Ordinary Differential Equations (noc 24 ma 78)

Dr Kaushik Bal

Department of Mathematics and Statistics

Indian Institute of Technology, Kanpur

Week-05

Lecture 30: Linear Boundary Value Problem

welcome everyone in this video and today we are going to talk about linear boundary value problem linear boundary value problem So what do we mean by this? See up till now we have talked about second order, let us say second order equations. So we stick with second order equations. But the thing is data was given only at one point.

So basically we are looking at y at the point 0, y prime at the point 0 or some other point. So essentially let us say up till now we have this sort of thing. Let me do it here. We have, let us say, the equation which looks like this, y double prime plus y equals to 0. And you have this initial data, y at the point alpha is, let us say, a, and y prime at the point alpha is b, right?

With this initial data, so the data is given at only one point. With this data, you know that there is a existence, you know, there is a unique solution, right? In whole of R, right? That is what we did. But the thing is here, the data is only given at a single point.

Okay, now what we want is we want to see that if let us say there are two points alpha and beta and you know what happens to y at the point alpha and y at the point beta or y prime at the point alpha or y prime at the point beta. You understand? So basically what I am trying to say is this you have the data or two points alpha and beta not only at the point alpha. Now, the thing is this, what can we say about the existence of solution or let us say uniqueness, that sort of questions, those sort of questions we want to answer now, okay. Okay, so let us start the class.

So, what we are going to do is basically we are going to consider the differential equation, so consider this equation, let us say, consider differential equation. y double prime so let us take p naught of x here p naught of x y double prime plus p 1 of x y prime plus p 2 of x y equals to Let us say for now let us just say R of X and let us just say that is your first equation. Now this is in some interval in some interval I which is alpha beta. Is this okay?

Yes. And of course where where P naught P1 P2 and r is assumed to be is continuous is continuous in i this is what our assumption is yes we are assuming they were continuous in i so together with one so essentially we are now putting some boundary conditions okay what are the boundary conditions so those are the these are the boundary conditions Boundary conditions. So, what are those?

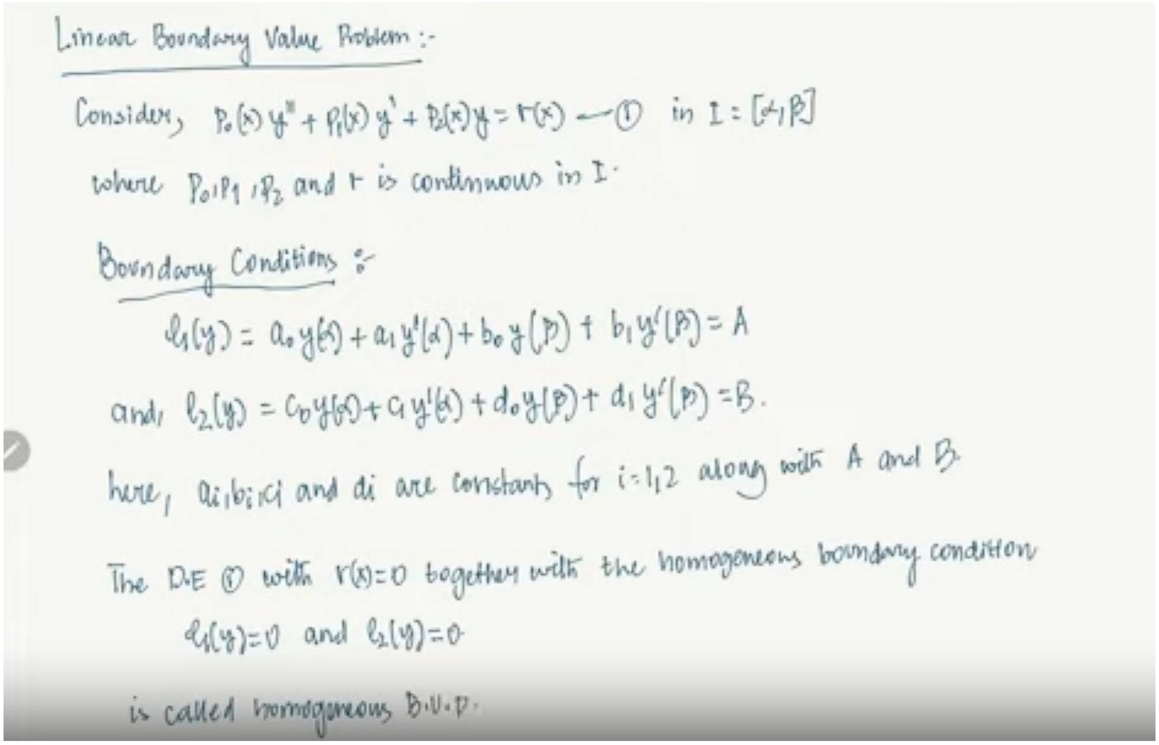
Boundary conditions are L1 of y. This is the first boundary condition. L1 of y. We will just denote it like this. So, please, I mean, we are going to use this notation L1 of y, L2 of y, this sort of thing. So, what is it? It is A0 of y at the point alpha plus A1 of y prime at the point alpha plus

plus b naught of y at the point beta plus b 1 of y prime at the point beta equals to a. L2 of y, we will define it as some a, b, e, so c0 of y at the point alpha plus c1 of y prime at the point alpha plus d0 of y at the point beta. plus d1 of y prime at the point beta. This is, let us say, b. So, a and b. Okay? Capital A, capital B. Now, here, if this is coefficient, here, all the coefficients, a0, b, I can write ai as a vector.

Here, ai's, bi's, ci's, and di's, and di's, R constants, R constants for I equals to 1, 2 along with A and B. I love it. A and B. It is okay. So, everything is constant.

Okay. So, now, see, let us say that. So, now, whenever we are working with a problem, basically, we are assuming that we have the problem 1. That is always there. Now, the boundary conditions change.

You see, if you, why we took it like L1, L2, these are the general boundary conditions. So, basically you can take different choice of constants and then you get different sort of boundary conditions, clear. So, first of all, let us just put it this way, that the boundary, the differential equation, okay, the differential equation 1 with r of x is 0 r of x is 0 together together with the homogeneous boundary condition homogeneous boundary condition. okay what's the homogeneous boundary condition l1 of y equals to 0 and l2 of y equals to 0 okay so these are the homogeneous boundary condition it's called it's called homogeneous boundary value problem.



This is okay, right? So, in this case, A equals to 0 and B equals to 0. This is, we are assuming here. Now, we will look at some interesting thing.

So, this sort of problem, we will call it a homogeneous boundary value problem. Obviously, we have the homogeneous equation and then the boundary values are also homogeneous. So, a and b are 0. Now, there are I mean many general boundary conditions, but these are the important ones. So, let us just put that important boundary conditions.

Boundary conditions. So, please remember these boundary conditions because we are going to use this sort of, you know, nomenclature even in the future also. So, important boundary conditions. Number A. This is called the Dirichlet. Dirichlet boundary conditions.

What is Dirichlet boundary condition? This is y at the point alpha is a and y at the point beta is b. Is this okay? That is your Dirichlet boundary condition. So basically you only have data at, see basically you need to have data at a and b, right? That is just given to you.

So a and b, those are the two points where data will be given to you. You have to have, right? The thing is you do not need to know anything about what happens to y prime. We only know what happens to y. Basically, we know what happens to y at the point alpha and we know what happens to y at the point beta. That should be the boundary condition.

Then you have mixed condition, mixed boundary condition. So, what are mixed boundary conditions? It is y at the point alpha is a and y prime at the point beta is b. Is this okay? So, this is the second boundary condition which is the mixed boundary condition. Or also one can see that it will be y prime at the point alpha is a.

And Y prime, Y at the point beta is B. So basically mixed means I have data at the point, for Y prime also and Y also. So basically at two different points, alpha and beta, I have data on Y and Y prime. So let us say that is B. That is your mixed boundary condition. Okay. And there is another one which is called C. This is called, for now we just call it as a Robin boundary condition.

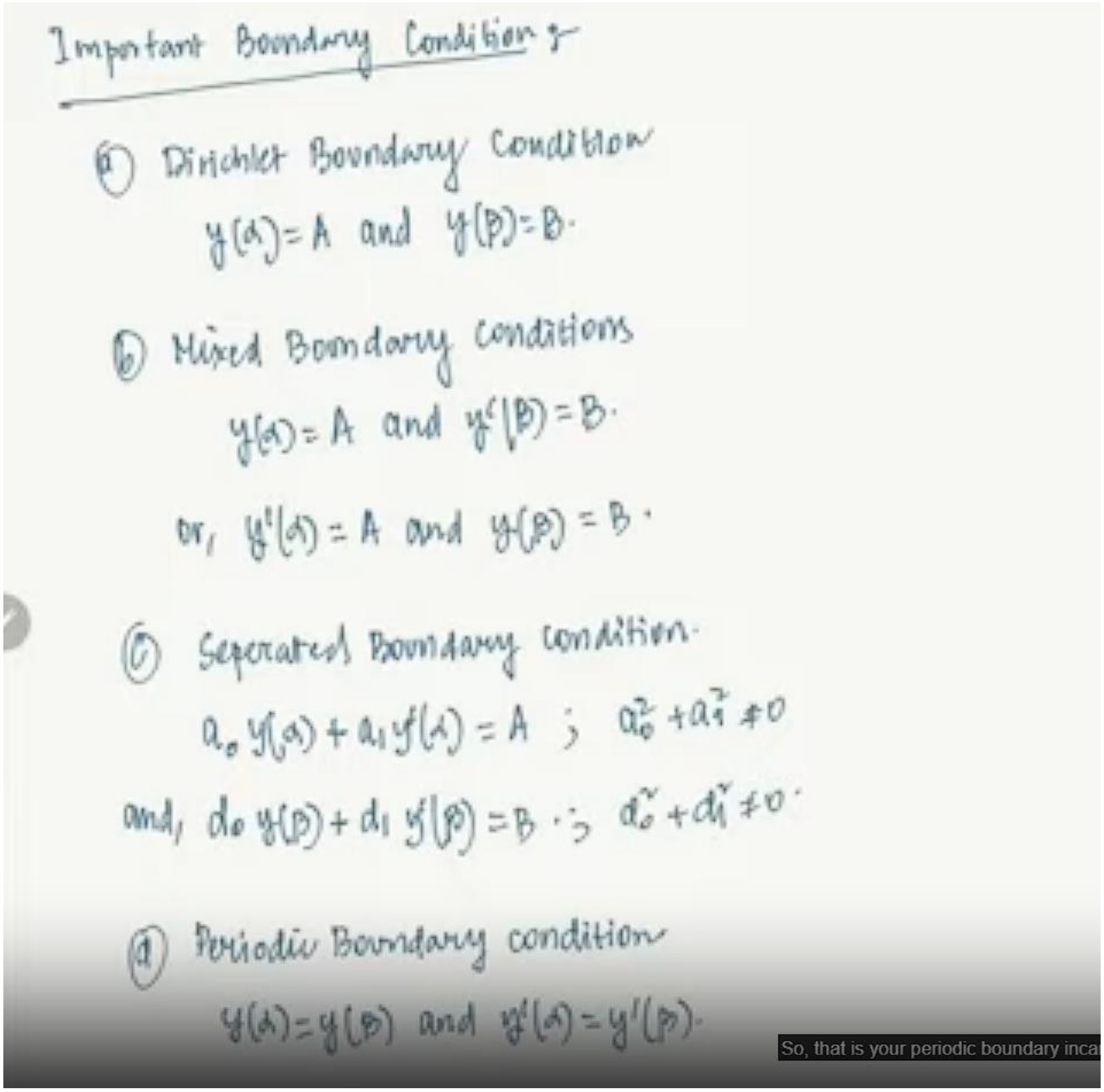
Okay. robin boundary condition okay and what is this this is uh maybe it is better to write it as separated boundary condition uh let let's not put robin because uh Separated, let us just put it separated. Separated boundary condition. What is separated boundary condition?

So, you have the data at y, a alpha and the data at beta. Like this, a naught of y at the point alpha plus a1 of y prime at the point alpha is a. And d0 of y at the point beta plus d1 of y prime at the point beta is b. This is okay. Now here you see the data is given like a0 square plus a1 square is non-zero. Again, d0 squared plus d1 squared is non-zero.

Why this condition? See, if both the coefficients a0 and a1 is zero, then you do not have any data on alpha. And similarly, same thing holds for beta also. So, you want that condition. And what happens if you do not have data at the one point, let us say alpha or at the point beta?

This seems to be a boundary problem, right? Boundary value problem. And then a very very important boundary condition this is called periodic boundary value problem. Periodic boundary condition. So what is periodic boundary condition?

It is given by y at the point alpha equals to y at the point beta. and y prime at the point alpha equals to y prime at the point beta. Is this okay? So, basically you know that at the end points, the curve comes back to itself such that a derivative is also coincides. So, that is your periodic boundary incarnation.



So, now we come to a very definition which we are going to call it definition 1. see we say that the problem one what is r1 this problem p naught y double prime p1 y prime p2 y plus equals to r of this problem okay and so uh maybe let me put it this way that this is 2 2 okay so the boundary value problem one plus two so you have the problem one and then two is a data right two boundary data it's called regular we call it a regular boundaries regular boundary value problem okay it's called regular if alpha and beta are finite qualifier beta are finite you see this data the problem this this uh interval alpha beta okay that should be finite okay and you need to have this p naught of x is either positive or negative it cannot have any zeros anywhere right so for without loss of generative we'll assume that p naught of x is positive you see you understood what i'm trying to say see

The thing is if P naught is taking 0 somewhere in that point, Y double prime basically does not exist in, right? P naught of X, let us say X naught where P naught is 0. So basically for that X naught, Y double prime, that first term itself is not. So basically this problem seems to be a second order equation and there is an issue there. So we do not want that to happen.

So we are basically assuming that P naught X is positive for our problem actually for the regular thing, yes? So, finite P naught of x is non-zero. Yes. So, it can be positive negative. Okay.

But generally speaking, we will assume it to be positive. Okay. For all x in I. For all x in I. Okay. Yes. So, this is A. Then P, the boundary value problem.

1 plus 2. This is called, the first one is called regular, right? Now, this is called, what is it called? The singular. Singular.

If, okay, C alpha is minus infinity. So, there is no lower bound. And, slash or, okay? beta is infinity so it doesn't have to be beta has to be infinity either one of them is infinite okay and or p naught of x is zero for some x in i this is okay yeah so then we call this problem as a singular problem is it okay so you see if p naught x is zero some point those problem points there is some issue about that second order equation and all we'll call those type of equations as a singular problem okay of course again if that you know

The interval which we are working with, that is also infinite. So, basically, minus infinity, infinity minus infinity alpha or alpha infinity, that sort of interval also, if you have the problem, then also we call it a singular problem. Is this okay? Right? So, now, we are only going to consider.

So, basically, in this whole course, singular problems are difficult to solve. To be very frank with you, singular problems are very difficult to handle. Okay? Generally speaking, we are only going to talk about regular problems. Okay?

So our focus is regular problems. So the first question is this. Does there exist an unique solution? Unique solution. So basically, first of all, is there a solution and whether that's a unique solution or not to a regular boundary value problem.

We'll just put it like this because it's a very big term. I don't want to write it. So this is called regular boundary value problem. Problem is this, okay. So, we want to know that whether there exists a unique solution or not, okay.

So, let us just look at it. See, let us just, I mean, you know, explore the situation with some examples. So, first example is this. So, example. See, let us consider this equation, our favorite equation, okay.

y double prime plus y equals to 0. And let us put some initial data on it. So, let us say y is 0. equals to 0 and y at the point pi equals to 0. So your alpha and beta is 0 and pi.

So this is the regular stuff you will want to solve problem. Absolutely no issues there. And what sort of problem is it? First of all, identify what sort of problem is it. This is called a Dirichlet problem.

You see? A and B, capital A and capital B, they are going to be 0. And you have data on y only at the 2, 8.0, sorry, 2, 8.0 and pi, right? So, this is a divisive problem, okay? Now, what is the solution of this problem?

I mean, let us do this thing. You see, a general solution y of x looks like ac1 sine x plus c2 cosine x, right? If you put the first data y0 is equal to 0. So basically you get c2 equals to 0. And when you put the other data y at the point pi equals to 0.

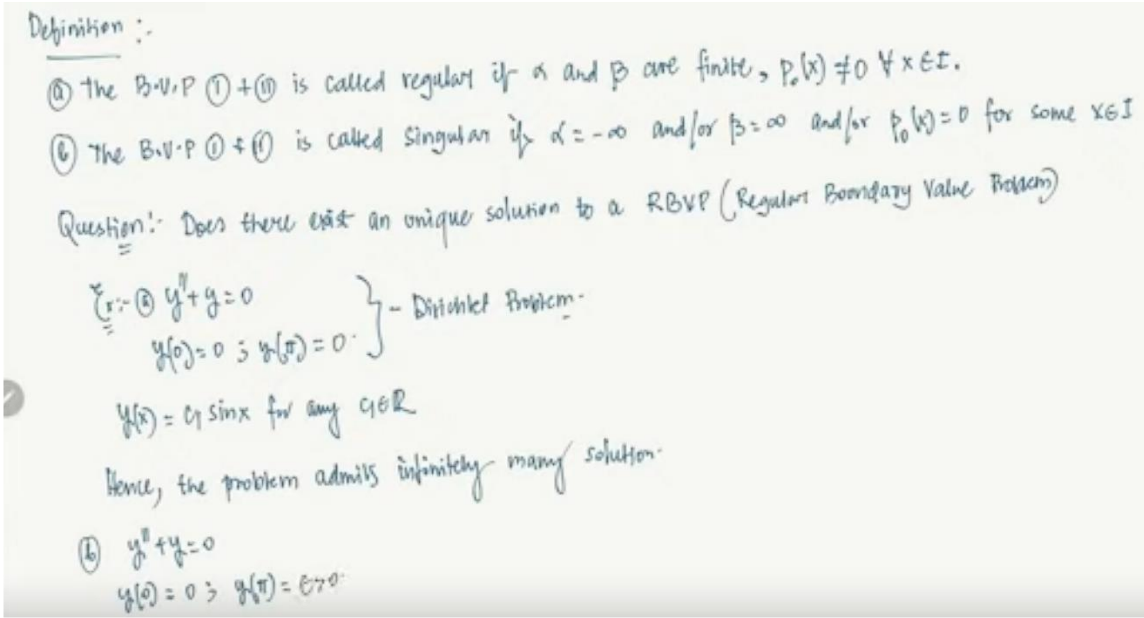
Then also you are going to get c2 equals to 0. So basically here for y. So what is the solution? The solution is this. Okay. For any c1 in R. Right.

Okay. Hence. the problem admits admits infinitely many solutions infinitely many solutions so see this is a like a huge departure from the theory of initial value problems right see initial value problems Once that initial data is fixed, right, data is given to you, these sort of equations, I know that there exists a unique solution in whole of R. But here what is happening is this two data, two point data is given and you have like the first example actually shows that they are like infinitely many solutions, okay. But the question is this, does it always have infinitely many solutions?

Yes, let us just look at that. So let us look at another equation. y double prime plus y equals 0. Same equation, right? So let us put this data.

y at the point 0 is 0. Maybe I can put it 0. y at the point pi, here I put it to be 0. Let us say I am putting it to be some constant c. Okay, if you want you can just put it to be some epsilon, doesn't really matter. So basically we are not assuming 0, that's all.

Epsilon positive. Okay, if we do that, see, it is easy to check, right? It is easy to check. So please check this pattern that yx, okay, has no solution.



No, you cannot find a solution. Okay. You cannot find a solution. Now. The thing is this, again, so basically you see two different datas.

We saw that with one data there are infinitely many solutions. With another data there is like no solution. And again, see, let us put this data down. y double prime plus y cos 0. And the second data which we want to put is y 0.

Because 0 and y at the point beta, some beta. I will specify what beta is. Epsilon 0 less than beta less than pi. If this initial data is here. Now, you can actually show.

One can check. One may check. Check that y of x. okay has a has an unique solution okay you can easily check this part right so please do that right there you know general solution put the initial data and see that they are like a unique solution i hope this is clear okay ah so please do that so you do realize that by just by changing the data you can actually see that anything can happen it can have zero solution it can of course not anything may it cannot have two three five solutions that cannot happen because it's just all of this theory is coming from you know linear equations linear system of equations okay so it can have zero solutions it can have no solutions uh or it can have unique solutions or it can have like um infinity minus those are the three possibilities which you can have right okay so now

And one thing is, see, for the homogeneous equation, let us say, this equation, homogeneous equation, you remember, r equals to 0 and l1, l2 is 0. This equation, homogeneous equation. You can always see that y equals to 0 is always a solution. That is always given to you, right? Homogeneous equation.

Okay. So, now, the thing is this. question is this for initial data initial value problem homogeneous equation if you are just taking it it has to have since the solution is unique it is only going to be zero solution but here is it the case okay so the question is question is can one find the necessary and sufficient condition condition okay so that 1 plus 2 1 second 2 is this condition the homogeneous case how do I put it I did not write anything this one 3 1 and 3.

So, Rx is there, but L1y equals to 0 and L2y equals to 0. This is what we are assuming. Okay. So, just an extra assumption that is. Okay.

What is it? So necessary condition so that 1 plus 2 admits has only, sorry, has only trivial solution. Okay, one thing, sorry, I should write rx equals to 0. This is the case. See, otherwise, if rx is of course not 0, then 0 is not a solution, right?

It will be a solution. So, 1 we are considering, but r equals to 0, basically homogeneous equation. with a homogenous data so basically this is homogenous boundary value problem that's all okay so our question is this homogenous boundary value problem always has a trivial solution but what is the guarantee that that is the only solution there does not exist any other trivial solution okay so let's look at the theorem What is the theorem? It says that you start with two linearly independent solution of 1.

So, let y1 and y2 are two linearly independent solution of 1. With r of x equals to 0. So, I am not going to write it anymore, but it is assumed, right? Then, the homogeneous boundary value problem HBVP has only trivial solution. Only trivial solution.

What is the trivial solution? If and only what is the condition? The condition is I will write it as delta. This is the requirement which I am going to use. L1 of y1, L1 of y2, L2 of y1.

L2 of y2, this will be non-zero. So, basically this C, we know what y1 and y2 is, right? Yes, let us say it is given y1 and y2 is given. So, essentially y double prime equals to y equals to 0 plus y equals to 0. Let us say that is your equation.

So, you know what y1 and y2 is. y1 is sine, y2 is cosine. You can consider that, right? Now, with those two linearly independent, you have the data also. So, L1 and L2, you already know how does it look like with those constants.

So, you put, you see what happens to L1 when you are putting the sine function. You look at what happens to L2 when you are putting the cosine function and vice versa. You do that, define that to be delta. If it is non-zero, then you know that there is a trivial solution. The only trivial solution is there.

If it is zero, sorry, it only has a trivial solution, then delta has to be non-zero. The proof. Very easy proof. How do we do it? See any solution.

Any solution of the equation y double prime, p naught y double prime plus p1 y prime plus p2 y equals to 0 takes the form, takes the form, what is the form? y of x to be c1 y1 of x plus c2 y2 of x. This is the only form which you can take. Now, this solves the homogeneous boundary value problem. If you see, this of course solves the equation.

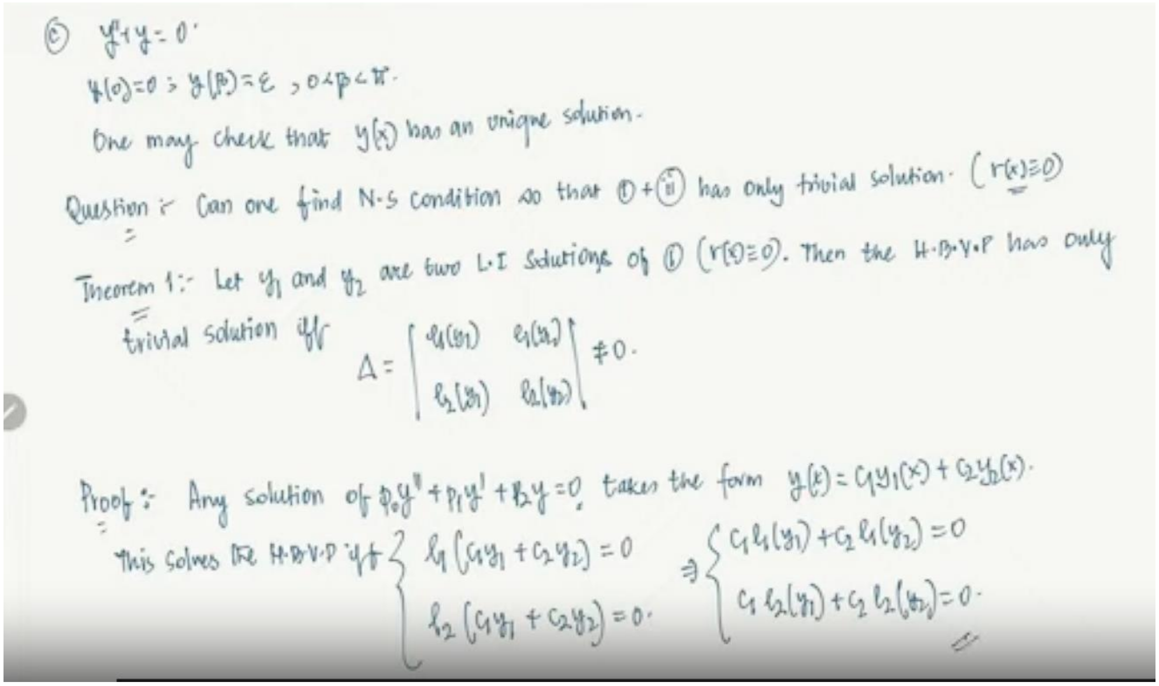
Now, this has to also solve the initial data, right? So, basically, if and only if this solves the homogeneous boundary value problem, if and only if L1 of c1y1 plus c2y2 is 0 and L2 of c1y1 plus c2y2 is 0. Yes, I think this is clear. Why this has to be a case? Because, you see, if L1 of Y has to be 0, L2 of Y has to be 0.

That is the condition. And if Y looks like this, then this is true. So, what does it imply? See, L1 of C1Y1 plus C2Y2 will look like this. C1 times L1 of Y1 plus C2 times L1 of Y2.

is 0 okay see l1 is basically a linear function if you think about it okay as a function of y so you can just break it up and again you can write it as c1 times l2 of y1 plus c2 times l2 of y2 is clear see here if c1 and c2 in y if c1 and c2 is 0 then y is trivial, okay? So, when does this system, this system, this system has solutions, right? When is that solution only the trivial solution? It can have many solutions.

When is it only the trivial solution? When do you think it happens? If the determinant is going to be non-zero, right? So, that is the condition. So, hence proved.

See, c1, there are many different, if the, let us say, if you are assuming that the, this, this system, okay, okay, this has a trivial solution, c1 equals to c2 equals to 0, if and only if delta is equal to 0. Delta is this, right, see, okay, the determinant, okay. So, essentially, the theorem is proved. So, basically, it says that the boundary delta has to be non-strong, trivial, very easy to see. Okay.



Now, see this one remark. You see, the theorem which we proved now, the theorem is independent of the choice of the choice of solutions y1 and y2. x and y to x see y 1 x and y 2 x are any two linearly independent uh you know solutions it doesn't have to be any specific solution so you don't have to worry about it right so so for convenience for convenience we will assume assume that You do realize y1 and y2 are any two linearly independent solutions.

So, they have to pass through some data, right? So, some initial data, right? So, what we are going to do is this y1 and y2. And one thing you have to remember here. See, whenever I am incorporating that they are linearly independent solution y1 and y2, we are assuming.

the initial conditions now right otherwise we do not get the individual independence thing right that that um theory we get it from the initial value problem so this is why to study boundary value problem also we have to study initial value problem see so so far for runway we assume this that y 1 at the point alpha is 1 y 1 prime at the point alpha is 0 okay And y2 at the point alpha is 0. y2 prime at the point alpha is 1. Is this okay? We can assume this, right? Yes.

See, initial value problem, right? So, see, how do we get y1 and y2? If you remember properly, it is corresponding to, so basically, the initial data 1, 0 and 0, 1, right? That is how we deduce the fundamental solution. This is why we can just assume it like this.

So, y_1 and y_2 satisfy this particular two data. This is what we are going to assume. So, let us look at a corollary now. what is it so the homogeneous boundary value problem the homogeneous boundary value problem okay let me write it like this homogeneous boundary value problem okay one with rx equals to zero with rx equals to zero that is plus two here has infinite has infinite number of non-trivial solutions, number of non-trivial solutions.

Can you tell us what happens when if and only delta is 0? That's trivial, right? See, in this case, what happens? If delta is 0, then you can have, there are like infinitely many solutions of this system, right? Then there are like infinitely many solutions c_1, c_2 such that y satisfies that equation.

So, you will have infinitely many solutions, right? So, that's if and only condition, if and only delta is equal to 0. That's trivial, okay? Now let us look at one example of what exactly are we talking about. What can we derive out of this theorem?

So let us look at this equation. Consider this equation, xy double prime minus y prime minus 4 xs cube y equals to 0. And the boundary data, and the boundary data, L_1 of y. What is L_1 of y? Let us look at how it looks like. For some certain coefficients, a_0, b_0, we will consider this.

y prime at the point 1 is 0. L_2 of y is y prime at the point 2 is 0. This is what we consider this equation. Now, since see the okay so now the thing is we want to find out if uh this equation has trivial solution or infinitely many solution or no solution that sort of question okay so this is basically look at this so you see uh i'm going to write two solutions out of it so you can check that whether this is the case or not right so check that y 1 of x

is equals to cosine hyperbolic x square minus 1. And y_2 of x is half sine hyperbolic x square minus 1. This is okay. These two solutions and y_1 and y_2 so check y_1 is this and y_2 is this are linearly independent let's just check that part i hope you can do that please do this part linearly independent okay so uh first of all we got two real independent solutions now the thing is this you see and

y_1 prime, okay, at the point x is nothing but 2x, the derivative of this, and cosine is sine hyperbolic x square minus 1, okay. And y_2 prime of x will look like this. It is 2x, x square 2x, so x, x, cosine hyperbolic x square minus 1. So, that is there.

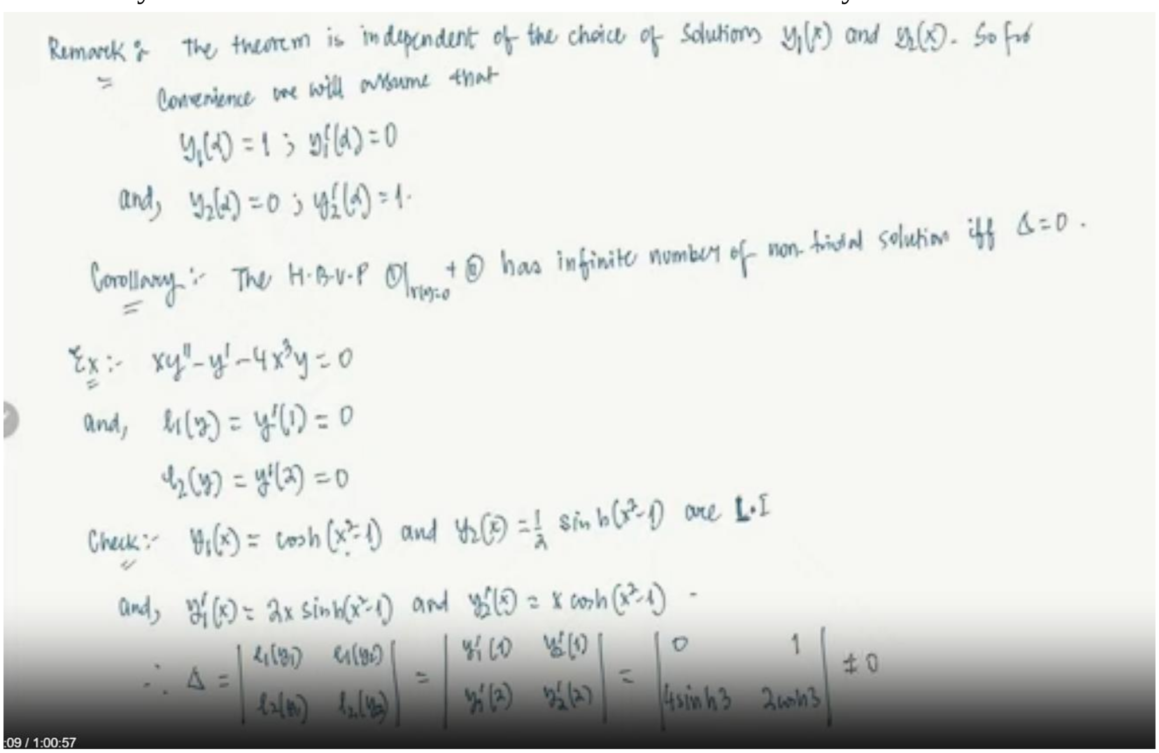
Now, the data is given that y prime at the point 1 is 0. So, that is, that will imply, this will imply that it is given y_1 prime at the point 1 is 0. So, we put those data here. So, therefore, what is delta? Delta is nothing but Y, this is Y1, L1 of Y1 and L1 of Y2, L2 of Y1, L2 of Y2.

So, what is L1 of y1? Let us just, so it is basically y1, y1 prime at the point 1, okay. What is L1 of y2? L1 of y2, it is y2 prime at the point 1.

Again, L2 of y1, it is Y1 prime at the point 2 and L2 of Y2, it is Y2 prime at the point 2. Now, if you calculate it, Y1 prime at the point 1 is going to be 0, right? And Y2 prime at the point 1, it is going cosine hyperbolic 0, which is 1. And this is at the point 2, this 4, 3, right?

So 4 sine hyperbolic 3. And this one is 2 cosine hyperbolic 3. So this is definitely going to be 4 hyperbolic. So this is non-zero. Okay?

So therefore, the problem only have this thing. Let us call it, I do not know, maybe 3. So 3 admits only trivial solution. There is no other solution. Is this okay?



So we can prove this part here. Okay, so now the thing is this, what about, so the homogeneous problem we understood. For a homogeneous problem, we have that depending on delta, you can say whether the problem has a unique solution or not. Okay, so now we have another theorem which actually tells us what happens if you have a homogeneous, inhomogeneous problem. Okay, so theorem 2, let us just call that.

So the non-homogeneous or inhomogeneous whatever you want to call it. But anyway, so the problem. Okay. Okay. Let me just put it this way.

The regular boundary value problem 1 plus 2 has a unique solution. Unique solution. This is r not equal to 0. Please remember 1 plus 2 has a unique solution if and only if the homogeneous boundary value problem. So r equals to 0.

So r of x equals to 0. Along with those two conditions. So what is it? 1 plus rx equals to 0. That is plus this data.

I don't know what I get. With this 3. I call this one as 3. So I should call this as This has 4 and this is 4.

So, 1 plus 3. This has only trivial solution. Has only trivial solution. In literature, sometimes this is called freedom alternative. Freedom alternative.

You can have similar sort of results even for PDE. So essentially what it is saying is this. There is a dichotomy between inhomogeneous equation and homogeneous equation. If the inhomogeneous equation. You understand?

If that has a trivial solution, sorry, if the homogeneous equation only has trivial solution, there is no other solution. See, it can have trivial solution, it can have infinitely many solutions, right? But what it is saying is, if it has only trivial solution, then the inhomogeneous problem has a unique solution. Okay. So, if you want to talk about whether the homogenous problem has a unique solution or not, all you have to check is if the homogenous problem has a trivial solution.

And from the earlier case, we know that when it has a trivial solution, right? So, it is all put together. Okay. So, let us look at the proof. Okay.

So, what is the proof? So, let y1 of x and y2 of x y2 of x are two linearly independent solutions, solutions of the, of 1 plus 2, 1 plus 2. Let us just, sorry, what am I writing? I should put it together, no?

of this equation y p naught because i we are dealing with so many equations okay so p naught y double prime plus p one y prime plus two y cost of zero let's take this equation here okay and and let z of x okay be a particular solution particular solution of the inhomogeneous problem okay of one is this okay right so then then the general solution general solution of one can be written as can be written as y of x is nothing but c1 of y1 of x plus c2 of y2 of x. You remember, this is that principle, right, that the general solution of the homogeneous problem is nothing but the general solution of the homogeneous problem plus a particular solution of the inhomogeneous problem, so plus z of x. z of x is this okay now see if this solution if this solves the regular boundary value problem right so and it has a unique solution so basically if it solves the irregular boundary value problem so the above y the above y solves 1 plus 2 if and only L1 of C1Y1 plus C2Y2, which is given by, if you break it up, it is C1 times, sorry, this is Z, I forgot to put that Z here, plus Z. It is nothing but C1 times L1 of Y1 plus C2 times L1 of Y2 plus L1 of Z.

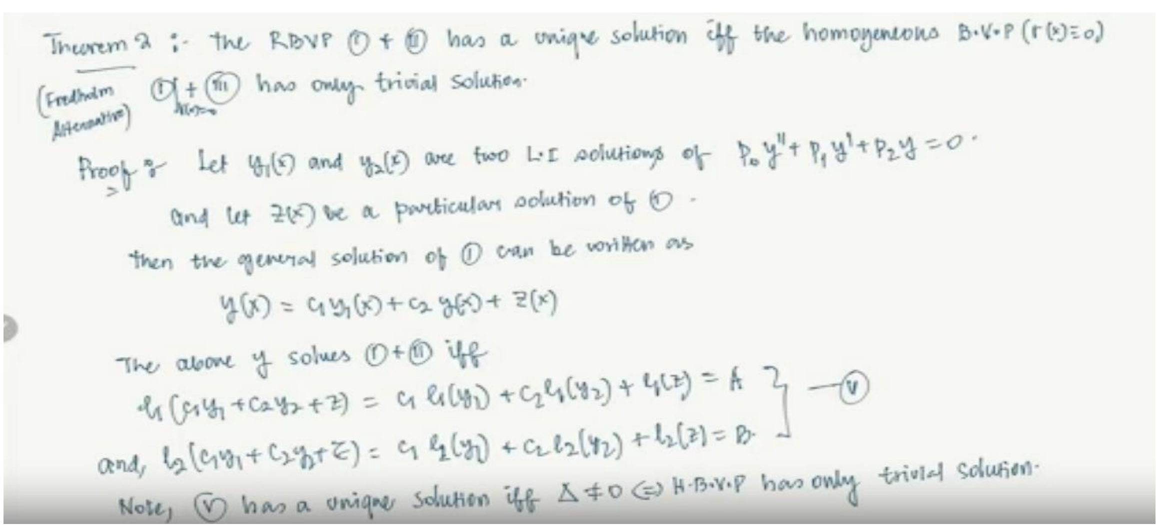
This is A and L2 of C1Y1 plus C2Y2 plus Z equals to this C1 times L1 of Y1 plus C2 times L2 of Y2 plus L2 of Z equals to B. Is it okay? Now, you see, this Z is a particular solution, right? Okay. So, this is again a homogeneous, sorry, this is a non-homogeneous system, right?

Okay. See, let us just call 4, 5. So, note 5. The non-homogeneous system 5, A and B are non-zero, right? It's a non-homogeneous system.

5 has a unique solution, has a unique solution, solution, if and only if delta is not zero. What does it mean? It means that if and only if, if and only if, the homogeneous problem, the homogeneous boundary value problem, okay, has only previous solution. This is by the earlier theorem, right? Solution.

Okay? So, you see... And hence, it is proved. So essentially, what we have is this. When is this homogeneous, sorry, system of equation, yeah?

When does this equation has unique solution? If and only if delta is 0, right? See, L1 of z and L2 of z, these are basically some numbers, right? For example, let us say L1 of y is given by y at the point alpha. Let us just say.



So, what is L1 at the point z? So, basically, it is nothing but z at the point alpha, which is just a number. So, this equation will look like this. C1 So the delta C1 L1 of Y1 plus C2 L1 of Y2 equals to you can write it as A minus L1 of Z. And similarly the other one is B minus L2 of Z.

Right? So this is some number. So non-homogeneous problem. When is this equation has a unique solution? If and only if delta is 0, non-0, sorry, delta is non-0.

Okay. And from the earlier thing, you know, the delta not equals 0, if and only if the homogeneous boundary value problem only has trivial solution. So basically what we have proved is this. If the inhomogeneous boundary value problem has a unique solution, that is an if and only condition. So basically, if it has unique solution, homogeneous problem has a trivial solution.

If homogeneous problem only has trivial solution, then the inhomogeneous problem has a unique solution. Now let us look at some example, one example. So, consider this equation. xy double prime minus y prime minus 4xs cube y equals to 1 plus 4x power 4. Clear?

Now, we also need the boundary values. What are the boundary values? So, first of all, L1 of y. What is L1 of y? It is given by y at the point 1. And this is 0, let us say.

And what is L2 of y? It is given to be y at the point 2, which is 1. Is this okay? So, this is the problem. Let us just call it 6.

Now, the thing is this. You can actually show. So, please check this part. Check. 6 has

Only trivial solution. Trivial solution. I hope you can check it, right? How do you check something like this that 6 only has trivial solution? You just check what delta is.

If delta is not equal to 0... Then you have trivial solution. So, it is very easy to check. So, please do that part. And how do you find y1 and y2?

For this problem, we already talked about y1 and y2. Where is it? See, y1 and y2. So, you can use that fact. It is not a problem.

Use that fact and show that this problem only has trivial solution. Now, the thing is this. So, if this problem has a trivial solution, what happens to the, you know, what is it, the inhomogeneous boundary value problem? Then, theorem 2 implies 6 admits a unique solution. Is this okay?

6 admits a unique solution. Okay. Now, the thing is this. How do you find such a solution? That question can be answered.

Okay. Let's just see how to do that part. Okay. See, the thing is using the theorem and claiming that there is a unique solution is very nice. But the thing is how do we get that solution?

Let's just try to find out. Okay. So, what we do is this. We start with two linearly independent solutions. So, the question is this question.

How does one find the solution? Unique solution is fine. But can you find that unique solution? That is the question. So, of course, we know that y1 of x, which is given by cosine hyperbolic x square minus 1.

And y2 of x, which is given by half sine hyperbolic. x square minus 1 are two linearly independent solutions right are linearly independent solutions of the homogeneous problem that is given we talked about it just before some time right so that is always there okay now you see you have to so you somehow have to you have to look at this problem and try to figure out if you can you know find out a particular solution or not for this problem as you can see that you can use So separation of variable right. So sorry not separation of variation of parameter or Duhamel's principle. So we may use Duhamel's principle.

I am not writing that part. You can use Duhamel's principle to get a particular solution or you can also note. So this is just observation observe. Observe that zx if it is minus x is a particular solution of particular solution of xy double prime minus y prime minus 4 xs cube y. If we look at it, this part is not there.

This is minus 1. So it is plus 1. And then 4 xs cube. So this is 1 plus 4 x to the power 4. So it solves this problem.

Is it OK? So you have a particular solution. clear so therefore what is the general solution so the sorry not the yeah what is the general sorry the unique solution the unique solution sorry what am i saying i'm really sorry about it y1 and y2 are two linearly independent solutions of the homogeneous form right solutions of the homogeneous equation. So, this equation, without any data or anything, with this equation, xy double prime minus y prime minus 4x raise to y equals to 0, right?

Subject to some initial data such that they are linearly independent. That is the thing. Now, the thing is, we also got a particular solution. So, what is the general solution? Let us just say y general solution of x. And what is the, this solution is the solution of this problem, right?

General solution looks like this. c1 times y1 plus c2 times y2 plus minus x, right? So, that is your general solution. Now, the thing is this, we know that there is a unique, you know, this system admits a unique solution. So, we have to find a unique C1 and a unique C2.

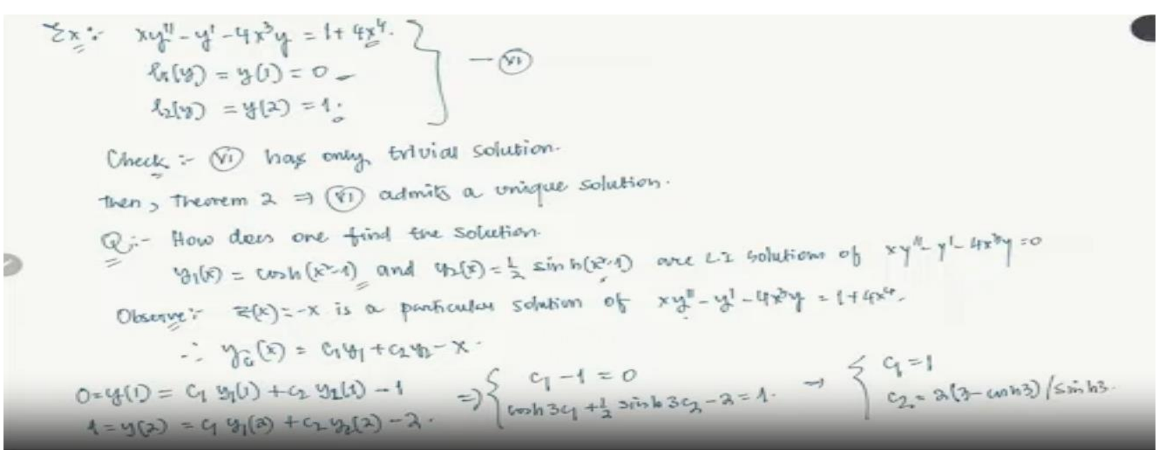
So, let us look at it. Yes. See, this data is there. y at the point 1 is 0, y2 at the point 1 is 1. So, see, if you put this data here, so we have

y0 equals to y at the point 1. What is y at the point 1? It is c1 y1 at the point 1 plus c2 y1 and y2 at the point 1. minus 1, clear? And another thing is 1 equals to y2, which is again c1 y1 at the point 2 plus c2 y2 at the point 2 minus x equals to 2, right?

So, 2. it okay now if you write down this equation this will imply this thing c 1 minus 1 equals to 0. this is what you are going to get right so this is y 1 at the you just evaluate what is y 1 at the point 1 y 2 at the point 1 so basically it is becoming c 1 minus 1 equals to 0 and the second part is cosine hyperbolic 3 c 1 plus half sine hyperbolic 3C2 minus 2 equals to 1. This is the two conditions which you are going to get. Now, you can of course solve this problem.

This is C1 equals to 1. From here, C1 equals to 1. So, the solution which you are going to get is this. C1 equals to 1. C2, you are going to get as 2C2.

3 minus cosine hyperbolic 3 by sine hyperbolic 3 this is what you're going to get right okay so therefore therefore y of x okay will look like Cosine hyperbolic x square minus 1 plus 3 minus cosine hyperbolic 3 by sine hyperbolic 3. Okay. Sine hyperbolic x square minus 1 minus x. So, this is what your unique solution looks like and this is how you conclude that the problem 6 has a solution and how does the solution looks like?



It has a unique solution and which looks like this. This is how you conclude it. So, with this we are going to end this study of linear boundary value problems.

$$\therefore \; y(x) = \cosh(x^2-1) + \frac{(3 - \cosh 3)}{\sinh 3} \sinh(x^2-1) - x$$