Ordinary Differential Equations (noc 24 ma 78)

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Lecture 29 : Oscillation Theory 2

So welcome students to this video and in this video we are going to continue our study of oscillatory equations. equations. So, essentially, what is an oscillatory equation? So, basically, if all solutions of this, we will start with this self-adjoint equation, right? P y prime whole prime plus q y equals to 0, let us say.

And P and q are smooth functions. P and q are smooth. It is in C1 of i, whatever i is, i can be r also, right? So, this is the problem. Let us call that 1.

And for P equals to 1, we have this equation, very special equation. Let us call that 2. Yes. So, we defined oscillatory equation as let us say if you have a solution which does not have a last 0, okay, then the solution is called oscillatory. And if your equation has all solutions such that they have no last 0, then this is called an oscillatory equation.

Very simple, right? And we have looked at two very important theorem. Let us just recall the theorem, okay? The first one is called the Stumpf's comparison. strong comparison okay so what does it compare it compared the solutions it compares okay theorem theorem so what does it uh says it says that let's say if so i'm not writing the exact statement okay that is already done this is just for a recall so if let's say q1 of x is greater than equal q2 of x if this is the case sorry q of x q of x okay and

and alpha beta and y at the point alpha equals to y at the point beta, okay, equals to 0. Alpha and beta are consecutive, right, consecutive zeros, okay, such that y satisfies satisfies y double prime plus qy equals to 0. So, let us say that you have a non-trivial solution of this problem y such that alpha and beta are consecutive zeros. Yes, if this is not equal, this is not equal, sorry, this is not equal.

If q1 is not equals to q2, then we can say, please remember this thing, it is said that every, not just one, every non-trivial solution, non-trivial solution z of x, okay, of which equation of y double prime plus q1 x z y equals to 0. This equation has a 0, has a 0 in alpha beta, right? That is the sum comparison theorem which we learned. Please remember this.

Two equations are there, right? y double prime plus pxy equals to 0. And another one, y double prime plus p1 of xy equals to 0. Okay? See, what it is saying is this.

If you can find just one solution, you understand? If you can just find one solution such that That solution, let us say y1 is a solution of this problem such that y prime of alpha equals to y prime of beta equals to 0. And alpha and beta are consecutive 0s. That is, there is no gap in between them here.

Gap in the sense that there is no 0. So, there is no gamma such that y1 of gamma is 0. That is what I am trying to say. Gap means this. Okay.

So, in that case, what happens? It is saying that in between. See, this is just one solution. Okay. We are assuming just one solution, non-trivial of course, because if this trivial solution, everywhere it is 0, it does not matter.

Now, what it is saying is, if you can just find one solution such that this thing happens, then any solution of this problem, any solution, let us say Z is any solution, it does not really matter what, if it is a solution, then Z will have a solution. gamma, let us say, between alpha beta where z has to be 0. Is this okay? Any solution z. Of course, again, it is non-trivial because if it is trivial, it is 0 everywhere. It does not matter, right?

So, this is true. When is it true? That you see, in the case that P1 should be at least at some point, it should be different than P. Is this okay? At least at one point, it should not be equal. And then, if you remember, we also looked at something called a Picon's identity.

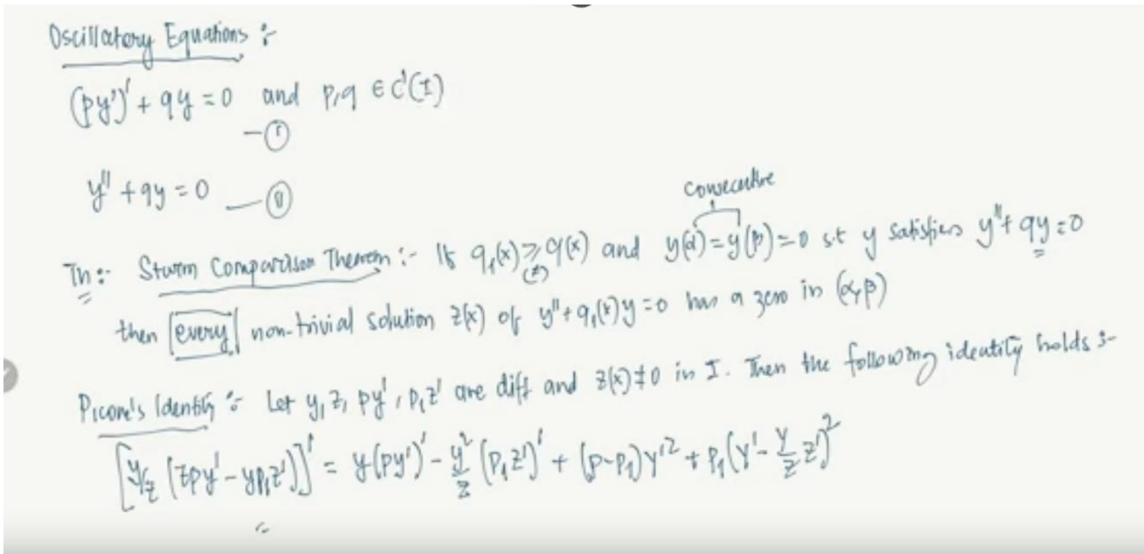
Picon's identity. And I have told you that this is very, very important to prove the theorem which we are going to prove today. What is Picon's identity? So, you know, all this. So, let the functions y, z, py prime.

This is a big statement. So, p1, z prime are differentiable. are differentiable. Again, and zx is not 0. This is very important in i. Again, let me tell you this.

See, Pecans identity as itself is not very important. It is just some identity which is satisfied by some functions. That is all. But the problem is this. You can use this identity to actually find, I mean, to actually, you know, in a lot of different applications.

then the following identity holds. Identity holds. Okay. So, what are these? See, this identity, y by z, yeah, again, zpy prime minus yp1z prime, the derivative of that, clear?

That is nothing but ypy prime, the derivative of that, minus y square by z, P1 z prime, the derivative of that, plus P minus P1 y prime square plus P1 y prime minus y by z, z prime y. square, right? That is the Picon's identity, we proved it, right? We just expanded this thing and we showed that this is equal to this, huh?



So, that is Picon's identity we are going to use today. So, now, see, the first term comparison theorem which you did is for this equation, yeah? This is not a general equation. The general equation is given by this, right? You remember, this is the self-adjoint form, yeah?

So, now we are going to do something called a Stern-Picon's theorem. Stern-Picon's So, let me write down the statement of the theorem, then I will tell you a little history about the problem. So, let us say if alpha beta in some i are the consecutive 0s, are the consecutive 0s, Of a non-trivial solution.

Please understand this. This is a non-trivial solution. Just one non-trivial solution. If you can find a non-trivial solution. Something like this.

Solution. Y of X of 1. Solution of 1. Yes. And if.

And this is the condition. And if. P1. X and Q1. are continuous with 0 less than p1 of x less than equal px and q1 of x is greater than equal q of x.

in alpha beta okay so that alpha beta consecutive is 0 in that interval alpha beta then then every non-trivial solution every non-trivial solution solution zx of the differential equation

What is the differential equation? P1 of x z prime whole prime plus Q1 of x z equals to 0 has a 0. 0 in alpha, beta. So, if you look at it carefully, see this theorem is exactly the same sort of statement which we did for the stump comparison principle, but that is for p equals to 1, this I am doing it for any p, smooth p that is, but any p. So, you see our condition was the

equation which we wanted our root to have 1 0 in between that coefficient should be bigger than the original equation so here also you see q1 is greater than q exactly fine now the thing is what is the requirement on p see here p actually flips so you have to have p1 should be less than equals to px and this should be strictly positive yeah this is very important it has to be strictly positive Okay, if something like this happens, then we can say that the exact same state matter that P1 z prime whole prime plus Q1 z equals to 0 has a 0 in closed alpha beta. Please remember, it is closed alpha beta. This is the difference here. It is not open alpha beta.

See why this requirement is there. I am not assuming that it cannot be same. So, here remark, remark. The difference with this as term comparison. You see here, here the 0, 0 may be admitted, okay, at the endpoints also, at the endpoints, at the endpoints.

Okay, do you see why it is the case? See, the thing is this, here I am not assuming that it cannot be same, right? So, P can be equals to P1. See, here P can be equals to P1 and Q and Q equals to Q1. If that is the case, then both the equations are basically same, right?

So, they will have the same zeros, right? So, in that case, essentially you have to have close A alpha beta. Of course, if you are doing that not equals to, then that is different case. So, we are assuming this thing. Now, let us look at the proof.

I told you that I will tell you something, some history about it. So the thing is this was actually done in around 1880s 1890s by Stern okay and at that time it was called Stern theorem okay and the proof is extremely lengthy it is like 4-5 pages proof very very complicated proof okay this theorem had okay. So, Stump used this Stump comparison theorem, that was very easy, right, which we have seen the proof of. But when that P is involved, okay, the proof was actually complicated. So, Stump proved it using basic analysis, but it was very, very complicated, 4-5 pages, as I told you.

Now, around 1901 or something, around that time, Picon. Came up with Picon's identity. And you will see that after using Picon's identity, the proof became just like half a page. So let us look at the proof. And it actually simplified the whole process a lot.

So what is it? So let us say that Z does not see. Here I have to show that there is a 0. Z has a 0 in closed alpha beta. So let us just assume that Zx is not equal to 0 in closed alpha beta.

Let us just assume that, right? Alpha beta, okay? Then, we will use Pecans, you see? Zx is not equal to 0, right? So, we can use Pecans here, okay?

So, we Pecans identity and find, then we use Pecans identity to get Okay, what is it? Let me just write it down. I will explain to you. y by z, zpy prime minus yp1 of z prime whole prime.

Okay, this is nothing but q1 minus q times y square plus p minus p1 times y prime square plus p1 y prime minus y by z, z prime square. Is this okay? This is what we are going to get. This is from that, see, from the picone identity, if you look at it carefully, p1 y prime minus y by z, z prime square, this term, okay?

p minus p1 y prime square, p minus p1 y prime square, okay? And then q1 minus qy square, you see? This expression and this expression. See. z solves this problem, right?

So, p1 z prime whole prime is minus q1 z. So, minus q, see, p1 z prime whole prime is minus q1 z. So, and this z is not 0. So, we can cancel it out. So, it is plus q1 y square. So, this is why q1 y square, okay? And from here, you can see it is becoming q of y square.

So, minus q of y square. So, minus q of y square, yes? So, very simple. Now, we integrate it. So, integrate this expression.

So, integrating the above we get What do we have? See if you indicate the above, you see there is a z here, there is a y here, right? And we know that y at the point alpha and y at the point beta is essentially 0, right? So what do we have?

We have alpha 2 beta q1 minus q y square, right? plus p minus p1 y prime square plus p1 y prime minus y by z z prime square dx. This is Going to be this expression if you remember. See this y is there.

So it does not really matter what is inside. y at the point. So if you just integrate it, it will become y at the point beta times this at the point beta minus y at the point alpha times this at the point alpha. This whole expression. Okay.

And y at the point alpha and y at the point beta is 0. So basically this is 0. Is this okay? Now you see this is a contradiction. this DAVAV let me write it this way DAVAV is a contradiction unless Q1 equals to Q2

And P equals to P1, okay? And y prime minus y by z, z prime is equals to 0, okay? Otherwise, it is a contradiction, right? See, these are all positive terms. Q1 is greater than equal Q. So, this is a positive term.

This is positive. P is greater than equal P1. See, it is given. P is greater than equal P1. So, this is positive.

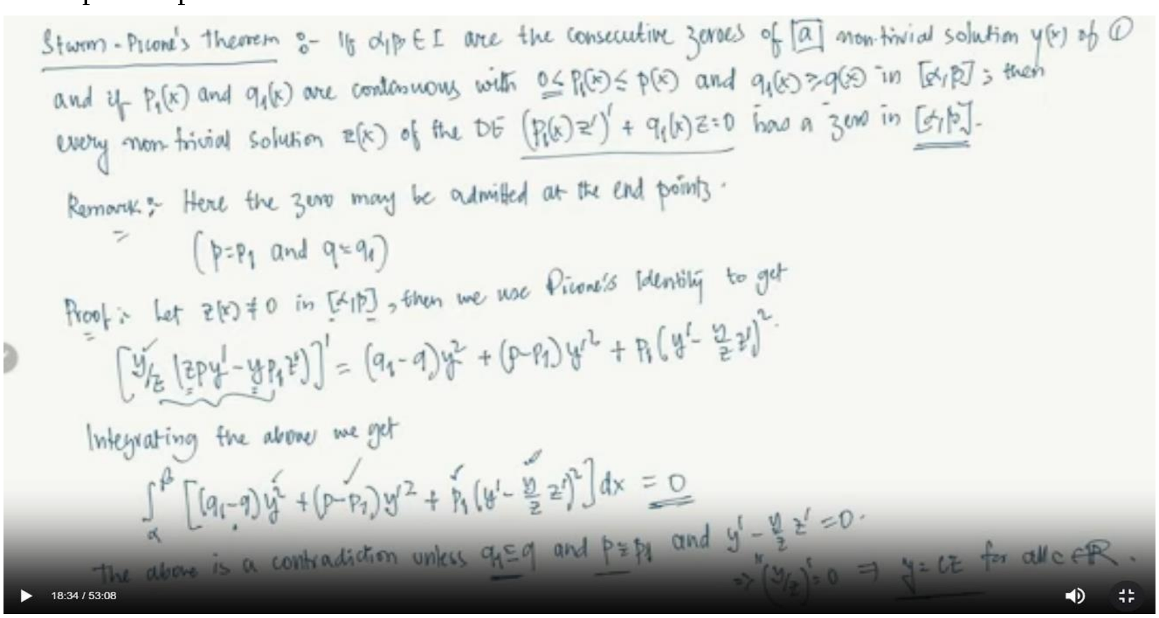
Yes. And this is a square term. So basically this is either greater than or equal to 0. So if this is whole, the whole thing is positive, essentially what is happening is this. P1 is given to be positive, right?

Okay. So the whole thing is positive. Okay. See this is the place where we are using it. P1 is greater than or equal to 0, strictly greater than 0.

So you see the whole expression is positive. So, positive expression integrated between alpha beta has to be positive. So, if it is 0, it means that all these parts has to be 0 identically. So, and in that case, Q1 has to be equal to Q and P1 has to be equal to P and y prime minus y by z, z prime equals to 0. See, this expression is nothing but y by z, the prime of that is 0.

So that will imply Y is some constant times Z. Right? Is this okay? Y has to be constant times. So you see, this expression is already a contradiction. It won't be a contradiction if this, this and this is true.

For all c in R, right? You understand what I am saying? Okay. But now, you see, what is y at the point alpha? That is 0.



Okay. So, that will imply that the constant c, see this constant c, this has to be 0. Right? Okay? Because you see Z is not 0 in whole alpha beta including the endpoints.

So Z at the point let us say alpha or Z at the point beta is never going to be 0. So C has to be 0. Right? Because Y at the point alpha is 0. So that will imply C has to be 0.

Is this okay? Now if C is 0 therefore Y of x by z of x has to be 0, right, or since z is not 0 in whole alpha beta, it means yx has to be identically equals to 0 for all x in closed alpha beta, okay. That is a contradiction again, okay, because we have assumed that y is a non-trivial solution, see, y is a non-trivial solution, okay, a contradiction again. So, this cannot happen essentially.

Hence, we have that it will have to have a 0 in closed alpha beta. Is this okay? Right. Now, as a corollary. Now, one thing, before we go on, you do realize it's like a two, three line proof, right?

I mean, that was like the original proof of Stump was actually four, five pages proof. Very, very complicated, very big. But this is very, it's like very easy once you know Pico's identity. Yeah. Okay.

So, now let's look at the Cordolari. Okay. Okay. So, what is it? Now, as a corollary, I am going to write down something called a Sturm's separation principle.

Sturm's separation theorem. Sorry. Separation theorem. So, what is it? It says that if y1 of x and y2 of x are two linearly independent, are two linearly independent

solutions, dependent solutions, okay, of y prime, p y prime, sorry, let me write properly, of 1, let us say. What is 1? 1 is this equation. Where is it? This is 1, p y prime whole prime plus q y equals to 0, okay.

in i, in some interval i, okay, then their zeros, their zeros are interlaced. Interlaced means, what does it mean? That is, if alpha, beta are consecutive zeros, are consecutive zeros, 0s of y1. Then there exists exactly 1 0 of y1.

y2 this is okay and vice versa so basically between any two consecutive 0 of y1 there is a 0 of y2 and we again between any two consecutive 0 of y2 there is a 0 of y1 okay so it interlaced so before we go on to with the proof of this let's look at a small remark let's say this equation y double prime plus y equals to 0 okay what are two linearly independent solutions sine x and cosine x cosine x to linear independent solution now you do realize that where is this zero this is at n pi and this is between n plus 1 pi right sorry pi by 2 okay so essentially what is happening is this see between any two zero of this there is a zero of this and again between any two zero of cosine there is a zero of sine right so basically sine and cosine in this case are interlaced okay so this is the kind of motivation you can think of of this theorem this is okay So let us look at the proof of this. I hope you understood what I am trying to say here. Okay.

So now, see, let me... Okay. So, since I am writing it as a call ready, so basically what we are trying to do is we are going to use Sturm-Picot theorem here. Yes. Okay.

Now, how do you use this theorem? What do you think can happen? See, the thing is, let us say since y1x and y2x cannot have Cannot have. Common zeros.

Common zeros. Okay. Right. Then. This theorem.

What is the theorem? Stump Econ theorem. Then. Stump Econ theorem. Theorem.

implies implies that the solution y2 y2 okay okay first of all you do realize what are the equations here right what is the equation Okay. So, what you do is this. See, first of all, you pause this video. Think about how to prove this statement.

See, the equation which I am comparing here, okay, Sturm-Picot theorem is basically a competition theorem between two equations. But in this case, what are the equations which you are comparing? See, we are basically looking at Linearly independent solutions of just one equation. Okay.

So, it is not two different equations. It is basically one equation. Yeah. And I am comparing two solutions of the same equation. Is this okay?

Yes. So, now, the question is, so, the thing is this. You have to, you know, note that these two, y1 and y2 cannot have common 0s. Is this okay? Why it cannot have common zeros?

Think about it. Okay. So, take some time, think about it. If you cannot do it, I will do it at the end of this proof. Okay.

So, it does not have a common zero. Let us just, I mean, assume that. Okay. Then the Sturm-Picot theorem, what does it imply? That the solution y2 of x has at least

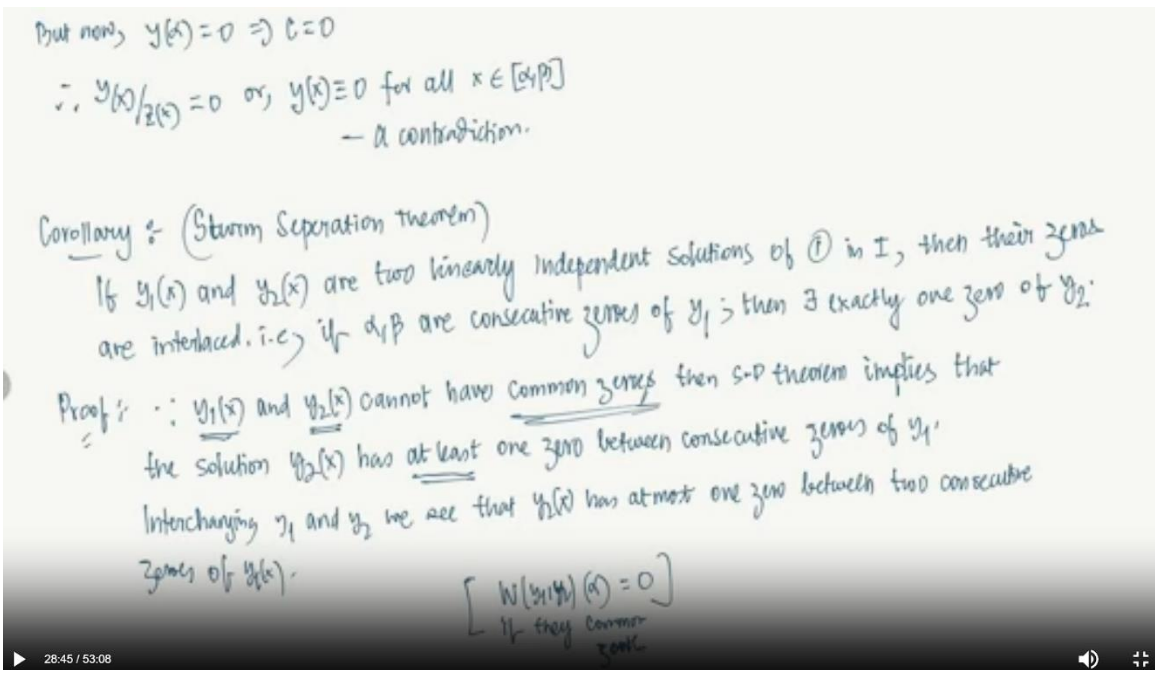
at least 1 0 0 between consecutive 0s between consecutive 0s 0s of y1. Yeah. See y1 and y2 between consecutive zeros of y1, y2 must have a 0 and between closed alpha beta. But it cannot have common zeros. So it has to be in between alpha beta.

Is this okay? Now if you interchange, interchanging y1 and y2. Okay. y1 and y2, we see that y2 of x, okay, has at most, has at most one zero between two consecutive zeros, two consecutive zeros, zeros, of y1 of x, right.

So, what did we prove? We proved that it is basically, it has to be, you know, interlaced. So, basically exactly 1 0 should be there. Here we showed that y2 has at least 1 0. Now, if you interchange y1 and y2, you can say the exact same thing

For y1 what you said about y2 right because it is not a problem y1 or y2 is just any two solutions okay. So in that case between any two consecutive zeros of y1 there will be at most one zero of y2. So if you interchange it will be exactly the same thing. So and hence it has exactly 1 0. So between 0 to 0 of y1 there is exactly 1 0 of y2.

Is it okay? Now I hope you have concluded why it does not have common roots. Let us say that alpha is a common root. Okay, see if it has to be common 0, it has to be either alpha or at beta. So let us just say that alpha is a common zeros.



So in that case, what is the Wronskian of y1 and y2 at the point alpha? Okay, you can check that that is going to be 0. And since at the point Wronskian is 0, so if they do not have common root, if they do not have common root, if they have common root, sorry. Common root. 0, 0. 0s. Okay?

Common 0s. Then, at the point alpha, the wrong scale is going to be 0. Abbas theorem says that then it is 0 for all points. And if it is 0 for all points, it has to be linearly dependent. But it is given to be linearly independent.

Hence, this is not true. Okay? I hope this clarifies everything. Okay. Now, the theorem is trivial.

Now, so, basically what we proved is the 0s are interlaced. Clear? Okay. So, now we come to a very, very important theorem and this theorem says that if you are looking, see here as I told you, I was just looking for some interval i, right. I did not specify what interval, it can be bounded interval, it can be unbounded interval, okay.

But now, I want to prove a theorem which will actually say that how many zeros can be there in a finite interval alpha beta, okay. The only solution, solution of one which vanishes, which vanishes infinitely often, infinitely often, often in alpha beta, in alpha beta is the trivial solution. So, if you have a bounded interval, right? So, what you can show is the only solution. So, in a bounded interval, let us say if you are starting with a non-trivial solution of that problem, right?

It cannot have infinitely many zeros. So, basically it is not going to be oscillatory. See, with finite zeros, there will be a finite number zero, right? I mean, if there are like finitely many zeros, okay? So, there will be a last zero.

It has to be. But in that case the solution does not stay oscillatory. So you can't say that solution is oscillatory. You understand? So in this case what we have proved what we are going to show is

only oscillatory if you are looking for oscillatory solutions of the equation one okay then the previous solution is the only one there is no other oscillatory solution okay if you have to look for oscillatory solution of this equation you have to look at it at whole of r here okay so what is the proof okay so let us assume let us assume let us assume that the solution, the solution y of x vanishes, okay, infinitely many zeros, so with vanishes, so basically, let me put it this way, has an infinite number of zeros, has an infinite number of zeros. Zeros in I. Let us just put it that way. Okay. See.

I. Is closed interval alpha beta. Right. Closed interval alpha beta. Some alpha beta. Does not have to be anything.

It is just some alpha beta. But it is closed interval alpha beta. And. You see. Let us say.

Let. This. The. This. There are like infinite.

Numbers. Right. Infinity such. Points. So basically.

Let. xm okay b so in alpha beta in alpha beta this whole interval is in alpha beta okay so for all m for all m in n be such that y at the point xm is zero this is okay for all m See, it has infinitely many zeros, right? So, what does it mean? It means that there are infinite number of points where y is 0.

Those points are in alpha beta. So, I can make a, you know, I can build a sequence out of it. So, xm is a sequence which always lies in the closed interval alpha beta such that y at the point xm is going to be 0. Is this okay? Right.

So, you see, xm is a sequence in a compact set. Okay, and what happens to the sequence in compact set? It converges. Okay, at least up to a subsequence it will converge. Okay, so I will just write this as the whole sequence xm.

Okay, so it will converge to x star. Okay, therefore xm converges to x star. Why? Because alpha beta is compact. Bounded sequence, Heine-Bodl theorem, right?

Bounded sequence in a closed and bounded interval, it has a convergent subsequence. So, that convergent subsequence, let us just call it again as xm. So, xm converges to x star with xm not equals to x star. Is this okay? Now, you see, we will show, we will show

that y at the point x star equals to y prime at the point x star is equals to 0. This is what we are going to show first of all. Let us just see if we can do something like this. What is y prime at the point x star? It is nothing but limit h tends to 0 y

at the point xm, sorry, not h, m tends to infinity, let us say, m tends to infinity, y at the point xm minus y at the point x star by xm minus x star, okay? Now, I know that y at the point xm are all going to be 0, right? It is always a register assumption. Okay, what happens to y at the point x at the point x star? See, xn converges to x star, right?

So, that will imply y of xn must converge to y of x star. Why is it true? This is by continuity. That is the very definition of continuity, right? If xn converges to x, f of xn must converge to f of x. f is continuous if and only if this happens, right?

Okay, so it is there. So, that will imply that limit m tends to infinity. 0 minus 0 by xm minus x star. See, xm minus x star is non-zero. So, this is going to be 0.

This is okay? Okay. So, what do you get? Therefore, y at the point x star equals to y prime at the point x star equals to 0. So, y is such a function such that y at the point x star equals to y prime at the point x star is 0.

Yes? So, uniqueness. Because existence uniqueness theorem. Uniqueness theorem. What does it say?

It says, you remember what does it say? That equation, right? The equation 1 along with this initial data. What happens to this? 0 is always a solution.

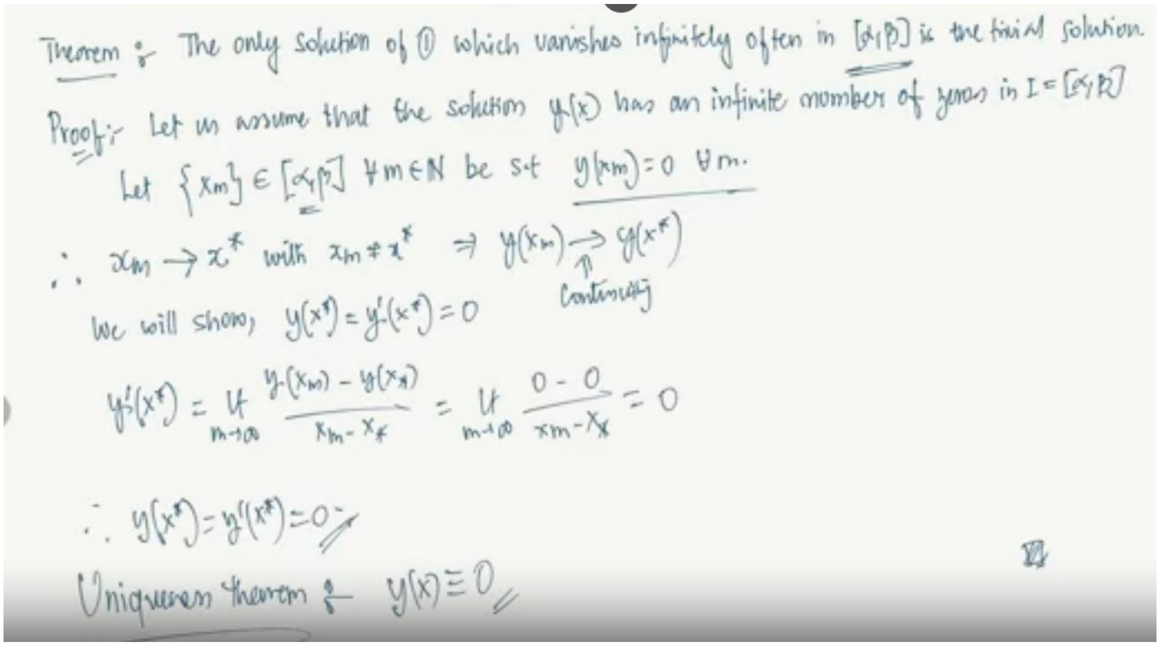
And it satisfies this y at the point x star. Some point x star is there. So, y at the point x star and y prime at the point x star is 0. So, that will imply yx has to be identically equals to 0. Uniqueness theorem.

Yeah. Okay. So it has to be the solution has to be trivial solution. I hope this is clear. Very very nice proof right.

It is very nice proof. Okay. And full of analysis. Okay. So what did we learn from this theorem?

Essentially what we learned from this theorem is this. See. in a finite set see up till now i did not see i was telling you that in some interval i i did not specify if the interval is whole of r if it is closed bounded set or not is this okay why i did not specify is because if if you have a problem one that sort of equation self adjoint equation in a closed bounded interval alpha beta close bounded interval that is very very important okay Then you cannot have oscillatory solutions. Is this ok?

It cannot be oscillatory. I mean if it is like infinitely many it admits infinitely many zeros it has to be trivial. There is no other way. Please remember this is working only because it is in the closed bounded interval alpha beta. Otherwise this is not going to work.



See Heine model is not going to work. So this is very important. So, now what we are going to do is basically how do you know that a differential equation 1 is oscillatory. So, how does one does one test if 1 is oscillatory where is where it is oscillatory in I which is 0 infinity okay see again i told you close bounded if it is close bounded cannot be so there is no point looking at it okay so it is always in the unbounded domain

Okay, so now we look at Rath theorem. Let us call this theorem. I do not know what he called this theorem. So, maybe Stampikon theorem. Let us call this one as theorem 1 for now.

Okay, and let us call this theorem 2. This theorem has also a name. This is very, very important theorem. This is called Leighton's oscillation. Oscillation theorem.

okay lightness oscillation theorem so for uh just for reference purposes let me put it in a red so that will you know identify it as a very very important theorem lightness oscillation okay right so what does that theorem says it says that if integral over infinity 1 by p of x dx if this is infinity okay, and integral of q of x dx till infinity is infinity, okay. So, basically, you have, you need to have two blow-offs, 1 by p and q. They should not be summable. Is this okay? They should not be summable.

Then, 1 is oscillatory, where oscillatory In 0, infinity. Is this okay? It is not a very difficult thing to understand. The theorem is quite clear.

It says that if 1 by p and q, if they are going to infinity, then the theorem is oscillatory. So, as an example, before doing the proof, let us just look at it. See, what is the one equation which we know is oscillatory? The most famous equation. y double prime plus y equals to 0 because y equals to sin x, y equals to cosine x is the solution of this problem.

So, we know that this is going to be oscillatory. Okay. What is p here? p is 1. What is q here?

q is 1. Okay. If you integrate 1 dx over infinity, it is going to infinity and again the same thing will happen with p. So you do realize that this is very natural assumption. This is okay. Right.

So basically what it is saying is this. If this assumption is satisfied, then 1 is oscillatory. Is this okay? It may happen that something is oscillatory but these two conditions are not satisfied or at least one is not satisfied. It may happen.

Clear? So please remember that. It is not like a necessary and sufficient condition. It is not necessary condition. So basically, sorry, this is a sufficient condition.

It says that if this and this two condition holds, then it is oscillatory. That's what. It is just sufficient. Okay? So let's look at the proof.

Now, I am not going to do the exact, the whole proof, okay? There are some parts which I want you to check yourself. Of course, I will put it in assignment also. So, because this actually involves a small trick, which is, you should at least know. So, let y of x, I have to show it is oscillatory, right?

Okay. So, let us start with a non-oscillatory solution of this equation. Let y of x be a non-oscillatory, non-oscillatory solution. Solution of 1. Solution of 1.

Right? Okay. Which we assume to be positive in x0 infinity. x0 is positive. Why we can assume this?

See, think about it. If yx is non-oscillatory, it means that it has a last 0. So, after some time, let us say 0, after some time, some xm is there, x0, let us say x0. After x0, there is no point where y will be 0. This is the last 0.

So, Since y is positive, this is the last point where y can take 0. After that, it has to be either positive or negative, right? Yes? But it can never touch 0 anymore.

This is the last 0, x naught. Okay? So, we are just assuming it is positive after that point. Okay? Right.

I hope this is clear. So, now, check this part. This part I want you to check. This is something called a Riccati's equation here. So, the Riccati equation.

See, this is a tricky proof equation. I will use the solution of this equation. z prime plus q of x plus z square by p of x equals to 0 has a solution z of x. okay in x naught 0 infinity sorry x naught infinity okay so you have to check this see the thing is this is manufactured stuff let's just put it this way okay there is no apparent how do i put it i mean It is not quite obvious where we are getting this from.

And how do I put it? The motivation behind getting this particular equation is not quite clear. Even for me also I am not quite sure how Leighton actually came up with this equation. But the thing is he used this Riccati's equation and what he did is basically once you have that you know that there is a Z. in this interval such that z satisfies this equation what happens now you see you integrate this problem what happens if you integrate it becomes z z prime becomes zx minus z at the point 0 that that will happen sorry z at the point x naught so it starts from x naught right z at the point x naught okay so let's make it equal minus integral q of t dt and what is the interval interval is x naught to x

Minus integral, you see, z squared t dt. t by pt dt. And the interval is x0 to x. Is this okay? If we do that. Now, you see, since q of t, the tail part of qt, this blows up.

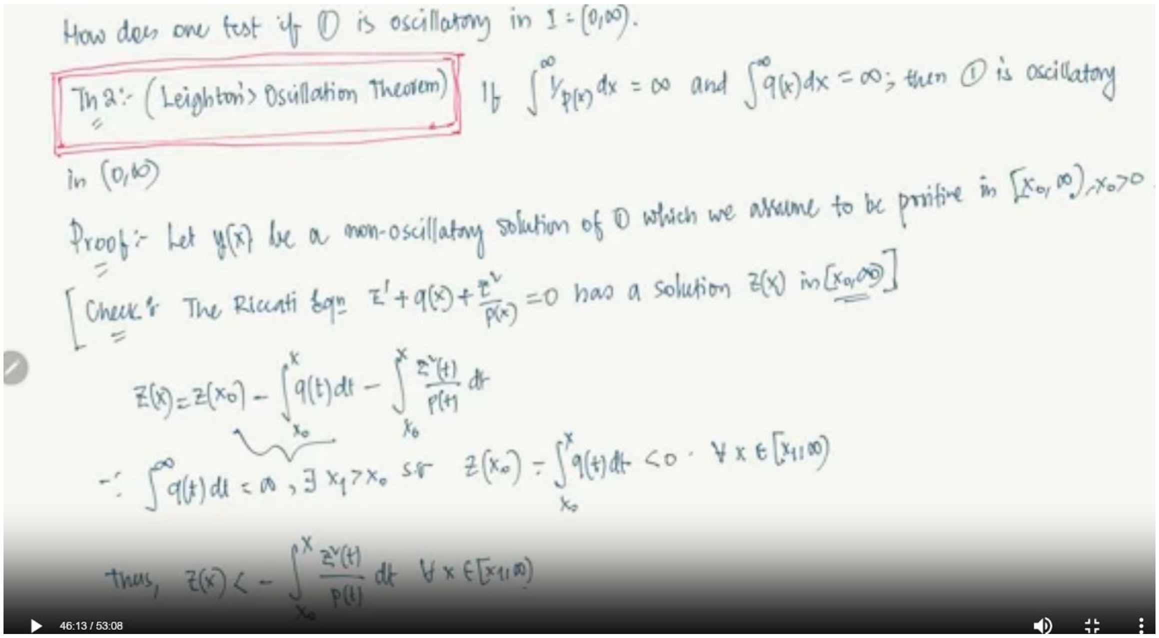
There exists x1, which is greater than x0, such that z of x0 minus x0 to x z square t by pt dt. So, basically these two terms minus x0 to x qt dt. This is negative. See z of x0 is some number 5, 10, 100, 1000, million does not matter.

But after x1, so, you know, you can always find x1 such that for x1 greater than x0, okay, what will happen is this, the integral, see, this is blowing up, right? So, after sometimes, the tail part will be very heavy. So, it will be very large, right? So, z of x0 will be dominated by this because of this negative sign. That is why it is negative.

Is this okay? So, this holds for all x in, in, x1 infinity. This is okay. For all x in x1 infinity, you can say this.

Thus, what do we have? We have zx. These two terms are negative. So, basically it means it is dominated by minus integral x0 to x z squared t by pt dt. This is okay.

pt dt. For all x in x1 infinity. Is this okay? Now, what we do is basically we write this as r of x, okay? So, define r of x, okay?



What is r of x? It is x naught to x z square t by pt dt. This is for x in x1 infinity. If this is the case, then z of x is dominated by minus r of x. Is it okay?

And what do you have? r prime of x this is okay, right? zx is less than equal minus of, see, r is this, right? So, zx is less than equal minus r of x. This is what I wrote up.

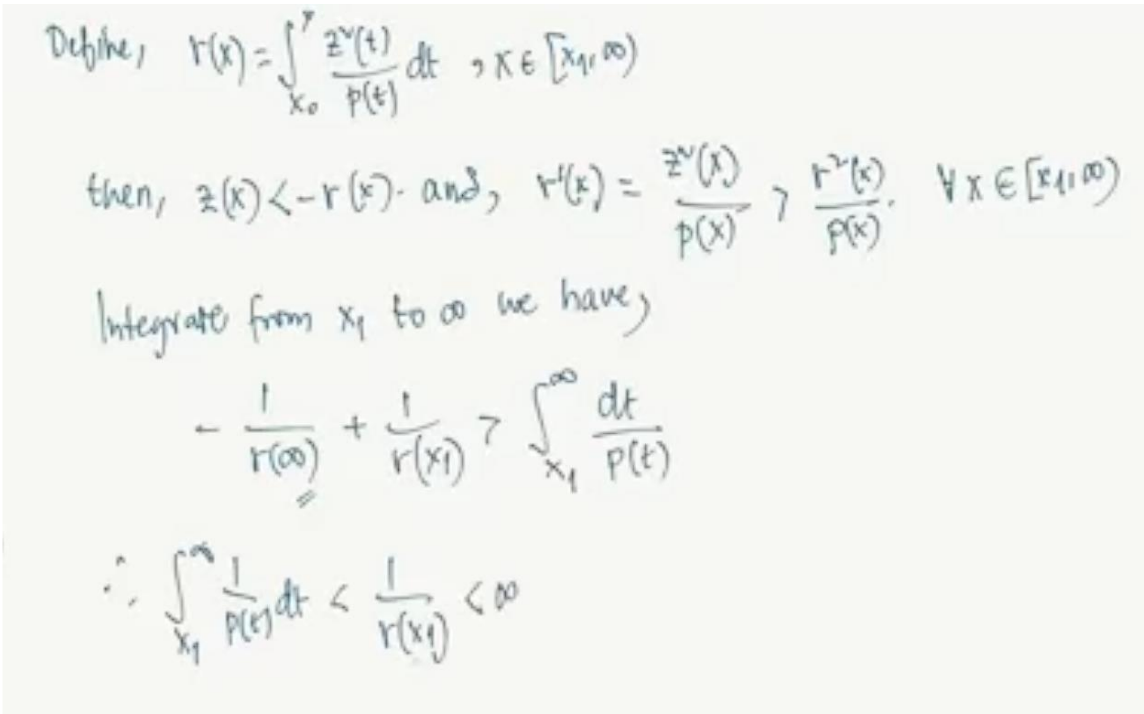
I hope this is fine. And r prime x, what is r prime x? This is nothing but z square x by p of x, clear? Okay. Now, that is greater than r square x

x by p of x is this okay see z plus r is negative right so z square minus r square so p is positive p is positive even right that's what we are also as you can see here this whole the this how do i put it uh The whole study of this here, we of course always assume P is positive because otherwise P can be 0 and then we have problem. You do realize everywhere we have checked, you see, we have always assumed that P is positive, you see, okay. So, P is always positive. Please remember this, very, very important, okay, right.

So, if positive, P is positive, of course Z square is greater than R square, right. Okay. So, if this is the case, what we have is, this is holds for all x between x1 and infinity. x1 and infinity. This is okay.

Now, if we integrate this thing, okay, integrate from x1 to infinity, we have, we have minus r evaluated at infinity. So, basically I should write it as limit r tends to t tends to infinity 1 by r of t, but I am just writing it like this, okay. I am just being lazy here. And r at the point x1 is dominated by x1 to infinity dt by p of t. Is this okay?

We can do that, right? We can just integrate it. See, r prime by r square is greater than 1 by px. So, then I can use this thing. Fundamental theorem of calculus to get this.



Is this okay? Therefore... Integral x1 to infinity, 1 by pt dt is dominated by 1 by rx1. Is it okay? See, this is a negative term.

So, you can just throw that away, right? So, it is dominated by 1 by rx1, which is finite. Okay? Now, that is a contradiction. Why it is a contradiction?

Because this is our assumption that the 1 by P is not summable. But here it is showing it is summable. You see the assumption is 1 by P is not summable. This is okay. So that actually concludes Leighton's oscillation theorem.

So let us look at a small application of Leighton and then we are going to finish this video. So let us look at a small example. So let us say that for All a, sufficiently a, there exist x0 large. Large.

Such that, you see, 1 plus 1 minus 4a square by 4x square. Clear? This is greater than half for all x greater than x0. We can prove that. Given any a, I can always choose a large n of x0 such that this happens.

If you are not convinced, please check this part. It is very easy, not very difficult. You can do it yourself, absolutely no issues. If you are not able to do it, what you do is you break everything up. You just break everything up and after that put it together.

You can get this inequality absolutely no issues. So you can show this that for any A, doesn't matter what A is, for any A you can actually find a large enough X naught such that this happens. And hence... Hence, integral till infinity 1 plus 1 minus 4a square by 4x square times, sorry, dx. This

is infinity because see this expression is always greater than half no matter what x is after x large enough this is always greater than half so if you are taking the integral up till infinity it is always going to diverge right that is just the case so therefore so this implies that the equation equation y double prime plus 1 plus 1 minus 4a square by 4x square times y equals to 0 is oscillatory. Okay. See by just looking by just by looking at it you can't really say if it is oscillatory or not. But the thing is on Leighton oscillation you can use directly and you can say this is oscillatory.

Okay. So this is a beautiful theorem which you can prove using Leighton. So with this we are going to finish this video.