Ordinary Differential Equations (noc 24 ma 78)
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Week-01
Lecture-03
Matrices

Students, welcome to this video and in this video we are going to talk about linear transformation and some properties of matrices.

Okay, so first of all, you see, we suppose that we start with two bases, okay.

$\{x\ \ , x\ \ , \ldots., x_n\}$ and $\{y\ \ , y\ \ , \ldots, y_n\}$, okay, are bases of vector spaces X and Y, okay, spaces X and Y, right.

Now, you see, you start with

So, for any A$\in$ $L(X, Y)$, right.

So, this determines, okay, determines a set of numbers, set of numbers.

such that you see A_{xj}, right, is equal to see.

So, for any j, let us say $\leq$ j $\leq$ n, right?

Okay.

So, you fix any, sorry, I should write it as a m. So, for any j between and n, A_{xj} is an element of Y, right?

So, you can actually write it in terms of $A_{xj} = \sum_{i=}^{m} a_{ij}\, Y_i$

So, it will be a linear combination of Y_i(s), Y and i lies between and m, right.

So, that we can always write it and this holds for all j between and m, right.

Okay, fine.

So, you see what is happening is this,

A if you want to write down okay so we can actually visualize this as a rectangular array of m rows and n columns called which we are calling as a m cross n matrix so therefore therefore the matrix A$\cong \begin{bmatrix} a & \dots & a_n \\ a_{m\dots\dots} & & a_{mn} \end{bmatrix}$

Okay, so this will be your m cross n matrix.

m plus n matrix okay so essentially what you have is if you have a linear transformation a to from x to y okay then a can be visualized as a set of m so as a matrix which is the m cross n matrix right okay so you see the uh so if let's say X= $\sum \quad c_j \, x_j$

Because you know I am starting from x in x and so x can be written as the sum of x_i(s).

The linearity of A, the linearity of A, of A combined with let us say let us call this .

. So essentially the representation of A shows that Ax=$\sum_{i=}^{m}(\sum_{j=}^{n} a_{ij}c_j)Y_i$

And therefore the coordinate of Ax the coordinate is what this sum part are

$\sum \quad a_{ij}c_j$ okay and this basically we write it like that so essentially this is our representation so given a linear transformation we can actually write it as a matrix okay so one question so check

Given a matrix, let us say, given a matrix A, which is a matrix, you know, m cross n matrix, can one write it as a, write it as a, okay, can I, can you represent it as a, let me put it, can one represent it, represent it as a matrix?

as a linear transformation.

Difficult question, right?

So, anyways, you please try and answer this question, if you can do that.

So, essentially what I am trying to say is this, given a linear transformation, there is a matrix or vice versa, if there is a matrix, there is a linear transformation, right?

So, they are basically the same thing.

Linear Transformation and matrices :-

Suppose $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_m\}$ are bases of vector spaces X and Y.

For any $A \in L(X,Y)$, determines a set of numbers such that

$$Ax_j = \sum_{i=1}^{m} a_{ij} y_i \, ; \qquad 1 \le j \le n \qquad — (1)$$

$$\therefore \; A \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

If $x = \sum c_j x_j$, the linearity of A combined with (1) shows that

$$Ax = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} c_j \right) y_i$$

$\therefore$ The coordinate of Ax are $\sum_j a_{ij} c_j$.

Check :- Given $A \in M_{m \times n}$; can one represent it as a linear transformation.

Okay, so now the question is this, why are you certainly interested in this?

Because matrix plays a fundamental role in our study of ODEs, right, which we are going to see later on.

So, okay, so first of all, anyways, you guys already know, so this matrix is of size m cross n means m rows.

So this is m rows and n columns, okay.

Now, the thing is this, we are going to, I mean, introduce some notations which we are going to use, okay?

So, definitions are notations.

Definitions, let us just write it as definitions, okay?

Definition A. So, essentially, if $a_{ij} =; \leq i \neq j \leq n,$

then then A is called a diagonal matrix right called a diagonal matrix so essentially every element

Of the set.

Sorry.

Every element of the matrix.

Except the.

Diagonal.

The elements.

Are.

Zero.

Okay.

And.

If.

And.

In addition.

Addition.

If.

$a_{ii} =$,

Okay.

Then we call it as a. Identity matrix.

Then.

We.

Call it.

identity matrix given by I_{m*n} . So, please remember this.

These are the notations which I am going to use throughout the lecture.

So, I am not going to explain all the time.

So, A_{m*n} ,if I am writing it like this, is basically with $a_{ij} =$, it is a diagonal matrix.

And if $a_{ii} =$, that is the identity matrix, which will be always written by I_{m*n}. Clear?

Okay.

So, without, if it is clear that the matrix which we are working with is A_{m*n} matrix, then it will be always written as I. Okay.

Now, the thing is, we will call a matrix this thing.

symmetric if A=A^T, and uh we will see the trace of the matrix okay so the trace the trace of a matrix

matrix A, right, we will denote it as Tr(A), okay.

Again, I am not going to repeat myself, Tr(A), is defined as the trace of the matrix and which is nothing but the sum of the diagonal elements a_{ii}, Tr(A)= $\sum_{i=}^{n} a_{ii}$, okay.

So, that is the

Okay, so that is more or less the, I mean all the definitions which we need.

Okay, and now I am assuming that you know some properties in determinants and all that we are not doing it right now.

Okay, right.

Now another simple fact which we need to know related to matrix is this.

Okay, so you see consider this system.

the linear system linear system of n equations n equations in n unknowns unknowns okay unknowns so uh so basically you are looking at set of equations how does the equation looks like $a \quad u \quad + \cdots . + a_n u_n = b$

$$a \quad u \quad + \cdots . + a_n u_n = b$$

And it goes on like this.

And lastly $a_n u \quad + \cdots . + a_{nn} u_n = b_n$

Let us just write it like this.

Okay.

Where a_{ij} this set.

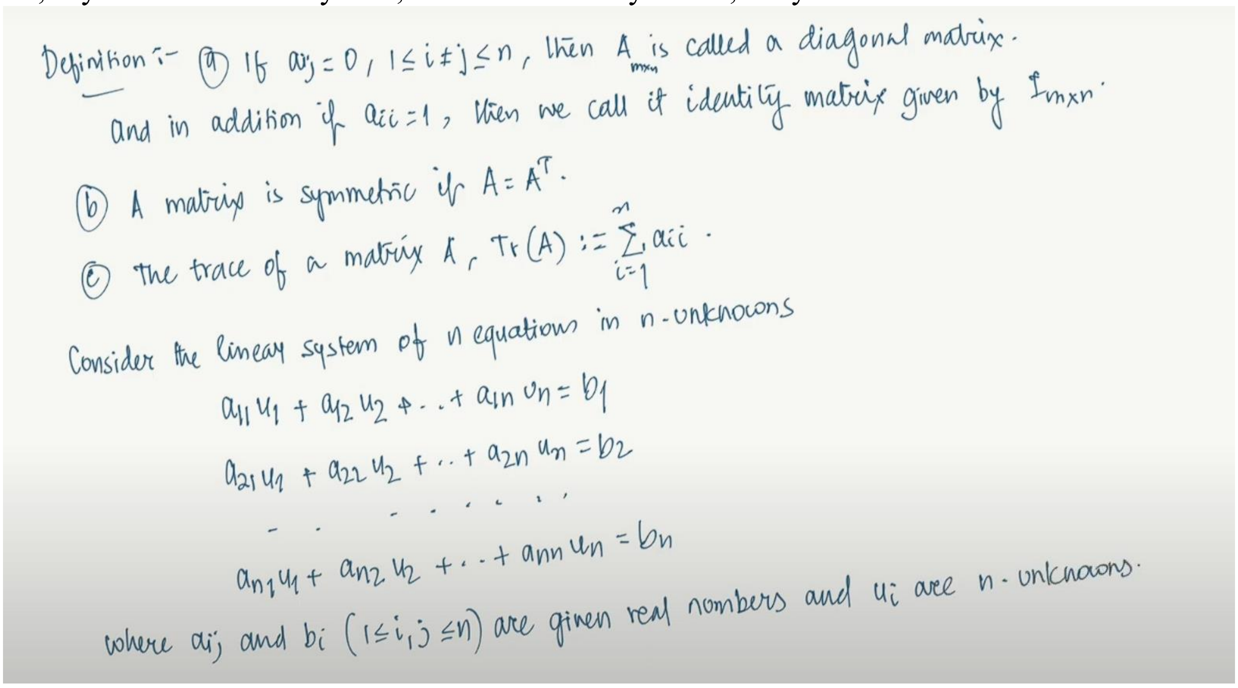
Clear?

And $b_i(\leq i,j \leq n)$

Are given real numbers.

Real numbers and $u_i(s)$ are n unknowns.

So, if you consider this system, let us call it the system *, okay.



Definition :- (a) If $a_{ij} = 0$, $1 \leq i \neq j \leq n$, then $A_{m\times n}$ is called a diagonal matrix.

And in addition if $a_{ii} = 1$, then we call it identity matrix given by $I_{n\times n}$.

(b) A matrix is symmetric if $A = A^T$.

(c) The trace of a matrix A, $Tr(A) := \sum_{i=1}^{n} a_{ii}$.

Consider the linear system of n equations in n-unknowns

$$a_{11}u_1 + a_{12}u_2 + \dots + a_{1n}u_n = b_1$$
$$a_{21}u_1 + a_{22}u_2 + \dots + a_{2n}u_n = b_2$$
$$\dots$$
$$a_{n1}u_1 + a_{n2}u_2 + \dots + a_{nn}u_n = b_n$$

where a_{ij} and b_i $(1 \leq i, j \leq n)$ are given real numbers and u_i are n-unknowns.

If you consider this system, now for this system, we call, we can write, you see, if we put it together, we can write it like this.

So, basically, if one defines, if one defines, defines A=$(a_{ij})_{n*n}$, right?

Then * may be written as may be written as Au=b; where,

b is a (n*l) vector.

And u is a (n*l) unknown vector.

unknown vector.

Here we can write it like this.

So basically this can be written in short form like this.

This we already know and the thing is if b=,we call the system as, we call it a homogeneous system.

Homogeneous system of equation

of equation if b≠, then it is called inhomogeneous, inhomogeneous system, right.

So, we are going to need a very important theorem here and this we are going to use in a lot of different places.

So theorem.

I am not going to prove the theorem.

You have to do it yourself.

The theorem is quite easy to prove.

So the system * has a unique solution.

has a unique solution, okay?

If and only if, if and only if det A≠, right?

Okay, so that is quite true, right?

And we can also say, so alternatively, alternatively, alternatively, the homogeneous system, the homogeneous system

So, v=, homogeneous system has only trivial solution.

So, trivial solution means u≡, identically equals to .

Then det A≠,

non-singular okay so A if a is non-singular then you have a unique solution and vice versa okay the proof i am not doing the proof this is very easy to do so so proof please check this proof please check check this okay and we are going to use this theorem okay so now let's look at a small uh property of uh determinants which you are going to need

So, you see, first of all, let us say that we assume, let us say, okay, so now, now, consider this, consider A(x). So, I am writing a matrix A, which depends on a variable x, right?

And this matrix is given by A(x)= $(a_{ij}(x))_{n*n}$.

So, you have that every coefficient is a function of x, clear?

So, consider this where a_{ij}: (a,b)→ ℝ , are continuous okay?

This is i am assuming right okay ah then the function is called the matrix valued function okay so then

A is called a matrix valued function.

So, a very simple example, you can think of a example.

Let us say A(x)=$\begin{pmatrix} x & \end{pmatrix}$

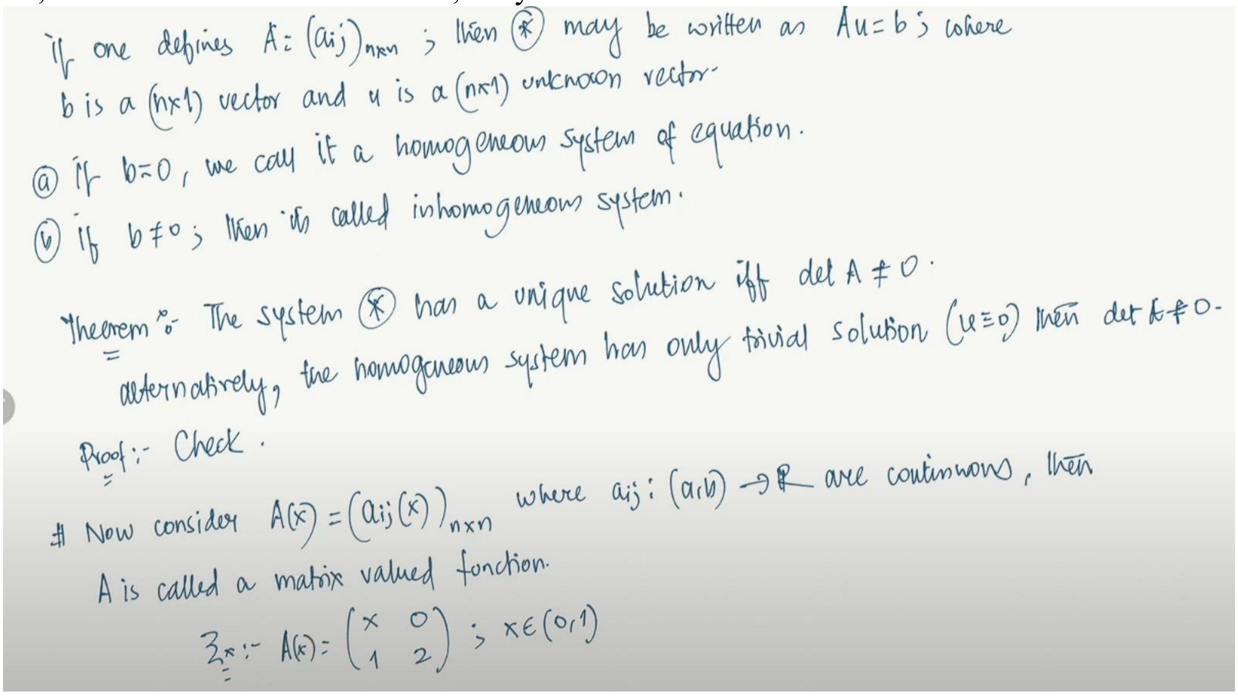
Let us say, okay?

So, you see, this is, it is not a constant function, constant matrix because there is a variable x, okay?

And let us say this x∈ (,), okay?

So, now, this matrix is not a constant matrix but a variable matrix A(x), right?

So, that is a matrix valued function, okay?



If one defines $A := (a_{ij})_{n\times n}$; then (*) may be written as $Au = b$; where b is a $(n\times 1)$ vector and u is a $(n\times 1)$ unknown vector.

(a) If $b=0$, we call it a homogeneous system of equation.

(b) If $b\neq 0$; then its called inhomogeneous system.

Theorem:- The system (*) has a unique solution iff $\det A \neq 0$.

alternatively, the homogeneous system has only trivial solution $(u\equiv 0)$ then $\det A \neq 0$.

Proof:- Check.

Now consider $A(x) = (a_{ij}(x))_{n\times n}$ where $a_{ij}:(a,b)\to \mathbb{R}$ are continuous, then

A is called a matrix valued function.

Ex:- $A(x) = \begin{pmatrix} x & 0 \\ 1 & 2 \end{pmatrix}$; $x\in(0,1)$

Now,

So, there are properties of this function.

Before doing the properties, let us just write it down.

See, the space of all continuous matrix valued functions, we are going to define it like this.

in (a,b) is denoted by, is denoted by, so we will write it as $C_{n*n}(a,b)$

So, this set of all continuous matrix valued functions on (a,b), right, okay.

And

If for all, okay, how am I defining it?

If for all A(x), B(x) in $C_{n*n}(a,b)$ and $\alpha \in \mathbb{R}$, we can do this.

You see (A+B)(x)= $a_{ij}(x) + b_{ij}(x)$, okay.

Clear?

And (α A)(x)= $\alpha a_{ij}(x)$. Okay?

So, that is the

So, with those three properties, we define a new space here.

The space of all continuous matrix value function, we denote it by cn and n. And what are the operations?

The operations are natural operations, but it just depends on x. Okay.

Right.

Now, one may define, one may define, okay, $C^m_{n*n}(a,b)$=space of all m (m$\in \mathbb{N}$) times continuously differentiable matrix valued functions in (a,b) and is given by $A'(x) = a_{ij}'(x)$

How do I write it?

What is m?

m is the mth derivative.

So, is this clear?

Now, you see, let us say that I want to write, I mean, you know, first of all, let us say I want to talk about the determinant of A(x).

So, basically given from matrix A(x)∈ C^m_{n*n}(a,b)

The notations are quite a little complicated, so we have to keep that in mind.

So, you see what specifically this means is we are looking at n cross n matrix, okay, where every entry is at least is m times continuously differentiable, that is what it says, nothing special, okay, and the functions all are defined on (a,b), that is what, okay.

So, if you are starting with A(x), which lies in this space, and then, and then,

What happens is det A(x)= $\sum_{j=}^{n} a_{ij}\ (x)\alpha_{ij}(x)$; where $\alpha_{ij}(x)$ is the cofactor of a_{ij}, okay?

Now once that is true you see we want to find out then we want to write down, $\frac{\delta det A(x)}{\delta a_{ij}(x)}$=$\alpha_{ij}(x)$.

It follows easily because every other thing is becoming zero and the only thing which is remaining is α_{ij} okay so and hence and therefore and so you see

$[\det A(x)]'$if you take the derivative of the whole thing that will be nothing but,

$$[\det A(x)]' = \begin{vmatrix} a\ \ '(x) \ldots.. & a_n'(x) \\ a_n\ \ (x) \ldots. & a_{nn}\ \ (x) \end{vmatrix} + \cdots + \begin{vmatrix} a\ \ (x) \ldots \ldots.. & a_n(x) \\ a_n'(x) \ldots \ldots.. & a_{nn}'(x) \end{vmatrix}$$

So basically you understand the we are doing like the derivative thing like a Leibniz rule kind of thing but only in the rows.

So the first row gets derived in the first thing the first row is derived a leaving intact all the rows.

below and then the second row will be derived every other thing will remain intact and we go on till the nth row this is what you do okay so this is how you take the derivatives of a matrix valued function okay so the determinant of the derivatives of a matrix valued functions okay so this is quite clear if you if you are still doubtful you please so check it

using a * matrix.

Then you will be convinced.

But this is quite clear, right?

This is quite clear.

So, with this we are going to end this video.

The space of all continuous matrix valued functions in (a,b) is denoted by $C_{n\times n}(a,b)$

if for all $A(x), B(x) \in C_{n\times n}(a,b)$ and $\alpha \in \mathbb{R}$

$$(A+B)(x) = a_{ij}(x) + b_{ij}(x)$$

$$(\alpha A)(x) = (\alpha\, a_{ij}(x))$$

One defines may $C^{(m)}_{n\times n}(a,b)$ = space of all m $(m \in \mathbb{N})$ times continuously differentiable matrix valued functions in (a,b). and is given by $A'(x) = (a_{ij}'(x))$.

Given, $A(x) \in C^{(m)}_{n\times n}(a,b)$ and then $\det A(x) = \sum_{j=1}^{n} a_{ij}(x)\, \alpha_{ij}(x)$ where $\alpha_{ij}(x)$ is the cofactor of a_{ij}.

then, $\dfrac{\partial \det A(x)}{\partial a_{ij}(x)} = \alpha_{ij}(x)$ and so, $[\det A(x)]' = \begin{vmatrix} a_{11}'(x) & \cdots & a_{1n}'(x) \\ a_{21}(x) & \cdot & a_{2n}(x) \\ \vdots & & \vdots \\ a_{n1}(x) & \cdots & a_{nn}(x) \end{vmatrix} + \cdots + \begin{vmatrix} a_{11}(x) & \cdots & a_{1n}(x) \\ \vdots & & \vdots \\ a_{n1}'(x) & \cdots & a_{nn}'(x) \end{vmatrix}$

[Check it using a (2×2) matrix]