Ordinary Differential Equations (noc 24 ma 78)

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Week-04

Lecture 27: Exact and Adjoint equations

So, welcome students in this video we are going to talk about something a little different. So, from now on for the next few lectures what we are going to do is we are going to talk about some equations right. So, up till now we talked about systems we looked at whether there is the existence in dictation what are the properties which you can expect out of the solutions right. Now we are going to look at some system which is given by this. So, consider the system sorry the equation consider the equation.

equation P naught of x y double prime x is a variable a right plus P 1 of x y prime plus okay so this is the system let us say this is equal to 0 given any again here what are we are doing is we are going to assume that p 0 p 1 and p 2 are smooth are smooth i mean most of the times continuity is enough but let us just say smooth smooth means it is at least c 1 okay continuous differential and that is fine let us just take it like this now you see the thing is for Let me just give you an exact idea of what we are trying to do here. See, we want to solve this problem. Now, why this problem?

Let us put it as a small remark. Now, for this problem, let us say p naught x, p 1 x, p 2 x, these are all constant functions. So, if p naught, p 1 and p 2 are constants, p 2 are constant function, constants, let us say, then we can solve the problem, right? One can explicitly solve the problem, right? We remember e power m x kind of thing, right?

Solve the problem. from the equation. Yeah. And what about the inhomogeneous part? Of course, if you have the solution of the homogeneous part, you know that you can use, you know, to find out the solution of the inhomogeneous part.

So, that is there. Yeah. So, we can actually solve this problem. But the thing is what happens if you have a, you know, variable coefficient. So, P naught, P 1, P 2 are like smooth C 1 functions.

Now, in this case, you see by, so this is A, this is B. If P naught P1 and P2 are smooth. Then Picard gives us Picard theorem. So how can you apply Picard here? Picard theorem.

you remember you can actually reduce it to a system right and that system and this equation are basically equivalent for that system even initial data you know that there is a existence and there is a uniqueness so basically this equation has a unique solution okay so then picker theorem guarantees guarantees unique existence right so basically there is we know that there is a solution provided the coefficients are smooth enough okay now the thing is this what we want to do is this is it possible to explicitly solve this is finding a i mean giving an existence theorem is fine but the thing is can you solve this problem yes and as you have seen the variable coefficient equations can be very very tricky okay even for simple you know coefficients so basically for simple functions also if you are just assuming then also it becomes very difficult okay but there are some instances where this is very easy to solve how to do that so here comes the idea of exactly so what we are trying to do here is see the thing is let's call this as one the equation equation one is is exact is exact if P0 of x, y double prime plus P1 of x, y prime plus P2 of x, y. This expression can be written as P of x, y prime plus Q of x, y, the prime of that. Okay, go ahead.

where P and Q are continuous, are differentiable functions, are continuously differentiable, so C1 functions. Why? You see, I want this expression, I want to write it like this. You understand? And so, of course, I am taking the derivative of that function.

So, basically, I need p and q to be at least c1, right? So, you see, if this is the case, then we say the function is exact. So, basically, the thing is, if you can find a p and a q, c1 function, such that you can write this expression like this, then we call it an exact equation here. Now, you see the necessary condition and also the necessary condition is this. Where can you write it, necessary condition?

Condition. So, when can we write it? You see, if this is the case, then if you break it up, it becomes P prime y double prime plus P, sorry, P y double prime plus P prime y prime plus q prime y prime plus again what am I doing q prime y plus q y prime this is ok. So, that will be equals to p naught y double prime I am not writing x all the time it is as you write.

So, we are just avoiding it. Now you see if you compute this thing essentially what is happening is this. Now hence one have that see this equation if you look at it properly it is what p minus p naught y double prime plus p prime, right, p prime, this equation, minus p1, y prime, okay, plus q, sorry, plus q is there, plus q, y prime, okay, and then plus q prime, minus p2 y equals to 0.

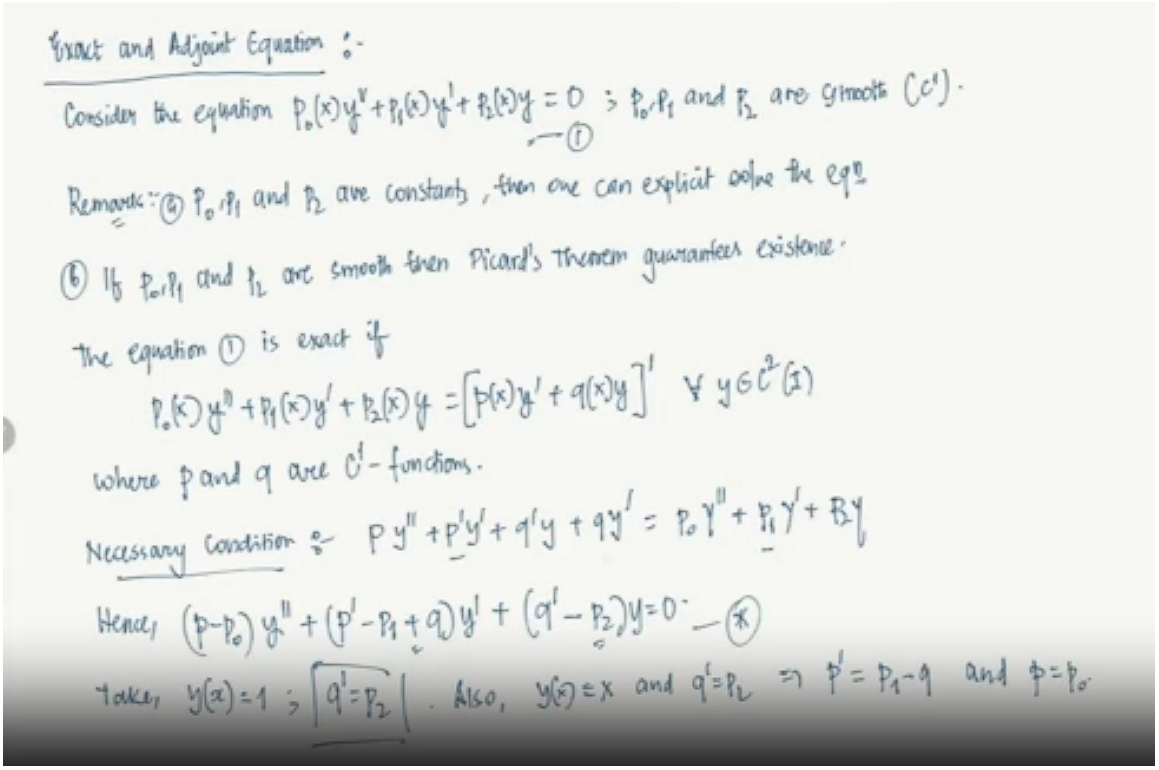
This is okay. Now you see, I want, I want my this expression to be equals to this. Yes. And this should hold for all y. It is not for a fixed solution y, right? This is, this is, this can hold, this should hold for all y. Is this okay?

I hope this is clear. See, we are not trying to solve the problem. Essentially, what we are doing is we are trying to say that this expression is equal to this expression. And for any y, essentially, it does not really matter what y is. So, basically, let us say at least such a y such that y double prime exists and it is continuous, right?

So, basically, we are looking for a y for this should hold for all y in C2. Is this okay? C2 of i. whatever the interval we are looking for in that interval, yes. So, you see now take take y of x to be let us say x, then y prime is equals to 1 and y double prime maybe I can take y to be 1, let us take y to be 1, let us take y to be 1 first of all, let us do that.

why do we want so if you do that then you see y double prime is 0 y prime is 0 so basically q prime has to be p 2 okay so you get this expression q prime equals to p 2 that will imply what ah so you have you know what is q so q is such that q prime equals to p 2 okay now also Also, if you take, let us say, yx to be x, okay, and q prime equals to p2, this is already given, right? So, I will see, I mean, the choice of p and q is on us. So, basically, if we choose q prime equals to p2 and yx equals to x, then also this expression, the star should hold So, then what happens is this, if y is x, y prime is basically 1.

right? And y double prime is basically 0. So, this part is gone. So, in that case, p prime will be p1 minus q. And similarly, you can also show that p has to be equals to p naught, okay? Choose a, I mean, nice enough, okay, such that y double prime is equals to 1, this is 0, this is 0, okay?



You can, of course, do that and then you can show that p equals to p naught. Is this okay? Once you do that, then what happens is you are going to get, I did some mistake somewhere. Yes, one second.

y prime, p prime minus p 1. So, p 1 is, p 1 is p prime plus q, no? p 1 is p prime plus q. Yeah, it is okay. Yeah. Okay.

So, now, you see, therefore, therefore, these equations imply, this implies that This implies, what does it imply? You see, P prime is P1 minus Q, P equals to P naught, right? So, P naught prime is P1 minus Q. That is what you are going to get, right? And then Q prime is P2.

So, basically if you put all of this together, see what do we have? Let us write it here. This implies, let us write it here. Otherwise, we will get confused. So what do we have?

P0 is P. P1 is P prime plus Q. And we have another one, right? P2 is Q prime. P2 is Q prime. So if you put it together, therefore, if you put all of this together, it becomes P0 double prime of x minus P1 prime of x plus P2 of x is 0.

This is okay? So, therefore, the differential equation 1 is exact if and only if, right? P naught double prime of x minus P1 prime of x plus P2 of x is 0. Is this okay? Yes?

Okay. So, this is the expression, right? Now, the thing is this, see. if this is exact what is so special about it let's say if you can show something is an exact what is so special about it so let's look at this expression x double prime y x square y double prime plus x y pi minus y equals to x to the power 4 and x is positive I want to solve this problem, yes.

Now, by the looks of it, of course, this is in Cauchy Euler form, right. So, it can be solved. It is not a very answer, it is not like an answer problem, it can be done, but it is going to be very messy. I will show you a very easy way of doing it. So, what we are going to do is this.

See, let us say here, here P naught of x is x square, p1 of x is x, p2 of x is minus 1. So, if you take p0 double prime minus p1 prime plus p2, if you take that, that is going to be 0. That will imply that let us call it 2, 2 is exact. Okay.

Now the thing is it is extract. So what? What do we get? See what you can do is essentially I mean you can have a P1 and P2. See P you can have a P and Q such that this you can write it like this right.

You can have a P and Q. And what is P and Q? P is P naught. and what is q q is q prime equals to p2 essentially okay so therefore we can write it as p is q naught right so uh sorry you see p is p naught okay so i can write it as p y prime plus q y whole prime so p naught p naught is here x square y prime okay and then i can write this plus q y whole prime okay what is q q is nothing but q prime is p2 what is p2 here p2 is minus 1 okay so p2 prime is minus x okay so x y this is what you are going to get right the derivative of this yeah this is equal to x we will get it like this is this okay you understood what i am trying to say see here if it is exact you can write it like this so basically you see here p naught is p And what is q?

p2 is q, q prime, right? What is p2? p2 is minus 1. So, q prime is minus 1. So, q has to be minus x. So, it is this.

So, this is fine. Now, you see, if this is the case, then I can write it as, we can write it as x double prime y prime minus xy equals to x power 5 by 5 plus some constant c. Very simple, okay. And then we can of course solve this. This is just a first order equation, right.

It is very easy to solve. So, please solve this and see that this is the solution which you are going to. So, this is the first order equation. first order differential equation yeah you can use integrating factor here very easy if you are integral x so basically ah you multiply it and after that you solve it and this is the solution which you are going to get x to the power 4 by 15 plus c 1 by x plus c to x. Is this okay?

This is the solution. So, you understood how the problems are solved, why exactness is so important.

$P_0 = P$; $P_1 = P' + q$ and $P_2 = q'$

$\therefore P_0''(x) - P_1'(x) + P_2(x) = 0$

$\therefore$ The DE (1) is exact iff $P_0''(x) - P_1'(x) + P_2(x) = 0$

Ex:- $x^2 y'' + x y' - y = x^4, \ x > 0$ — (II)

Here, $P_0(x) = x^2$; $P_1(x) = x$; $P_2(x) = -1$.

$P_0'' - P_1' + P_2 = 0 \Rightarrow$ (II) is exact.

$\therefore (x^2 y' - x y)' = x^4$ ($P_0 = P$ and $P_2 = q'$)

$\Rightarrow x^2 y' - x y = \frac{x^5}{5} + C$ — 1st order DE

$\Rightarrow y(x) = \frac{x^4}{15} + \frac{C_1}{x} + C_2 x$

Now, let us look at an expression. What happens if the equation is not exact? So, let if the, if 1, if the equation 1, 1 is not exact. Not exact what happens okay so can you guess anything what is happening what we will do is see then then we see an integrating factor if integrating factor okay so what is the integrating factor it's a positive uh function which you multiply with your original equation and that will actually give you a exact equation okay integrating factor let's say z of x That makes one exact. Makes one exact. So, you understood the idea. The idea is this.

See, one is not exact. But can we make it exact? So, basically what we are doing is we are assuming. We do not know whether such thing happens or not. What we are doing is we are assuming that let us say there is a Z. Such that when you multiply the equation with Z, what happens?

Then the equation will become this, right? P naught of Z Y double prime plus P1 of Z y prime plus p2 of z y equals to 0. Now, let us call this thing as 3. Now, 3 is exact if when?

What is the expression? See, this is, let us say this is nothing but q naught. This is q1, this is q2. So, when is it exact? If p naught z double prime, q naught double prime, you see, this is expression.

minus p1 of z prime plus p2 of z this is going to be 0 yeah that's your expression okay this will actually give you this will actually give you so i i mean i will write it i will write this equation as p naught q naught right so i will write it as q naught sorry q naught of x z double prime plus q1 of x z prime plus q2 of x z equals to 0 okay where where what is q naught of x q naught of x is nothing but this expression p naught of x q1 of x is nothing but 2 p naught of x prime minus p1 of x and q2 of x. So, nothing but p0 double prime of x minus p1 prime of x plus P2 of x. Is this okay? Yeah.

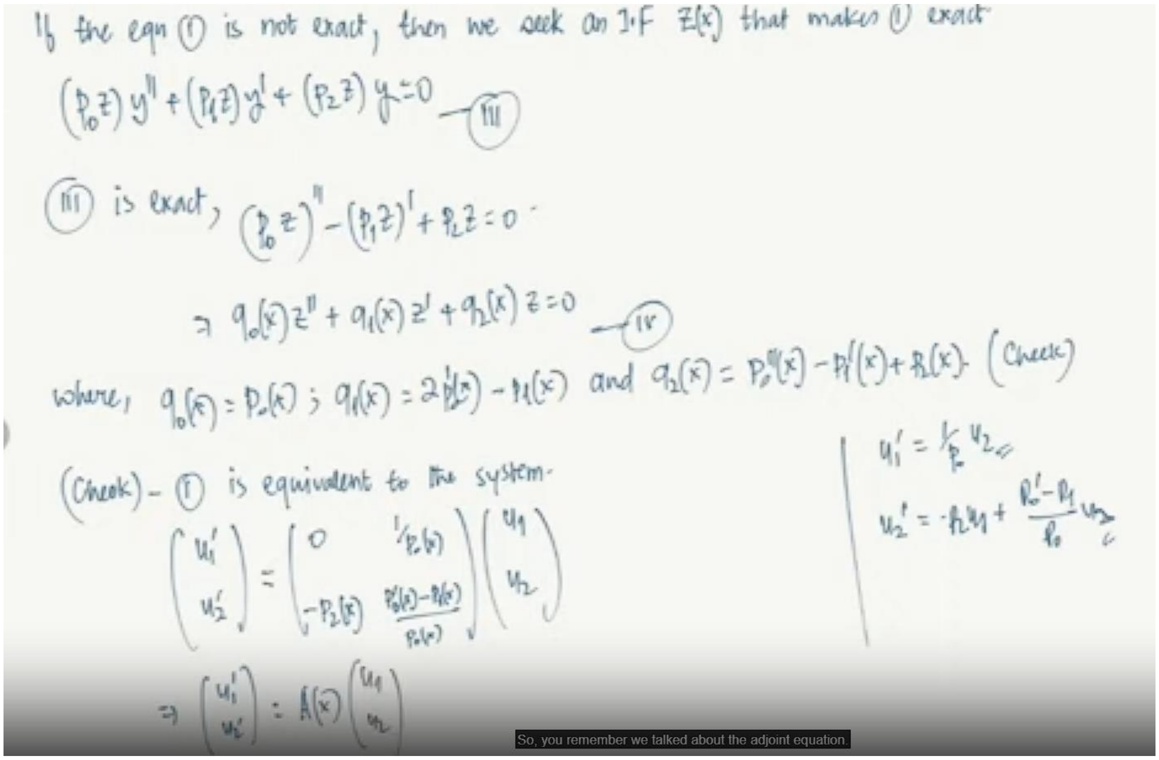
You can just solve it. So, please check this part. So, you can solve, just solve this problem and just write it like this. Yeah. Okay.

Now, you see this expression, this expression, let us just call it 4. Okay. So, if you think about it, see that if you write this properly, so this also you please check this part. Check this. 4 is equivalent to this system.

Equivalent to the system. What is the system? Let us just call u 1 prime, u 2 prime, which is nothing but 0. Sorry, I should… no i should i don't know which one i wrote here ah okay this one okay so you see this one if you write it in terms of a system it will look like this no u1 prime u2 prime is 0 1 by p naught of x u1 u2 okay i'm just writing you see this is not 4 this is not 4 this is 1

1 is equivalent to this system and then minus p2 of x and p0 prime of x minus p1 of x by p0 of x. Please check this part. How do you check it? You see, first of all, what is from here? What is u1 prime?

It is nothing but 0 times u1, 1 by p0 times u2. And the other part is u2 prime is minus p2 times u1 plus p0 prime minus p1 by p0 times u2. right. So, you put this expression here and after that calculation you are going to get it is very easy yeah. So, please check this part this is for you to check also. So, if this is the case so you see you can write it as let us say u 1 prime u 2 prime equals to some function a x times u 1 u 2.



So, you remember we talked about the adjoint equation. What is the adjoint equation? This is the original equation. So, adjoint will be, let us say this is x equals to a t times x. If this is the equation, the adjoint will be minus a transpose, right?

So, you see, if this is the equation, this also I want you to check. Then, its adjoint equation is Advanced system is V1 prime, V2 prime is 0, P2 of x. Okay, minus 1 by P0 of x minus A transpose, right? So, exactly the same thing minus A transpose that is P0 prime of x minus P1 of x by P0 of x.

and then v1 and v2. Is this okay? This is what we are going to get. Now, if you write this, you see this is basically v1 prime is nothing but p2 of x v2 and v2 prime is nothing but minus 1 by p0 of x v1 minus p0 prime of x minus p1 of x v2. by P0 of x V2 is what you are getting.

Now, if you put everything together, okay, so this expression, if you put it here, therefore what you get? You are going to get P0 of x, okay, V2 double prime, V2 double prime, plus p0 prime of x v2 prime equals to minus p2 of x v2 plus minus, sorry, p0 double prime of x minus p1 prime of x v2, sorry about it, v2, okay. So, this is little you know calculation, but we have to go through it. So, please bear with me here.

Minus p 1 of x v 2 prime. I hope this is the calculation is correct. So, please check this part. If you are not convinced, please check the calculation. Yes, I hope this is clear.

So, if you put it together, what happens? This is nothing but p naught of x v 2 double prime plus p 2 P0 prime of x minus P1 of x V2 prime plus P0 double prime of x minus P1 prime of x plus P2 of x V2 is 0. Yes. Now you see this equation, this equation, this equation and this equation are basically same.

You see, this and this are basically same equations, right? So what is happening is this, that when you multiply your z and basically that equation becomes this thing, exact equation, that equation is nothing but the adjoint of the original system. You understand the original equation. So if you are given an equation, so basically what it means is this, if you are given an equation, right? And then you multiply it with an indicating factor.

And if this equation is, after multiplying it, if the resulting equation is exact, then that equation is nothing but the adjoint of the original equation. Now let us look at a definition. We will put a definition here. You see, the thing is, 1 is said to be said to be Okay.

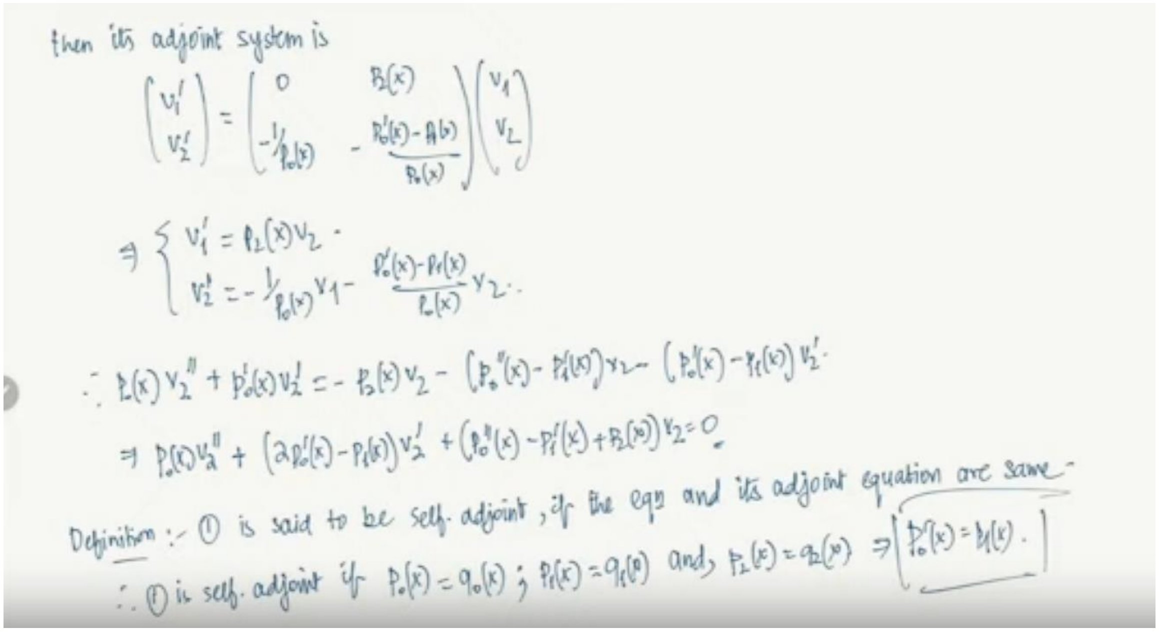
Self-adjoint. Self-adjoint. So basically when it is the same as the equation is same as its adjoint. Okay. So if the equation and its adjoint equation are same.

I hope this is clear. You see, it may happen that the equation of the adjoints are basically same, right? In that case, we call it as self-adjoint equation. What is the relation? Therefore, 1 is self-adjoint, self-adjoint if, what is the relation here?

If P naught of x equals to Q naught of x. And the other expressions p1 of x has to be equals to q1 of x and p2 of x equals to q2 of x. Is this okay? See, adjoint equation, this is... This is the adjoint equation, right? The original equation is this.

So basically it is P naught, it is Q naught, P1 is Q1 and P2 is Q2. That has to be the case. Okay. See, P naught and Q naught are already same. If this is equals to, if Q1 is equals to P1, then basically that will say that P1 equals to P1, then it says that P naught prime has to be P1, right?

That's what it is saying. So therefore, which implies that P naught prime of x has to be equals to P1 of x. Okay. So, that is the relation which we have to satisfy for this to become a self-adjoint equation. Okay.



So, let us put it. Therefore, 1 is self-adjoint. Self-adjoint. If. Okay. You see, it is 1, right? P1 of x y prime plus P2 of x y equals to 0. Now, you see, if it is self-adjuvant, P0 prime is P1. So, it is, you can write it as this, P0 of x y prime whole prime plus p2 of x y equals to 0.

This is a self-adjoint form, okay. Please remember that it is extremely important. You must remember this thing. We are going to work with it a lot. This is a self-adjoint form of 1.

Form of 1. Is this clear? Okay. So, this is the form, okay. So, you see, hence what did we find?

Let us just understand that. So, you see, we multiplied, let us say, sigma x with 1. We multiplied sigma x with 1. And if it is self-adjoint, it is self-adjoint. What is the expression?

See, 1 is given, right? Yes. And now I want 1 to be self-adjoint. Okay. What is the expression if P0 prime is P1?

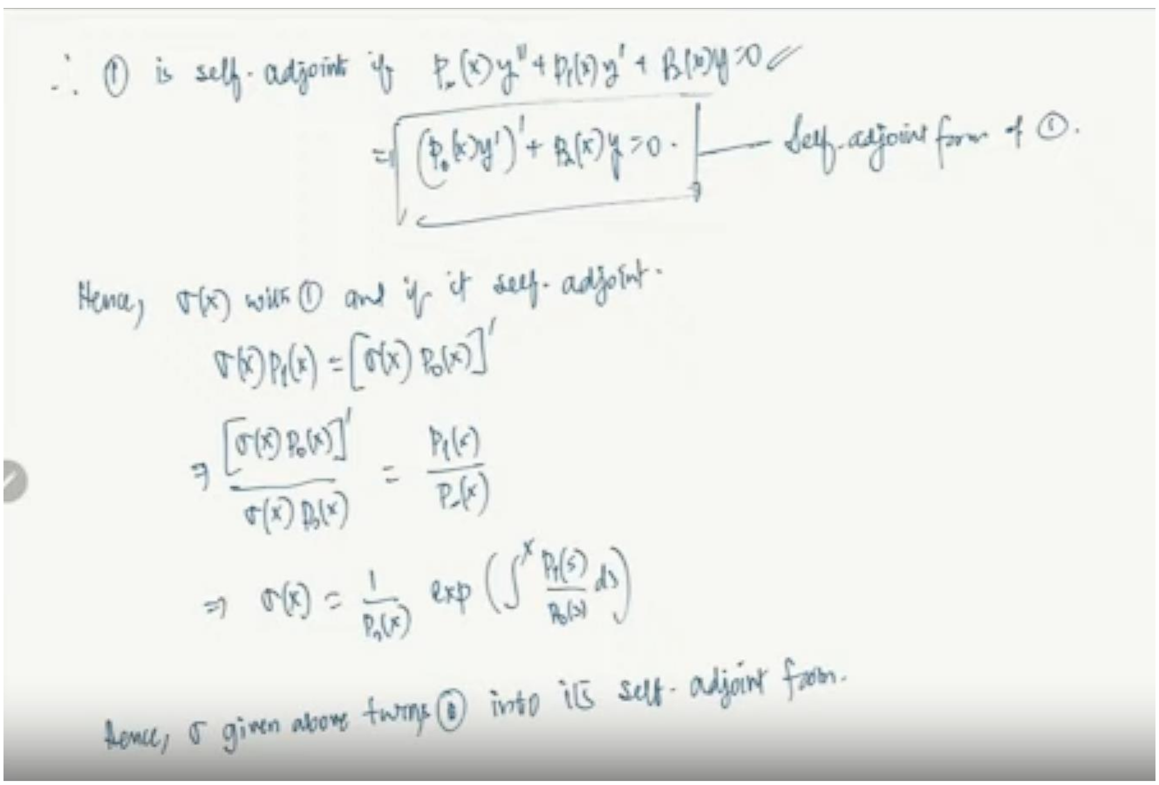
Okay. So 1 is given by this equation. I multiply it by sigma x. And then if it is self-adjoint, what happens? That P0 prime is P1. So in that case, it will look like this.

that sigma x P1 of x should be P0 prime, right? So, in this case, P0 is sigma x P0 x. The derivative of that is equals to sigma x P1. This is same as nothing but sigma x P0 of x the derivative of that by sigma x P naught of x. Let us do it this way. So, then that will give you P1 of x by P naught of x, right.

So, that will give us, that will give us sigma x is nothing but, you see if you indicate both sides, what is going to happen? It will be 1 by P naught of x exponential integral over x p1 of s p0 of s ds. This is what you are going to get, right? So, what happens is, you see, if this is, this is the integrating factor, which will actually, when you multiply it with your equation, will actually make one exact.

So, hence, sigma given above Trans sign equation. Trans 1 into its self-adjoint form. So I hope you understood what I am trying to say. See what I am trying to say is this.

if you are given one and if you want to put it in a self-adjoint form what you do is you multiply it by sigma what is sigma sigma is essentially this okay so if you multiply it with this equation it actually becomes its self-adjoint form okay so that is how you transform an equation in the self-adjoint form okay now the next part is very very very important okay so let me put it very very very important okay yeah please understand what is happening because we are going to use it to solve many a problem okay fine so you see the thing is ah one second what did i put it Okay, let me write it. See, sigma naught of x, let us say sigma, sorry, not sigma naught. Sigma of x is such that it makes itself adjoint, right?



So if you take that sigma, so sigma is an integrating factor which makes the equation self-adjoint. Times p naught of x, y double prime, plus sigma 1 of x, sorry, what am I writing? p1 of x, y prime, Plus sigma of x b naught of x y equals to 0. This equation is self-adjoint.

It is self-adjoint. Let me write it this way. Self-adjoint. Yeah. Okay.

So I am going to deduce something. Yeah. And I am using a little facts here. So please understand this. See this equation.

So let us call this as star. Star is self adjoined. Yes. And its solutions. Solutions.

Okay. Are also its integrating factors. Integrating. factors, right? This is quite evident.

You see, if you solve, this is a self-adjoint equation, right? So, if you solve this equation, the solution, that will also be an integrating factor, right? So, but since star is what? Star is essentially 1 multiplied by sigma x, okay? So,

But since star is 1 multiplied by sigma x, what does that imply? That will imply, hence that will imply that the solutions of star, of star, okay, is same. The solution is very negative, that is fine. And star is nothing but sigma x times 1, right?

That is what it is. Yeah. So, you see, okay. So, hence Sigma x times the solution of times the solution of 6.1 sorry 1 sigma x times the solution of 1 of 1.

Okay. Are the solutions are the solutions of the of star are solutions of star. Okay. Which are also the integrating factors. Because solutions are integrating factors.

So sigma x times the solution of 1 are the solutions of star. But since the integrating factor of 1 are solutions of the adjoint equation. Of that joint equation. Equation. Okay.

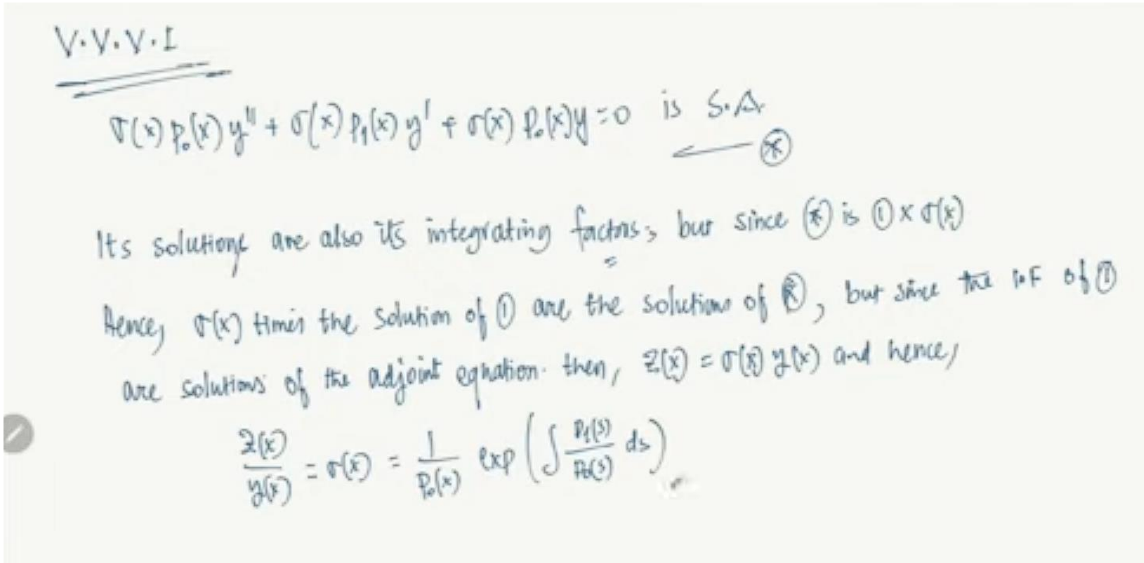
Then. What do you have. Is. C. We will write. How do I put it.

Maybe. Maybe. One second. Okay. So, you see, let, what is that equation called?

I do not know. This is a so many equation, it is confusing. Ah, okay, 4. Okay. So, you see, this equation is nothing but the equation multiplied by the integrative factor, right?

So, and the solution is given by z. Okay. So, what happens is then z of x can be written as z of x can be written as sigma x times what is it y of x okay okay fine yeah and hence hence zx by y of x is nothing but sigma x which is again given by 1 by P0 of x exponential integral P1 of x by P0 of x ds, okay. Is this okay?

Sigma x is given by this, right? So, I am just writing it this way, yeah, okay, fine. So this is basically the thing which we need. So essentially this is the expression which we need.



Now the thing is this. I just wanted to deduce this part that you know Zx can be written Zx can be written as sigma x times yx, where sigma x is even. That is the important part. Now, why it is important? Let me just do this part and then we are going to finish this video.

So, with an example. Why this is important? You know, this makes our life very easy. Why? Because let us look at this problem.

x double prime, xy double prime plus 2y prime plus a square xy equals to 0. Let us say a is x is positive here. So this is not exact. Why it is not exact?

It is not exact, let us say. The equation is not exact. Why it is not exact? Because if you look at p0 double prime minus p1 prime plus p2, this is 0, this is 0, so ultimately this is just an x square x. which is definitely positive, which is definitely positive, okay.

And what about its adjoint? Its adjoint equation, adjoint equation is, what is the adjoint equation? It is given by x z double prime, okay, minus 2 z prime plus s square x, z. That is the adjoint equation. See, this is the adjoint equation.

Where is it? I wrote it somewhere, no? Where is the adjoint equation? Ah, this one, the 4. 4 is the adjoint equation, okay?

So you just write that one. So given an equation like this, first of all, write down the adjoint. So that is, what is the adjoint equation? It will be given by x, z double prime plus a square z equals to 0. Here x is positive.

So that will imply that z double prime plus a square z equals to 0. Yes, this is the adjoint equation, right? Now the general solution of this adjoint equation, what is the general solution? That is given by c1 times cosine ax plus c2 times sine x. That is your z of x, right? Okay.

And what is sigma x? Sorry. Sigma x. So what is sigma x? It is 1 by p naught, right, which is x. And then exponential. integral p1 2 by p0 t, tt.

So, if you calculate it, this is log, right? Log square, so basically it is x. x square yx which is x, okay? So, hence what is the solution of our original problem y of x? It is nothing but zx times sigma x, right? Sorry, zx by sigma x. zx by sigma x. I hope this is correct.

It is c1 times cosine x by x plus c2 times sine x by x. Is this okay? This is how you solve the problem.

Example: $xy'' + 2y' + a^2xy = 0$; $x > 0$

the eqn is not exact, $p_0'' - p_1' + p_2 = a^2x \neq 0$.

Its adjoint equation is $(xz)'' - 2z' + a^2xz = x\left(z'' + a^2z\right) = 0$.

$$\Rightarrow z'' + a^2z = 0.$$

$\therefore$ G.S $\Rightarrow c_1\cos ax + c_2\sin ax = z(x)$

and, $\sigma(x) = \frac{1}{x}\exp\left(\int \frac{2}{t}\,dt\right) = x$

Hence, $y(x) = \frac{z(x)}{\sigma(x)} = \frac{c_1\cos ax}{x} + \frac{c_2\sin ax}{x}$.