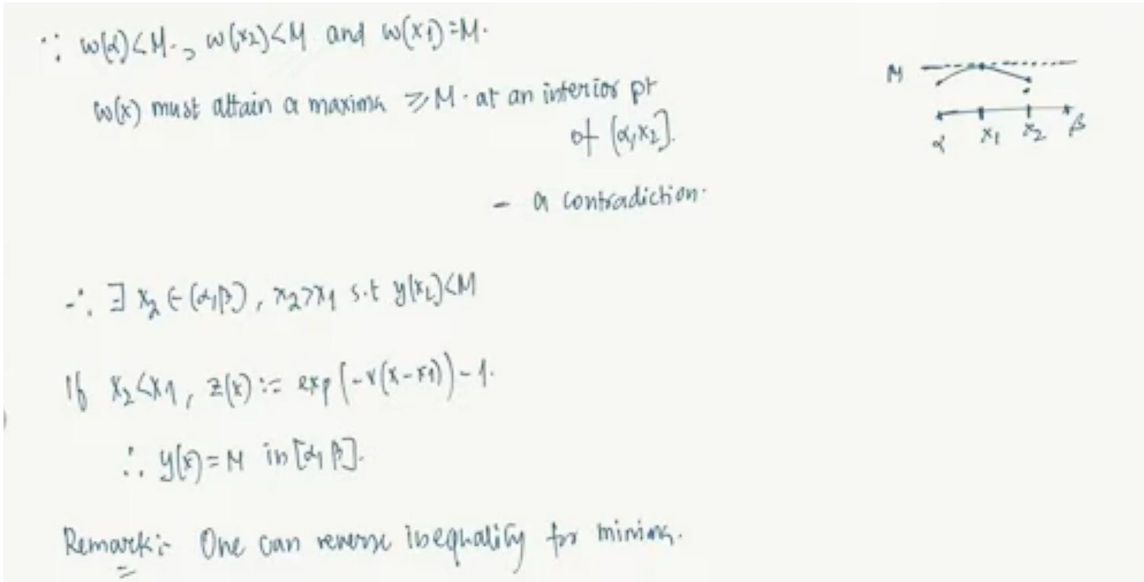
And here we are showing it is taking a maxima. It has to have a maxima between alpha x2. Okay. So, that is a contradiction. A contradiction.

Hence, what can we say? Hence, therefore, there does not exist a x2 between alpha beta, clear? Between alpha beta, x2 greater than x1 such that y of x2 is less than m, okay? Now, what happens if x2 is less than x1? Then you, so if

x2 is less than x1 we can do similar sort of analysis but z of x we have to set a little bit differently it is will be exponential minus gamma x minus x1 minus 1 okay and you can do the similar sort of argument and you can show therefore yx has to be identically equals to m in alpha beta Is it okay? So, basically what it is saying is this, if you have y such that it takes the maxima at the interior point and y double prime is positive, then it has to be identically constant. Please remember this thing, okay? Right.

Now, the thing is, so this is for any solution, sorry, any function, right? It does not have to be a solution or anything. It is just function, okay? Now, you see, You can have the reverse inequality.

So let me put a small remark here. One can have reverse inequality for minima. So what happens then? You can say that if you have a y which is c2 such that y double prime is negative, non-positive essentially, less than or equal to 0. Then it will not be attaining its minimum at an interior point.



And if it does, then it has to be constant. This is okay. Now what we are going to do is look at some equations. We will use this and we will look at some equations from here. So first of all,

See, the thing is, we want to generalize the idea. We want to use maximum principle. Use maximum principle. For what sort of problem? Let me write it down.

Principle for this sort of problem. y double prime plus px. y prime plus qx y is greater than equal 0. This is okay. We want to write down a maximal principle for problems of this particular form.

Okay. Now, you see, let me give you an example to show that one cannot expect something like that. So, let us say that, you see, the function, let us say yx goes to sin x. Okay, we know that solves this equation, right? Solves y double prime plus y equals to 0.

We know that, right? Okay, now in 0 pi, we know this, right? Yes. And why do you think the maximum is attained? The maximum is attained in the interior pi by 2.

But maximum of y It goes to y at the point pi by 2, right, which is 1. So, it is attaining at the interior, right? So, basically, maximal principle, maximal principle, let me just write it, does not, does not hold. clear so basically you cannot expect this also basically see we did it for y double prime positive greater than equal zero now if you want to you know add some lower order terms so basically y double prime plus py prime plus qy greater than equal zero if you want to do similar sort of thing for this sort of equation it does not work that's what it says

Is it okay? But the thing is, but should we, you know, give up? But you should not. Let me give you another example. See, let us say that b. You see, if you change the equation and then you have this sort of equation.

Let us say yx equals to e power x minus, let us say minus, minus e power minus x. Both are minus. Minus e power x minus e power minus x. solves this problem, right? Solves y double prime minus y equals to 0. Is it okay?

And the maximum, maximum of y is attained at, if you can check it, it is attained at minus, sorry, attained at 0, which is minus, attained at 0, which is minus 2, okay? And this is also in Minus 1 1 so basically it is not that you know only that particular equation has a issue There are other equations also, which you write there are issues. Okay, so it can happen now The thing is this but we can actually you know work around this problem. Okay, so we look at a theorem theorem and What does this theorem say it says that let Y X satisfies

satisfies the differential inequality. Differential inequality of dI, let me just write it down, given by y double prime of x plus px y prime plus qx y greater than equal 0 for x in alpha beta. Clear? Where Px and Qx are non-positive and bounded.

Positive and bounded. Please remember, non-positive and bounded. Very, very, very important. We need these particular conditions. Non-positive and bounded.

In every closed subinterval of alpha beta. Of alpha beta. Okay. So, I mean, you can change boundedness to continuity also.

It is not a problem. Then this will work. Okay. So, basically, you just assume that P and Q are continuous, but they are non-negative. Sorry, non-positive.

Now, if yx assumes that If yx assumes a non-negative maximum value m at an interior point of alpha beta point of closed alpha beta. Then, then yx is identically equals to n, okay. So, we have the maximum principle. So, what does it say?

It says that if you look at this equation, y double prime plus px, y prime plus qx, less than or equal to 0, and both are non-positive. So, this is very important. p and q, they have to be non-positive, okay. So, it is less than or equal to 0. And they are continuous.

For now, let us just assume it is continuous. You do not need continuity, but we can assume that. Now, you see, what it is saying is this. Beforehand, before we were not saying anything, before we just said that If the function at n is maximum at an interior point, it has to be constant.

Here we are not saying that. Here we are saying that if yx assumes a non-negative maximum, not any maximum, but a non-negative maximum value, m, at an interior point, then it is constant. Is it okay? So let us look at the proof. Proof.

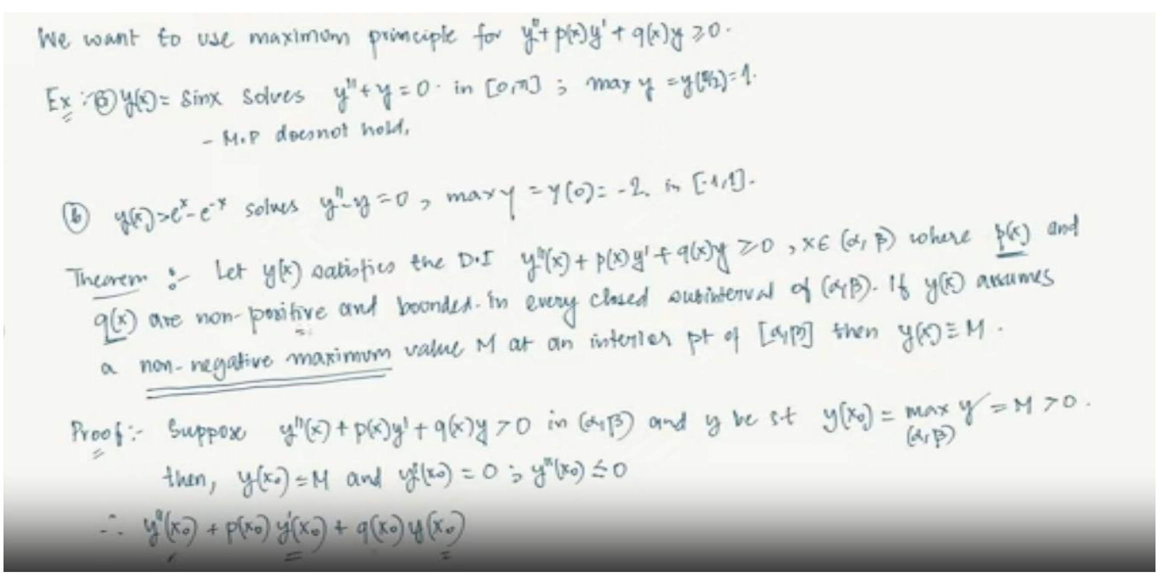
So, you see, the thing is, let us just assume as we did it earlier, right? So, first of all, suppose, suppose y double prime x plus p x y prime plus q x y, okay, is strictly greater than 0 in alpha beta. Let us just assume that. Now, you see, we will also assume that and it as and y attains a non-negative maxima, this is very important, non-negative maxima, not any maxima, but non-negative maxima, non-negative maxima m, maxima m, let us say, So let us write it like this.

So y be such that y at the point x0 equals to maximum of y here in alpha beta and what is it? equals to m. So, m we will just, so let us say that is positive. This is what we are assuming. Is this okay? Then what we can say is, see, then y at the point x0, that will be capital M here.

And y prime at the x0, this is the maximum, this is the critical point. So, it has to be 0. What happens to y double prime at the point x0? If this is your second derivative test, maxima, so it has to be less than equal 0. So, therefore, y double prime at the point x0 plus p of x0, y prime at the point x0 plus q of x0.

y at the point x naught, what happens to this? See, this term is not there because y prime x naught is 0. This is m times q, okay? q is less than or equal to 0, right? It is non-positive.

q is non-positive. y double prime is non-positive. This is always strictly less than or equal to 0, right? That is a contradiction. Why contradiction?



Because it is given that this is strictly positive, okay? So, it is a contradiction. contradiction. Is this okay? So, we basically have that if it is strict positive, then I mean, there is something fishy going on and we cannot have it, have these assumptions.

So, basically what it is saying is yx has to be strictly constant. So, basically it is identically equals to m. Is this okay? Now, let us look at the other part. What is the other part? If, you know, it is greater than equal to 0, not just power 0.

So, suppose, so suppose, y double prime plus pxy prime plus qxy. Let us say that this is greater than equals to 0. And there is a x1 such that yx1 equals to m. So, the maximum is attained at m for some x1 in alpha beta. Is it okay?

And we suppose, suppose there exists a x2 in alpha beta. such that y at the point x2 is strictly less than m. Yes, why? Because see, we are assuming, we are starting with the assumption, this is not identically equals to m. So, basically if that is not the case, there must be a x2, right, which at where y should be less than m. m is a maxima, it cannot cross m. So, it should be less than m, Now again from the earlier thing you remember if let us say x2 is greater than x1. Now we will just use what we did earlier same sort of ideas we will use it.

So if x2 is greater than x1 then what we will do is we define then we define a function z of x which is exponential Okay, gamma times x minus x1 minus 1 and gamma positive of course, gamma positive it is a value to be determined, to be determined. And you see the z of x and z of x satisfies those conditions, right? What we did, you know, earlier that z of x is negative in alpha x1. You remember, we just did in the earlier theorem z of x1 because it is basically the same function, right?

Nothing changes, does not depend on the equation. And z of x is positive in x1 and beta. x1 and beta. Is it okay? Right.

Sorry. It has to be close beta. Up till beta. Okay. So, z has that same property.

It does not change anything. Now, since qx is negative, this is given non-positive essentially. Okay. It follows that you see z double prime plus px times z prime Plus qx times z. Okay.

What happens to this? Let us just look at it. You see, it is gamma square. If you just calculate it. Yes.

So, please check this part. It is gamma square plus px times gamma plus qx. Okay. 1 minus exponential minus gamma x minus x1. Okay.

and then exponential gamma x minus x1. That is what we are going to get. Now, this is always dominated by gamma square plus px times gamma plus qx times exponential gamma x minus x1. Is it okay? We can always do this, right?

1 minus exponential. So, that is why we can, you know, we can show that this is always dominated by. So, this Q can be replaced by this, right? Yeah. Okay.

So, now the thing is, you see, we choose. See, we have assumed that gamma is to be determined later, right? So, we choose gamma such that gamma square plus p of x gamma plus q of x is strictly positive in open alpha beta. We can choose it, right? Gamma is some given function.

Yes, gamma is nothing special. Gamma is some given function. So, we can choose gamma such that gamma square plus px gamma plus qx is strictly positive, okay? Now, you see, if this happens, you see, gamma is such a function such that this is strictly positive. So, that will give you that this is strictly positive times exponential.

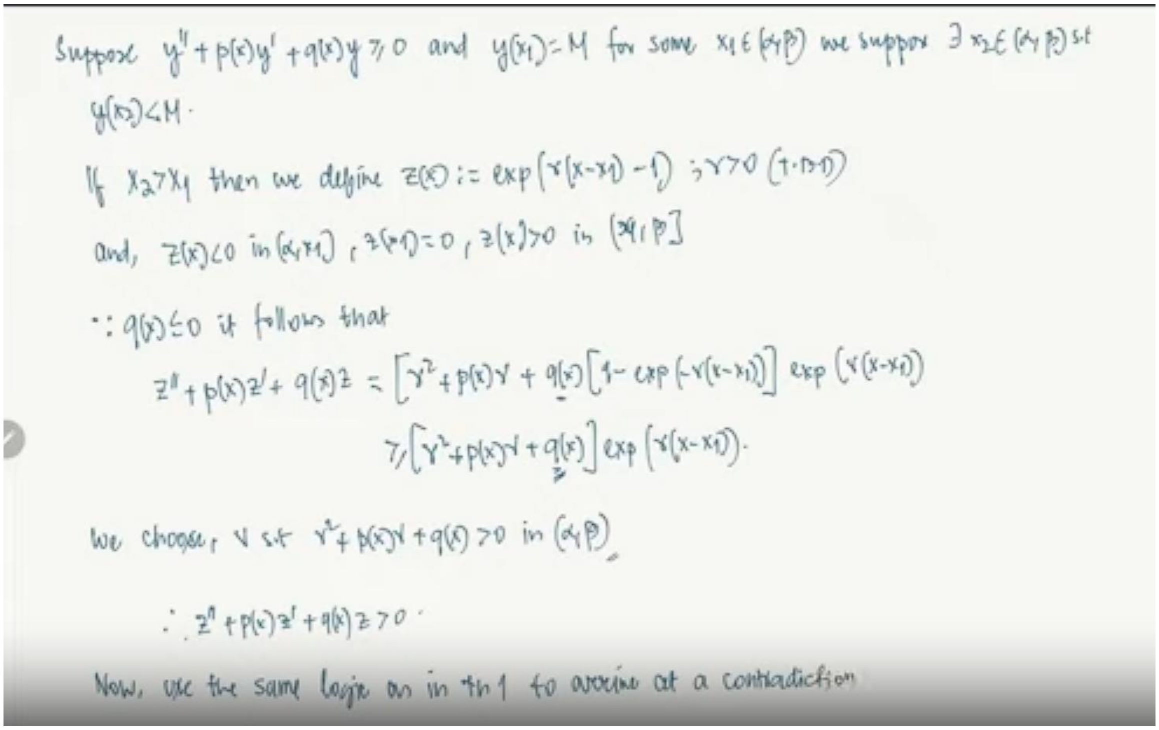
That is also strictly positive. So, basically, therefore, z double prime plus pxz prime plus qxz will be strictly positive. Yes, we can do that. Now, what we do is then we can say that, you know… one second, I missed something, I do not think so, yeah, this is fine, right, yeah, so now, once this is positive, then this expression containing z, that is also positive, so we can use this particular thing, right, you see, if it is positive, if this expression is positive, then it has to be identically equals to m, if there is like, if it assumes a non-negative maxima, that is, okay, so we can use that, and then, you know, use the

you know how do i put it this part yeah see this this uh this logic which we used here okay see we showed that w double prime is strictly positive and then it attains a maxima cannot attend some cannot attain a maxima this you can say using the other part the first part of the theorem and this then you use this uh you know uh logic use this logic exactly the same sort of logic works and you show that it cannot uh work i mean the maxima cannot be attained so basically you will get a contradiction okay so use the same logic now what you do is sorry now use the same logic same logic As in theorem 1. Theorem 1. To arrive at a contradiction.

I hope this is clear to you. Arrive at a contradiction. Contradiction. Is it okay? Fine.

So. You can. We can do what. That we can do. You can use this actually theorems.

To actually you know. prove some better results about solutions of some equations that we will do I will put some problems in the assignment where you will see that how Maxwell principle can be used to talk about you know uniqueness or to give some sort of estimates upper bounds or what bounds on some solutions. So, with this I am going to end this video. Bye.


Suppose $y'' + p(x)y' + q(x)y \geq 0$ and $y(x_1) = M$ for some $x_1 \in (\alpha, \beta)$ we suppose $\exists\, x_2 \in (\alpha, \beta)$ s.t. $y(x_2) < M$.

If $x_2 > x_1$ then we define $z(x) := \exp(\gamma(x - x_1) - 1)$; $\gamma > 0$ (to be fixed)

and, $z(x) < 0$ in (α, x_1), $z(x_1) = 0$, $z(x) > 0$ in $(x_1, \beta]$

$\because q(x) \leq 0$ it follows that

$$z'' + p(x)z' + q(x)z = [\gamma^2 + p(x)\gamma + q(x)[1 - \exp(-\gamma(x - x_1))]] \exp(\gamma(x - x_1))$$

$$\geq [\gamma^2 + p(x)\gamma + q(x)] \exp(\gamma(x - x_1)).$$

We choose γ s.t. $\gamma^2 + p(x)\gamma + q(x) > 0$ in (α, β)

$\therefore z'' + p(x)z' + q(x)z > 0$.

Now, use the same logic as in th 1 to arrive at a contradiction