Ordinary Differential Equations (noc 24 ma 78)

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Week-04

Lecture 28: Sturm Comparison Theory

So, hello everyone. In this video, we are going to talk about oscillatory equation essentially. So, basically the theory of oscillation. So, what do we have? First of all, we are starting in the last lecture, we have seen that let us say you given a equation which looks like this p0 x y double prime plus p1 x y prime plus p2 x y equals to 0.

Let us say you are given this equation, right? Now, we know that there exists a function, there exists sigma x, let us say, such that if you multiply, let us say 1, such that sigma x times 1, let me write it down like times the equation 1, will convert 1 into a self-adjoint equation, convert 1 into 1. a self-adjoint equation and go you remember that self-adjoint right okay so you see basically what i am trying to say is this even any equation you can actually convert it into a self-adjoint equation right so from now on i am only going to talk about self-adjoint equation because those are very nice equation i will tell you why that form is very important okay so uh consider the equation this so in the from now on i am going to consider this particular form consider This equation P of x y prime whole prime. If you remember, this is the self-adjoint form, right?

Q x y equals to 0. Yes. And so we will look at this problem. Let us call this problem as 2. And if P x is identically equals to 1, okay, then this equation looks like this.

Y double prime plus Q x y equals to 0. Q x y equals to 0. Okay. So this is 3. When Px is essentially 1.

Yes. So, here we assume that P and Q is in C1 of I. Whatever interval I it is defined on, it is C1. So, continuously differentiable. This is what we are assuming essentially. Yes.

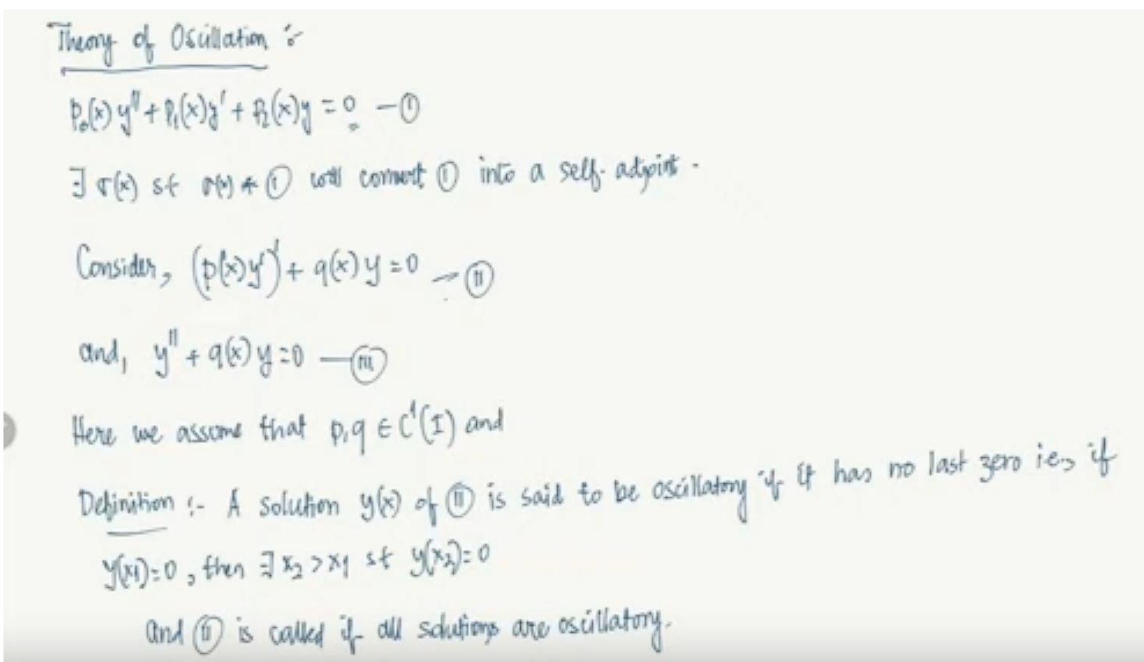
Okay. So, you see here into and into of course, you see if P is C1, Y prime is continuous and differentiable. See Y is C2, right? For any solution of this is at least twice continuously differentiable. So, Y prime is continuous.

okay and and it is differentiable p is differentiable y time is differentiable so i can talk about the derivative so all of this makes sense okay right now okay now definition definition what do we mean by oscillation so a solution y of x okay solution of what let's say solution of this equation 2 of 2 here is said to be oscillatory is said to be oscillatory okay so we are we are calling the solution as oscillatory for now okay if it has no okay last 0. Is this okay? Last 0.

So, what it means is, that is, let us just put it this way, that is, if y of x1, let us say, if you can find x1 such that y of x1 is 0, then there exists a x2, okay, which is greater than x1, such that y of x2 will also be 0. Is this okay? So, this is where we call the solution as a oscillatory solution. And 2 is called oscillatory.

See, the solution is oscillatory if there is no last 0 of the solution. And 2 is called oscillatory if all solutions are oscillatory. Is this okay? Are oscillatory. I hope you, it is clear here.

See, if all solutions are oscillatory, then we call the equation as oscillatory equation, yes, and one solution, if you are just looking at one solution, that is oscillatory if it does not have any last 0. very clear right let's and of course if a solution if you have a i mean a solution y of x which is not oscillating we call it a non-oscillatory so that's just um i mean nomenclature right so let's look at an example see consider the equation consider y double prime plus y equals to zero So, you see any solution y of x will look like this, c1 times cosine x plus c2 times sine x. And this is valid in whole of R, right? Whole of R. And do you think there is a last zero?



Of course not, right? Because sine and cosine, they may have, I mean, if you take any solution. So basically, let's say any solution of this, y1 of x, let me put it this way. This will become more, I mean, it would be. sine x these are the two linearly independent solutions for this problem and they form of the basis for the solution space so basically and you can see that all solutions so this solution doesn't have any last zero this doesn't have any last zero right so basically the combination also you can actually see that there there is no last zero so basically what you can say is this is an oscillatory equation is oscillatory the equation itself is oscillatory okay right now a but

If you change the sign a little bit, y double prime minus y equals to 0, then you see that the solutions y1 of x you are going to get is exponential x and y2 of x what you are going to get is exponential minus x. None of them has any 0. So, it is non-oscillatory, oscillatory. Yes. So, now the thing is this, we want to prove a result which is like a fundamental result in this direction. So, this is called.

So, essentially the idea is this. See, as we have seen that for a variable coefficient equation, yes, even linear. So, let me put it this way. Remember this. Linear equations are easy to solve if they are constant coefficients.

If they are constant coefficients. If the coefficients are constant. Now if the coefficients are not constant they are not easy to solve. They are not easy to solve. coefficients are variable right now in the last videos we have seen there is some way of solving it like right if you can solve the adjoint equation and then of course you can do it but generally speaking it is not also easy to solve adjoint equation in most of the cases okay so in those cases we want to know if without solving the equation we can actually deduce something more out of the solution of the equation yeah

So, in this direction, the first fundamental result which we are going to prove and this is one of the most important results in this course. This is called Stump's Comparison Theorem. Comparison Theorem. Clear? So, what does it do?

It actually compares. Compares with what? We will look at it now. So, let us say if alpha beta. in i okay some interval i see i can be r also but some interval i are the consecutive zeros consecutive zeros of a non-trivial solution okay of course non-trivial solution because if it is trivial solution then they everything the i mean every number is zero right so it doesn't matter so we are looking for a non-trivial solution

So, let us say yx, clear? Of, please remember this thing. We are talking about this equation 3, not 2, 3, okay? 3. For now, let us just look at 3 and then we will talk about what happens in 2. And if, if q1 of x, okay, is continuous, there is a function q1 such that q1 of x is continuous.

And q1 of x is greater than equal qx. Is this okay? Of course, they are not 0. So, they are not equal in alpha beta. They are not equal, but q1 of x is greater than equal qx.

For some point, they may be equal, but generally, they are not equal. And see, both are continuous. See, q is this, q is this. Since both are continuous, if they are non-zero at one point, they have to be zero in some neighborhood of that point, right? So, that is the only assumption.

Then what you can say is, then every non-trivial solution, non-trivial solution zx of x, Y double prime plus Q1 of X times Z equals to 0 has a 0 in open alpha beta. Okay. Fine. So, let us understand what it is saying.

You have two equations. Y prime plus PX or QX in this case. qx, y equals to 0. And you have another equation, y double prime plus q1 of x, y equals to 0. Let us say you have a solution here.

Let us call that solution to be y, such that y at the point alpha equals to y at the point beta, equals to 0. So basically, it is a 0, right? Now what this is saying is this. See that between alpha beta if there is these are consecutive alpha beta there is no other 0 of alpha beta right other than alpha beta then there exists a gamma between alpha beta such that. Any solution of this, okay, every non-trivial solution, you see, any solution of this, of course, not the zero solution.

But any other non-trivial solution, let us say z is any other non-trivial solution, z of gamma will be zero. That is what it is saying. So, the bigger coefficient, okay, is actually forcing the smaller one to actually give up a root. Let us just put it that way. So, you remember this?

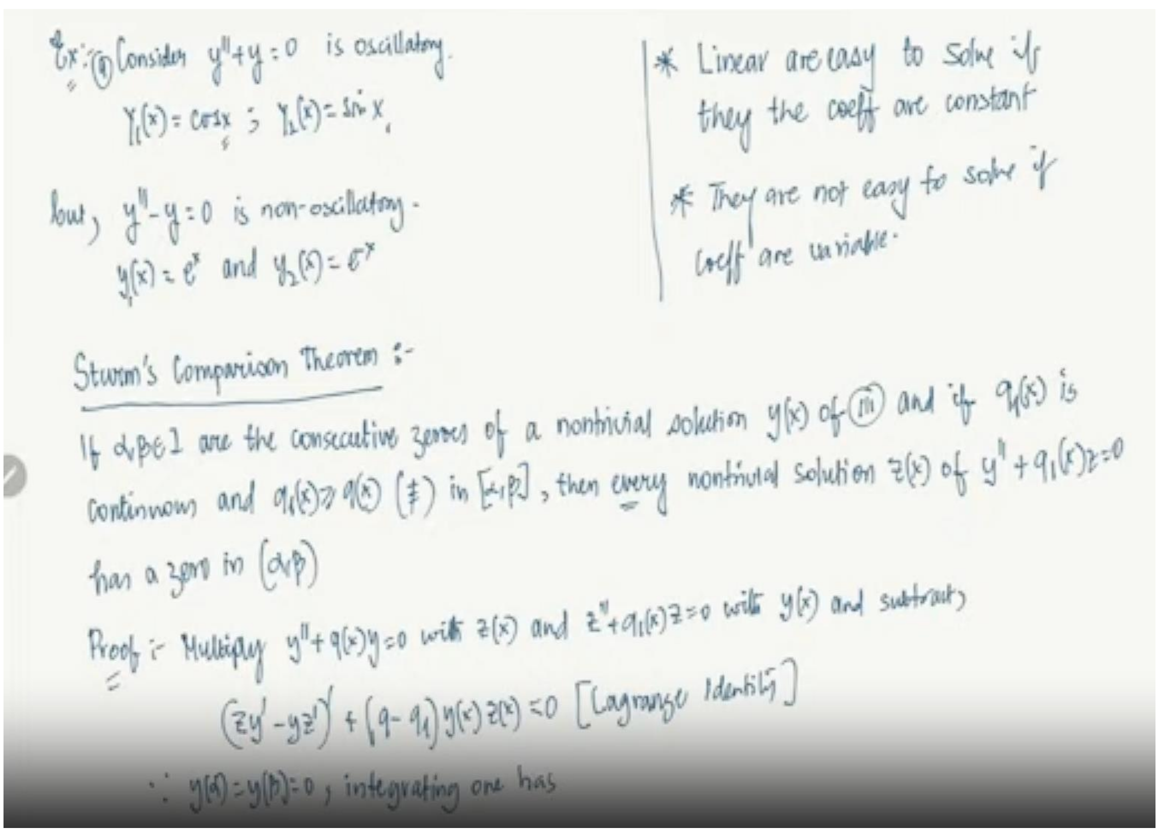
You see, the smaller coefficient has two consecutive zeros. then the bigger coefficient will have 1 0 in between that clear okay so let's look at the proof proof The proof is not very difficult. So, what you do is this. So, first of all, maybe, okay, so multiply, let me write it, otherwise it may get confused.

Multiply y double prime plus q x y equals to 0, this equation with z of x, clear, with z of x. And this equation z double prime plus q1 of x z equals to 0 with y of x. This is okay. Now once you do it and subtract. and subtract. You know, this is the exact same thing which we did while we did use the Lagrange identity, if you remember from the last class.

So, once you do that, what you have is z y prime minus y z prime whole prime plus q minus q1 y x z x equals to 0. How do you get this? This is exactly the Lagrange identity if you remember. This is Lagrange. Lagrange or greens, whatever you want to call it, doesn't matter.

But anyway, so exactly the same thing. I mean, if you just take the integration part, then you can call it greens. But anyways, it's okay. So, this is the Lagrange identity, okay? The proof of this, we did it in the last class, so I'm not discussing this part, okay?

Now, once this is true, now, since, now you see, y at the point alpha equals to y at the point beta equals to 0. This is given, right? Consecutive zeros, you remember? then integrating what you get integrating one has one has you integrate this thing what will happen you are going to get z at the point beta y prime at the point beta minus z at the point alpha y prime at the point alpha that is what remaining because you see What happens to y at the point alpha, y at the point beta?



That is becoming 0. So, those terms are not there. This is the only term which is remaining. And then integral of this particular thing. So, plus alpha 2 beta cube of x minus q1 of x y of x.

Z of x dx. I hope this is clear. Very simple. Lagrange identity, after that you integrate that. This is basically the Green's identity, if you want to call that.

So, we have this. Now, you see, the thing is, what we are going to do is we are going to use a linearity. So, basically, we assume that y prime of x is positive in alpha beta. See this is where we are using the consecutive test.

See essentially what we are saying is. alpha beta y is zero here y is zero here these are the consecutive zeros so basically it can either do this or it can do this any one of them okay so we are just assuming it is positive we can assume negative also it doesn't matter okay it is okay so basically let's say if it is positive this is alpha beta what is y doing y is zero here so it is going up somewhere and then it has to come down and match here that is beta Yes. Okay. So, y, so we are, we are, sorry, we are assuming y of the, y at the point x is positive.

That is what we are assuming. In open AB, in open AB, y at the point alpha, y at the point beta is 0, y is positive in open alpha beta. Okay. Then what happens to y prime at the point alpha? This is strictly greater than 0.

Okay. Why? So, what I want you to do is this. You see, you do realize that it has to be increasing, right? So, y prime at the point alpha, you see, y prime at the point alpha should be greater than equals to 0, right?

See, it is increasing. It can be greater than equals to 0, okay? So, but the thing is, but that is not the case. y prime of the alpha has to be strictly greater than 0. Why?

Because if y prime at the point alpha is 0, And it is also given that y at the point alpha is 0. That is our assumption. That will imply y of x is identically equals to 0. This is Picard's existence uniqueness, right?

Picard's existence uniqueness. Existence uniqueness. This is the only solution. Okay. That actually violates the fact that y is a non-trivial solution.

That is what we assumed. So, this has to be true. Okay. So, similarly, we can also show that y prime of beta has to be strictly negative. Clear?

So, if that is the case, you see, now, If let us say what we have to show, I have to show z has a 0 in between alpha beta. If this is not the case, so if let us say z is positive in alpha beta, then what happens? Then let us call this a star. Then from star, what do we have?

See y prime of beta is negative. z of beta is positive. So this number is going to be negative. Again, z of alpha is, z of alpha is positive. y prime of the alpha is positive with a negative sign.

So, this whole term, this whole term is going to be negative. Is this okay? Right. Now, let us look at y. y is positive. This is what we assumed.

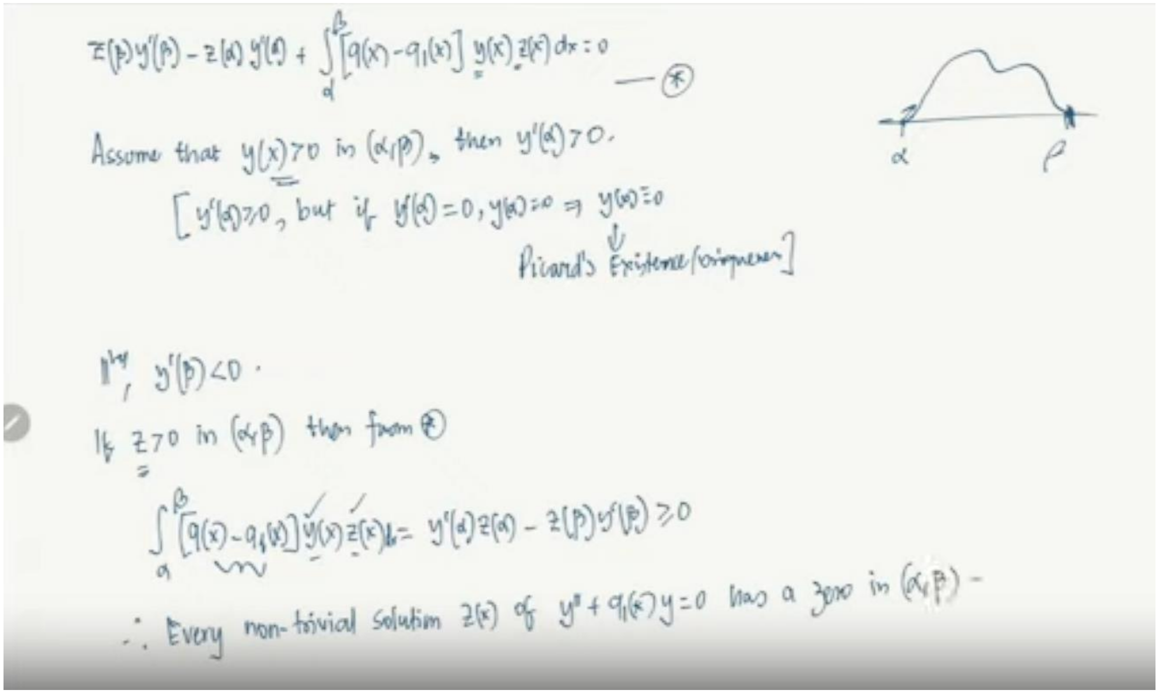
y is positive in alpha beta. What about z? z is positive in alpha beta. What about q minus q1? q of minus q1 is going to be

Negative. This is okay. Q, see, q1 is greater than equal q. That's what given to us, right? So, what does it mean? It means that alpha to beta q of x minus q1 of x, y of x, z of x should be

y prime at the point alpha, z at the point alpha minus z at the point beta, y prime at the point beta. This is the case. And this, you see this y prime at the point alpha as I already explained, right? So, this is greater than equal to 0. This is greater than equal to 0.

But Q minus Q1 is negative. This is okay. You see Q minus Q1, this is negative, right? Because this is given. You see Q1 is greater than equal Q. So Q minus Q1 is less than equal 0.

This is negative. y is positive z is positive so this is a negative term and that the integral of a negative quantity basically we are writing that that is going to be positive right so this cannot happen until unless see y and z they are non-trivial so until unless q is identically equals to q2 q1 okay but that is also not the case because you see we say that this is not identically equal okay so hence Our assumption is wrong and therefore, every solution, every non-trivial solution, trivial solution z of x of y double prime plus q1 y, q1 of x y equals to 0 has a 0 in alpha beta. i hope this is clear okay so uh and by the way the very very very important theorem this is a stump comparison so please remember this okay so let's do a quick corollary of this theorem it says that if qx okay is greater than equal one plus epsilon by four x square okay epsilon positive for all positive x Then y double prime plus qxy equals to 0.



This equation is oscillatory where is oscillatory in 0, infinity. Is this okay? So basically what I am saying, what we are trying to prove here is this. So y double prime plus qx y equals to 0, right? That is the equation given to us.

Can we say whether this equation is oscillatory or not? What is the condition? So this actually gives us the condition that qx has to be greater than or equal to 1 plus epsilon by 4x square. Where did we get this thing? Let us just understand that.

Let us look at the proof. Now, I am not going to do the exact, the whole proof one. What I want you to do is, I mean, I am going to leave some gaps. I want you to fill those gaps up. So, for epsilon greater than 0, okay, let us look at all non-trivial solutions, you see.

All non-trivial solutions. Solutions of y double prime. plus epsilon by 4y equals to 0 are oscillatory. Right? This is trivial, right?

Cosine, sine. Yes? The solution. So, they are going to be oscillatory. No problem.

Okay? Now, define this thing. Change of variables. So, t equals to e to the power x. Okay? Equivore x in this equation, y double prime plus epsilon by 4y equals to 0.

Okay. If you take this change of variable, we have, you can actually convert it as a function of, from function of x to function of t. And that will look like this. t square y double prime plus t y prime plus epsilon by 4y equals to 0. This is okay. Okay.

Now, again using, using the substitution, the substitution y equals to z by root t. We obtain obtained z double prime of t plus 1 plus epsilon by 4 t square z equals to 0 this is what we get okay now since z t equals to e to the power x by 2 y as a function of t so e power x this equation is oscillatory right i don't know how to put it maybe double star double star is oscillatory It has to be right because this equation is oscillatory. So, every solution is oscillatory.

y is oscillatory. So, every solution of this is oscillatory. Now, I changed it to another equation. Now, for this equation also, it has to be oscillatory because z is e to the power x by 2 y power e power x and y is oscillatory. So, z has to be oscillatory, right?

Therefore, therefore, between any two solutions of this equation, what is the equation? This equation, star. So, therefore, ah, between any two zeros any two zeros of which one the this equation of star star okay there is a there is a zero of every solution solution of y double prime plus px, sorry, qx, y equals to 0.

I hope this is clear. So, therefore, the equation See q is greater than this. So you understand what I am trying to say. Therefore the equation y double prime plus q x y equals to 0 is oscillatory.

Some way you can compare that oscillatory. So this is fine. Now the thing is this. Suddenly the question is this. Why did we change it like this and what is the main motivation of doing all this?

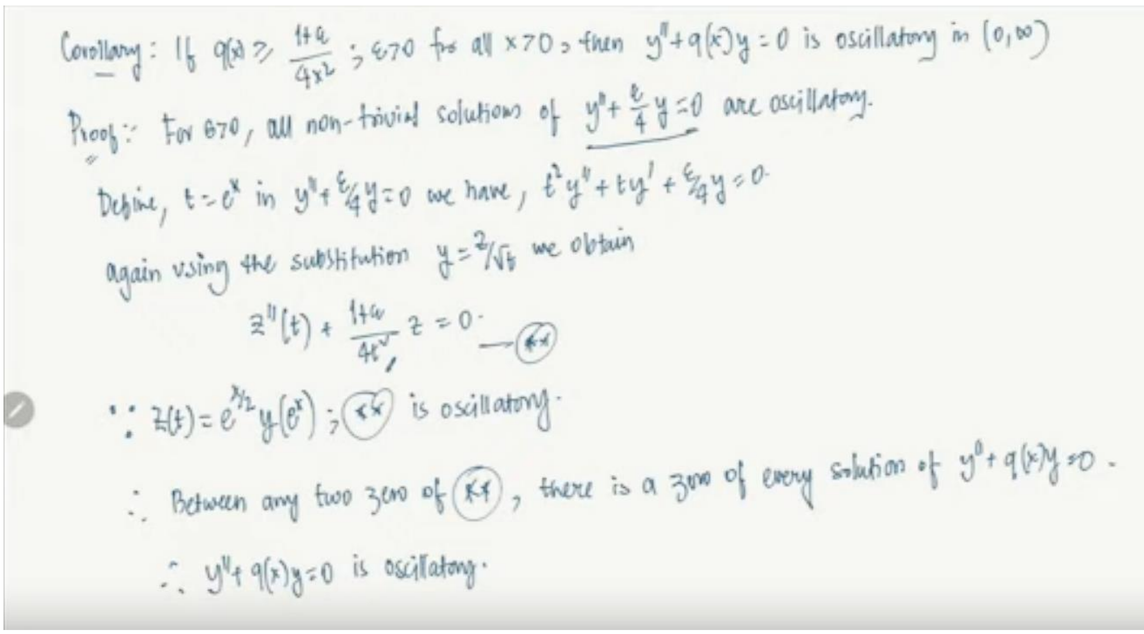
Okay, let me just clarify first of all. This expression t equals to if your x are changing into this, if you have a equation like this and if you want to change it to this sort of equation. So basically the equation, first equation doesn't contain any y prime. You have to check that. First equation doesn't contain any y prime.

You see, this is the change of variables. If you are, if you did, one second, sorry. If you did your Cauchy-Weiler equations, you know that this change of variable actually converts this equation to that equation, the e to the minus x change of variable. So basically, we are doing this in an inverse manner. So basically, if we are taking t equals t power x, then we can change this thing to t square y double prime plus ty prime plus epsilon by 4y.

So this expression, this change of variable is coming from the Cauchy-Weiler form. okay fine now you see what we want to do is basically this problem again we want to write it in terms of another function okay uh i mean this sort of function z double prime plus a function let's say q1 of q1 of t times z equals to 0. we wanted to write it like this and to do that what we are going to do we did is this y equals to f of t times z this is the i mean substitution which we took Now once you do it and f is smooth enough, f is smooth, okay? You see, I am giving you an idea.

You have to check this part. f is smooth. Now you see y prime exists, y double prime exists. You have to find it in terms of f. Once you find it, you put it there, okay? And you assume that you will have some coefficient, so you will get some equation, right?

With z double prime, z prime and z. The coefficient corresponding to z prime, you make it 0. When you make it 0, then you get f of t, which is 1 by root t. This is the idea. So, once you get it, now why do we need this form? Why do we need this particular double star form?



Because then we can use your strong comparison principle to get our result. I hope this is clear to you. Now, let us look at some examples also. Let's look at some example. See.

And this example, how do I put it? Let me put it in red. I am going to use this example in a lot of different places in this course. Okay. So I want you to understand this example.

Example. Maybe you can also find some questions in your assignment and in exams also corresponding to this particular example. Okay. Right. So, the differential equation y double prime equals to 0 is non-oscillatory, right?

Non-oscillatory. Why it is non-oscillatory? Because any solution, because you see any solution of this problem yx looks like ax plus b, right? ab is in R, some constant, okay? It will look like this.

So, you can see that these solutions, I mean, there is only one 0, so it cannot have infinitely many 0s kind of. So, it is non-oscillatory. Thus, if qx is less than or equal to 0, clear? Yes.

And not identically equal to 0, of course, in i, whatever i is. Then, Sturm comparison theorem, that will imply This is a small one, right? Okay. That y double prime plus q of x, y equals to 0 cannot have more than 1, 0.

Imply that. No, each solution, I should write it. Imply that. Each solution of this equation cannot have more than 1 0. Is this okay?

Why? Why is that true? Let us look at it. Again, I am not writing it. I want you to check it yourself, but I will tell you why.

See what happens. Let us say y double prime plus q x y equals to 0. This has two roots, alpha, beta. q x is less than equal to 0, right? So basically, if there are two roots, consecutive roots, two roots,

Then, in between those two, there is another root of this 0 thing, right? Because Qx is dominated by 0, some compression of the inverse says that then y double prime equals to 0 will have one root in between. Is this okay? But this cannot happen, right? See, this will...

Yx equals to x that one solution may have root but you see yx equals to let us say 1 also solves this problem right. But that solution does not have any root in between alpha beta. But what does term comparison piece work say? It says that every solution of that equation will have one root in between. So, but a constant solution is not going to have that root.

Clear? So, it is this equation y double prime plus q x y equals to 0 is non-oscillatory. Please remember. Again, let me put it in this way. Very important result.

y double prime plus q x y. Okay. Maybe I can put it this way. Q, let me give an example of Q. So, let us say x square y, yes, equals to 0 is non-oscillatory. You can say that. I can take it to be p, it does not matter.

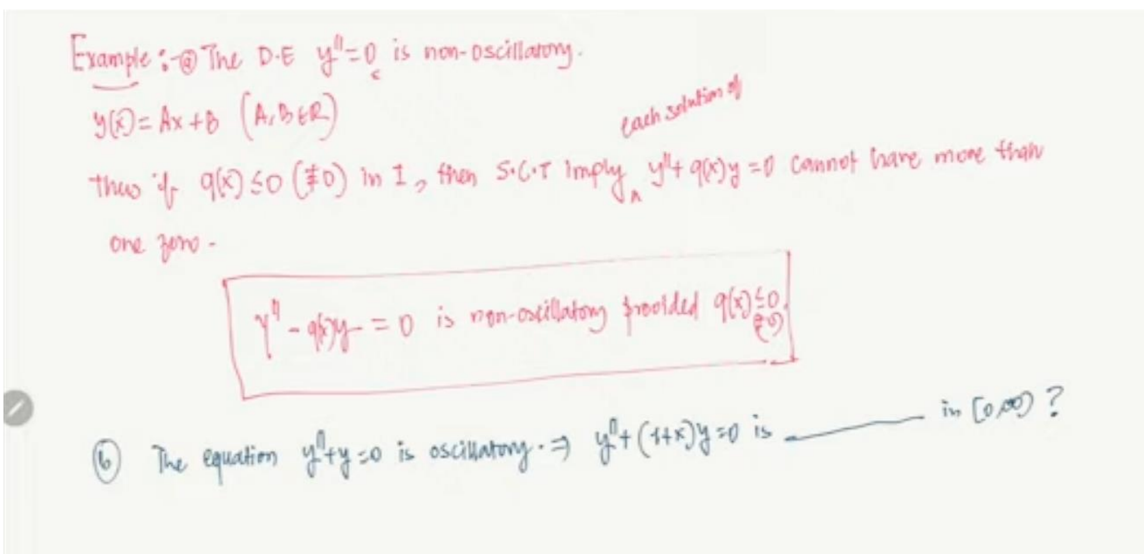
So, Q x y equals to 0 is non-oscillatory, provided, provided. Qx is less than or equal to 0. Is this okay? And of course not equal to 0. Is this okay?

It is not. Very very important result and we are going to use this result a lot of time. This is first example. Now let us go back to our original thing. Second example is this.

See the equation equation y double prime plus y equals to 0. See, what we are doing, if you look at it carefully, you see, we are implying something about the solution of a variable coefficient equation using our knowledge of solutions for a constant coefficient equation. Constant coefficients we know how to solve. So, those solutions we know how it, you know, behaves. We are using that behavior to give us what is the behavior of this sort of solution.

Clear? Okay. The equation y double prime plus y equals to 0. That is oscillatory. This we know.

oscillatory okay so that will imply that y double prime plus 1 plus x times y equals to 0 okay is what oscillatory or non-oscillatory think about it in 0 infinity this is i want you to check what exactly is this see think about it 1 plus x is always greater than equal 1 in this interval in 0 infinity right so what can we say about it so any solution of this will it have a 0 in between y double prime plus y any solution is oscillatory right so what can we say about any solution so here uh let me just put it this way you will have oscillation here okay but please you do it yourself think about it why it is true this is okay right So, yeah, this is fine. So, now what we are going to do is going to prove a very important theorem, which is called the Picot's theorem. And we are going to do here.



But before that, let us just do the Picot's theorem first. So, Picot's identity, sorry, Picot's identity. identity again the identity which i am proving now because identity it may look like a very crazy kind of thing i mean you may get the idea that how are you getting this thing and So, but what is the use of it? This is, let me tell you, this is a very, very important identity as far as this ODE is concerned.

But in PDE, this is like a very, very vital identity. There are a lot of things which you can prove using Picot identity, very important. So, we are just doing it the ODE part of that. So, first of all, let the function, what are the functions? y, z, p, y prime, p1, z prime, okay, b, smooth, smooth means c1, smooth, smooth means c1, clear, and z of x is not 0, in i, clear, i is some interval i, you can put any interval u1, doesn't matter, then the following, following,

Identity holds. Let us look at what is the following identity. It says y by z, zpy prime minus yp1z prime. whole prime this particular quantity is y now please understand there is nothing to understand in this this is just some technical identity okay but it is very very useful to prove theorems that's what we are doing y p y prime whole prime minus y square by z plus p1 z prime whole prime plus and you do not need to remember this identity also.

So, do not worry about it. We are just going to use this to prove this theorem which we will call as Trump-Picot theorem. Very important. We will do it in the next video. minus y by z z prime whole prime this is okay so how do you prove something like this the proof it's a no brainer basically what you do is you break it up and then put it together that's all okay so what we are going to do is i'm going to start with this right hand side so we have this expression one second

sorry let's write the left hand side and we go to the right hand side from there okay that will be better because it is just an expansion right so expanding the right hand side what do we get expanding the left sorry left hand side one gets y by z right z py prime whole prime plus z prime py prime okay minus y p1 z prime whole prime minus y prime p1 z prime this is what we are getting okay i don't need that this this is what we are going to get plus y prime by z minus y by z square z prime z p y prime minus y p 1 z prime. This is what we are going to get when we expand it, right. Now, see this is nothing but y p y prime whole prime minus y square by z

P 1 z prime whole prime, right? Okay. Plus P y prime square. Okay. And one second.

Let me just check. It is okay, right? Yeah, fine. Yeah. Minus this part.

One here, one here. So, 1 y y prime P 1 z 1 y y prime this one, this part and that part. So, basically 2. 2 y y prime. p1 z prime by z that part is there plus another by z square that part is there right so basically this is this one z square part so y square what do you have y square one second let me just check y square

P1 y square P1 z prime square by z square. That's what we have. Yeah, this way. Okay. So, now you see we are already here.

So, basically what we have is y P y prime whole prime. This is this y square by z P1 of z prime whole prime plus. Now, I will just write it like this. P minus P1. y prime square.

Is this okay? Plus P1 y prime minus y by z, z prime square. See, this is the cost of this. I did not do any calculation because I know this is going to this. So, please check this part.

Okay. So, this is the end of the proof. With this, I am going to end this video. Thank you.