Ordinary Differential Equations (noc 24 ma 78)

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Week-04

Lecture 26 : Asymptotic Behavior of Linear Systems III

welcome students in this video we are going to talk about asymptotic behavior solution to the linear system this is the third part eventually okay so uh we talked about what happens when we have a cost and coefficient equation right so in this part we are consider this problem consider the linear and the homogeneous problem the problem x prime equals to a t times x. It is looking simple equation here. But here this a is variable right depends on t where a is given by a i j let us just call it a i j okay n cross n and has continuous entries. Continuous entry is clear. Continuous entries.

Right. So now let's look at some what is the behavior of the solution. So now the question is this question. What is the behavior? It's the behavior.

Of the solution. solutions of 1 1 as t tends to infinity that is the question yes very very simple question yes now here what we will do is this we see that if we are to study this sort of thing we need to involve the i n values okay of some particular matrix so what we are going to do is see first of all let's look at this theorem we start with this theorem Theorem 1. Yes. So, let the matrix A of t is continuous on 0, infinity.

This you have always assumed, but let us just write it in 0, infinity. And mx here and m of x be the largest eigenvalue. be the largest eigenvalue, okay, eigenvalue of the matrix, of the matrix A transpose T plus a of t so basically i am looking at the matrix a t plus a transpose t and i we are assuming that m of sorry this is m of t m of t this is the largest eigenvalue of that matrix clear of course you see a depends on t right so the eigenvalues will also depend on t we are assuming mt is the largest eigenvalue clear then Every solution of 1 tends to 0 as t tends to infinity.

Clear? So basically, every solution, what I am saying is this, they are not only bounded, they tend to 0. Look, as t tends to infinity, provided, what is the relation? Provided, 0 to infinity.

This does not have to be 0. It can be from any x naught or t naught. It does not matter. So, basically what it is saying is the tail part, this should be minus infinity. Okay.

The integrability. Okay. Now, let us look at it. What is essentially what we are trying to do here. So, proof.

So, let x of t solves 1, right? Solves 1, clear? Then, x of t squared, how can you write it? We can write it as x transpose of t, x of t. Right?

See, let us say that for 2 cross 2, let us just do it for 2 cross 2. x of t is nothing but x1 of t, x2 of t, right? So, the square if you take, okay, so, and the modulus of square, that will be x1 square plus x2 square, okay? Now, again you can break, you see, x1, x2, right? And x1, x2. Okay.

If you just do that, then you can see that if you multiply it, it is going to be this thing. Okay. So, this is what we wrote. Sorry. This is square.

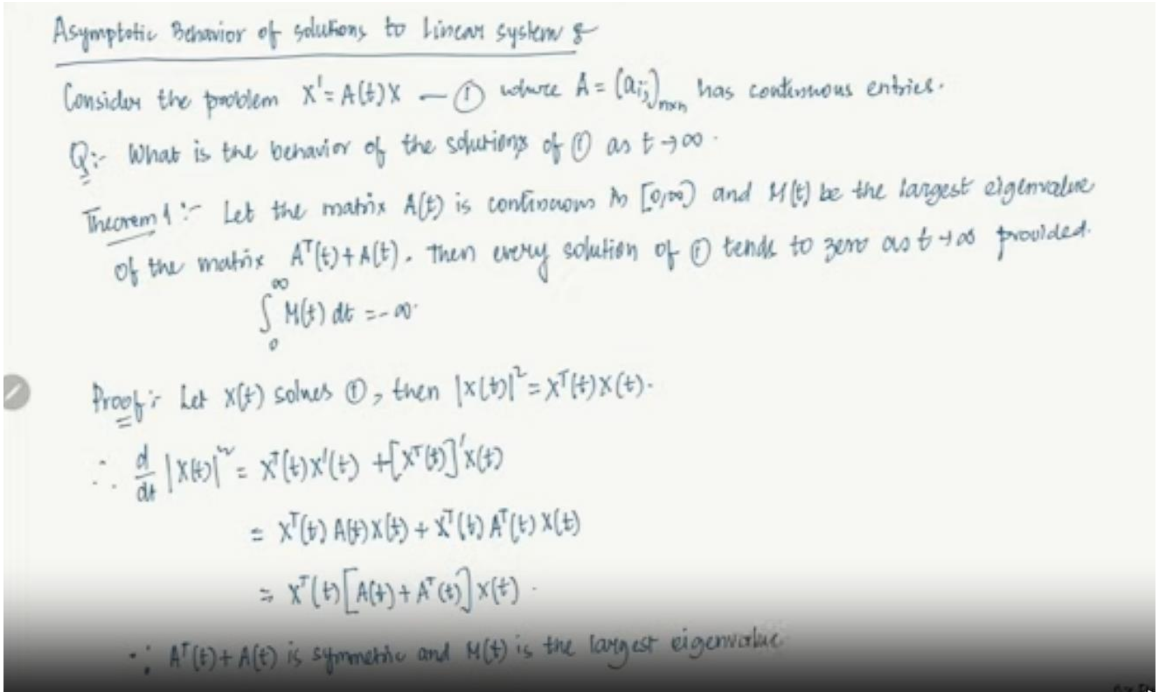
The root is not there. Root is not there because this square is part. It is there. Right. So, you can see that this is happening.

Okay. So, this is true. Sorry. Now, therefore, d dt of x3 squared, what happens to this? This is nothing but x transpose t, x prime of t plus...

x transpose t the prime of that times x of t is this okay and then that will imply that x transpose of t you see x prime of t is nothing but 80 times x right so it is 80 times x of t plus this part X transpose of T prime, right? So, this is X transpose of T, A transpose of T, right? And then X of T, okay? So, now this can be written as x transpose of t and if you take this part common, it is a t plus a transpose of t times x of t.

This is okay. Now you see, this matrix, one beautiful thing about AT plus A transpose T is it is symmetric. Since A transpose T plus A transpose, sorry, plus AT, AT is symmetric. And what is given? And m of t is given to be the largest eigenvalue, right?

Is the largest eigenvalue. Eigenvalue of this matrix, right? a plus a transpose. Okay. So, what can we say?



So, therefore, therefore, x transpose of t A transpose plus A t plus A transpose of t. This times x of t gets dominated by m of t times x of t square of that. This is true. Therefore, Since this is the eigenvalue that is going to happen, right?

So, for t greater than or equal to 0, what do we have? 0 is dominated by mod xt square, okay? Equals to is when x t equals to 0 and if x t is not 0, then it is greater than or equal to 0, greater than 0 strictly. And then this will be dominated by x at the point t naught square plus 0 to t m of t x of t. t square d t right i'm just replacing this thing see see x x t square okay x t square this is this part okay x t square okay i am just writing it like x t d dt of x t square is this right

Okay, so x t square, I will just write it as integral. If you take the integral on both sides, then x t square minus x t naught square is dominated by 0 to t m t x t square dt, right? I will just take the derivative on both sides, okay? d dt of this is there, then fundamental theorem of calculus I will use. Now, therefore,

Now you see this is in the granule form, right? Xt and this is, sorry, I should write it like, one second, let me change the variable at least. I should write it as ds. This is okay. Because the integral is from 0 to t, right?

Okay. So I have to change the variable a little bit. Sorry about the variable, I need to change it. So, image. I hope this is fine here.

Now, by Grunwald inequality, we have that norm of x t square gets dominated by norm of x 0 square. What is t naught less than theta 0? exponential 0 to T ms ds. Is this okay? Now, therefore, if exponential 0 to T ms ds

Okay. See, the thing is, if the tail part of this, 0 to infinity mt dt, if that part is basically minus infinity, right? So, this goes to 0 in that case. This goes to 0, yes? And then xt will also go to 0, right?

Because this is basically bounded. x at the point 0 is some fixed number. Okay. So, if this is going to 0, then xt goes to 0. So, basically, you do not have to do anything.

This the results the result concludes by the given condition yes Okay. So see here you do not have to be 0 to infinity. It can be large enough.

So basically you know from some t naught to infinity also for large n of t naught. Does not matter. You do not have to put any, you know, condition on this lower bound. So, basically what it is saying is after some time the integral should be, because m is continuous integral, okay. So, after some time it has to, the integral has to be minus integral, okay.

I hope this is clear. Very, very simple, I mean it is not a very difficult theorem to prove actually, okay. Now let us say that now a small remark let us put out a remark. If let us say if let me put it this way msds if we write it if this condition now this is important right this condition can get that xt goes to 0. Now if we are replacing this condition

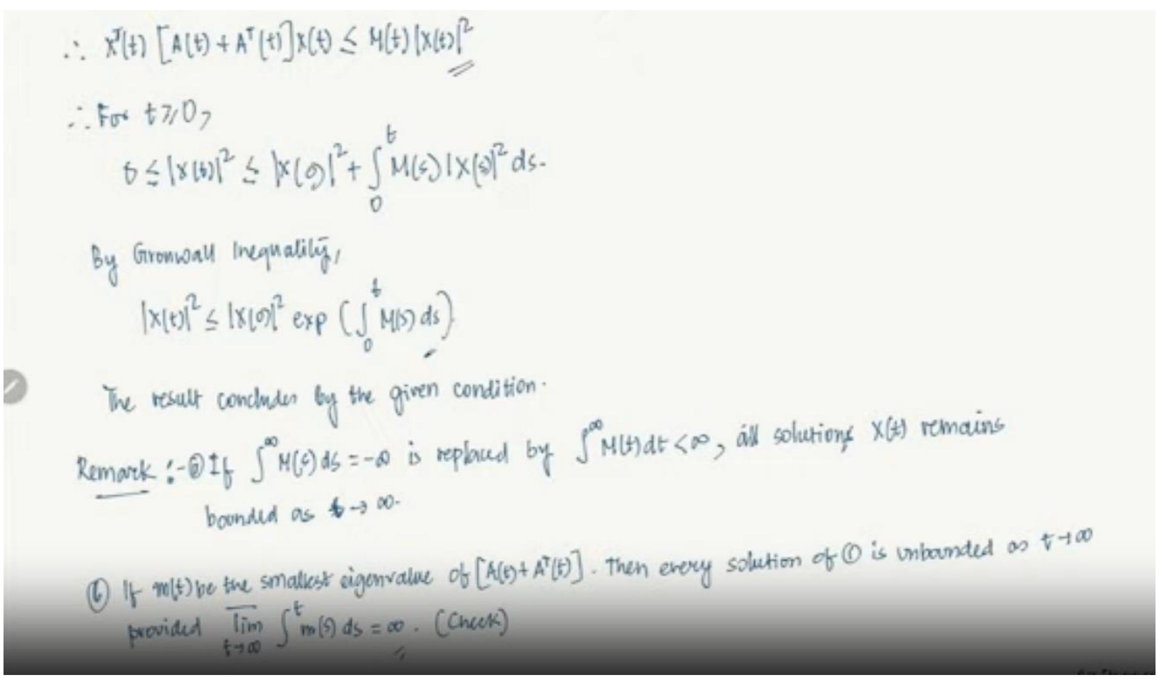
This condition, mt dt is finite. Yes, it does not have to be minus mt, but it is basically finite. If you have that, then what can you say? From exactly this thing you see, you can say that if this is finite, then exponential that part is going to be finite. So basically, all solutions x of t remains bounded.

Right? That is what is this thing. All solutions... x of t remains bounded as x tends to infinity, sorry t tends to infinity, as t tends to infinity. Is this okay?

See, if this is bounded, exponential of that function is going to be bounded and then x of t is going to be bounded. So, basically this is a very, very simple, you know, remark from this part. We can also say more. Yes, we can also say more that this is a. right now b this remark yes i am doing it i want you to do it yourself please do this part okay so the thing is let's say that if you see capital m of where is it uh yeah capital m of p is the largest eigenvalue right if you have small m of p be the smallest eigenvalue, let us say, be the smallest eigenvalue of A t plus A transpose of t. Yes, if this is the case. Then a beautiful thing happens, you know. Then every solution, every solution of 1, okay, is unbounded, is unbounded as t tends to infinity, clear, provided t tends to infinity. provided limb soup, okay, as t tends to infinity, that is, t, m of s ds, this should go to infinity. Is this okay?

See, how do you, so this part I want you to check it yourself. Check. So, this is the condition under which all solutions are unbounded. How do you prove something like this? Exactly the same proof which we did here.

You see, here, since capital M is the largest upper bound, we got this. So, in this case, you will get, here it will change, it will be greater than or equal to small m times x0, xd squared. And then you do exactly the same thing, same sort of analysis you have to do. And once you do it, you can actually show that if this limit supremum, what is it? limit supremum is infinity then we have also it's not a bound unbound okay you have to check it yourself but exactly the same sort of proof works so you do not have to worry about that particular point okay so let's look at some examples now example one let's say so let's say it is a matrix given by 1 by 1 plus t square and this is t square, right? And then minus t square and minus 1. Is this okay? Therefore, let us say this is A of t. A t plus A transpose of t, what is it? It is 1 plus 1, okay?



So, 2 by 1 plus t square, right? And then t square minus t square, so this is going to be 0. Again, t square minus t square, this is going to be 0. And this is minus 1, minus 1, minus 2, right? So, this is what we are going to get, clear?

And therefore, what is m of x? m of x, define m of x to be the maximum. You take all IMLs, you take the maximum of that, right? Okay, so... So for this, the eigenvalues are 2 by 1 plus t. See, this is like a diagonal matrix.

The eigenvalues are 2 by 1 plus t squared and minus 2. So what is the maximum here? It is 2 by 1 plus t squared. I hope this is clear. See, this is the diagonal matrix, right?

Diagonal values are basically the diagonal elements, okay? And between minus 2 and 2 by 1 plus t squared, maximum is 2 by 1 plus t squared. So, m of x is 2 by 1 plus t squared. That's just definition, right? And 0 to infinity, m of s d is what happens to that?

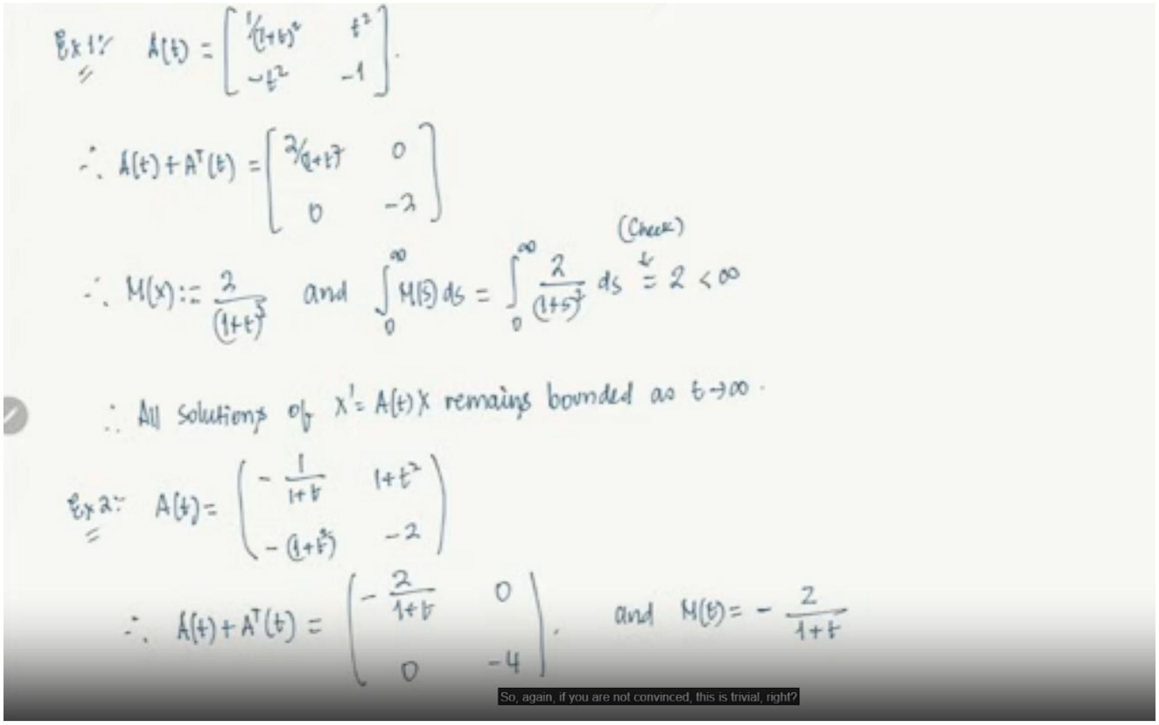
It is 0 to infinity, okay? 2 by 1 plus a square ds this is okay right now you can calculate this thing okay and please do this this part you have to check it yourself it's not of course this is very reasonable please do that so if you check that this is going to be 2 which is bounded Okay, so what can you say if you can, if you can see that this is going to, what can you say? You can say you use this first part of the remarker. If this integral up to infinity mt dt is finite, then all solutions remain bounded as t tends to infinity.

Is this clear? Okay, so therefore all solutions. or solutions of x prime equals to a t times x that remains bounded as t tends to infinity is this okay very simple okay so this is the first example let's do another example okay example two So, you see, let us say a t is minus 1 by 1 plus t. Okay. And this part is 1 plus t square.

The second part is minus 1 plus t. And the third part is minus 2. Is this okay? So, therefore, let us look at what is a of t plus a transpose of t. So, this is nothing but of course again a plus a transpose a i i plus a i i. So, it is 2 by 1 plus t. This and this. Let us write it as t square.

So, it gets cancelled out and becomes 0. So, we are just making our life easier basically. And again this term is going to be 0. And this is minus 2 minus 2 minus 1. So, now we have to calculate that what is m of x.

And what is m of x? Sorry, m of t. See, out of these two, it is very clear that minus 2 by 1 plus t is the 1 plus t. The maximum is given by this, right? m of t. Okay? So, again, if you are not convinced, this is trivial, right? Of course, you can see that.



Okay. So, since this is the case, therefore, what do you have? Therefore, 0 to infinity m of t dt, if you calculate that part, that will be nothing but 0 to infinity m of t dt. minus 2 by 1 plus t dt right and that is going to be minus infinity if you calculate it yeah therefore all solutions by the theorem all solutions of the differential system x prime equals to 80 times x goes to 0 as t tends to infinity, okay?

So, very simple. Yes, you just have to calculate what is the eigenvalue and then you are more or less done, okay? So, you do not have to worry about any other thing to be very frank with you, okay? So, now what we are going to do is basically we are going to look at a different kind of problem, okay? So, you see, up till now we have looked at this problem, x prime equals to

AX, okay, and then what happens? This is the purely homogeneous case. And then we looked at A plus BT, okay, a part of system X, yeah. Then we looked at this problem, X prime equals to AT times, sorry, AX, the homogeneous problem, right, AX plus B of T. Now, in this video, we talked about this problem, A t times x. Now, what is the most natural thing to do here?

Now, we will talk about, again, the part of system. So, x y equals to A t plus B t times x. And then what happens? Let us just do that part. Once we do it and then we will look at some other important theorems. So, first of all, we start with a theorem.

This is theorem 2. So, let all solutions solutions of the system, x prime equals to 80 times x, clear? Are bounded in 0, infinity. Okay.

So in the first part we have seen, right, that when are all solutions bounded? When integral mttt is finite, right? That's what we have seen. We just looked at this. Okay.

So this is the assumption. Okay. Now, now the boundary is infinity. And This is the condition which I am assuming, infinite norm of bt dt is finite.

This is what we are assuming. If this happens, then all solutions of This equation, x prime t equals to a t plus b t times x, the part of system, yes, okay, are bounded in zero infinity provided. provided limit infimum trace of a matrix A of t. This is a little different. And this is a very, very important theorem.

Please remember, trace of A t greater than minus infinity or if trace of A t equals to 0. Yes, very important. So, here what it is saying is this, if you have a part of system, you want to see whether all solutions of the part of systems are bounded or not. What you have to do is you just find out if all solutions of the homogeneous systems are bounded and integral up till infinity. So, basically up to norm of bt dt, you take up till infinity, that is finite.

So, before I do that, let me do a small remark. In case you do not understand see why i am not putting a lower bound here see let's say if i put 0 to infinity norm bt dt bt dt is finite okay this saying this and saying this is and if I am not putting anything, so basically it says that you can start from anywhere. So, let us say I can start from any t0 to infinity, norm of bt dt, bt dt is finite, okay.

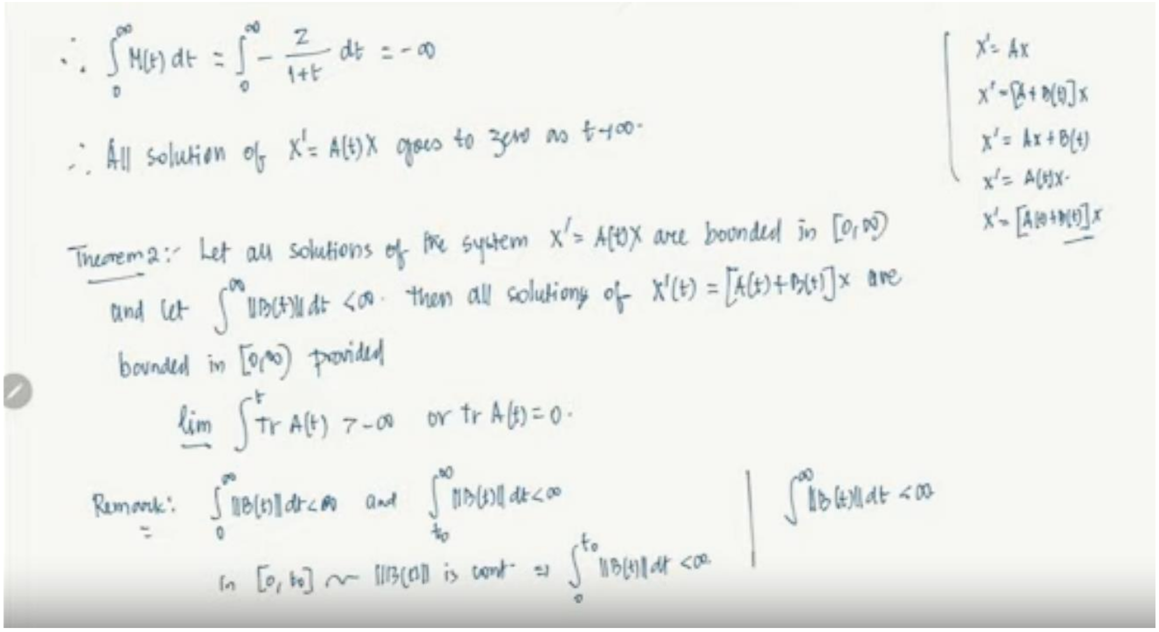
See, in 0 t0, okay, Norm of bt, this is continuous function, right? It is continuous. So basically what happens is that interval is always bounded, okay? So if you are taking the integral on a finite set, so basically that will imply the integral between 0 to t0, norm of bt dt, that is going to be always finite, okay?

So you do not have to worry about it. You just take from t0 to infinity. That is what matters. So hence, we write it like this. So you do not put any lower bound here.

You just put it like this. The d is finite. Is this clear? You just put it like this. You do not put any restriction.

Okay. So let us look at the proof of that. Proof. So, what is the theorem saying? The theorem saying that these are the two conditions.

If the trace of A t is 0 or if the trace of A t, the limit is going to minus infinity, then we are done.



How do you prove something like that? So, let us look at this. Let phi of t be the fundamental matrix. Of the system x prime equals to 80 times x. We start with this fundamental matrix.

Is this okay? Now you see all solutions. All solutions. Okay. Of x prime equals to 80 times x is bounded.

Bounded. Okay. that will imply that the fundamental matrix is bounded matrix is bounded okay okay how is it why is it true okay if you are not convinced here what i want you to do is please pause the video think about it why this is true okay so let me tell you why this is true see Maybe I will not put it here. Let me do it in the border boundary here.

See x prime equals to 80 times x. x0 equals to let us say ei. ei is the ith unit vector. Ith unit vector. That has a solution. What is the solution here?

The solution is given by xi of t less than. Okay, now what is x? x is nothing but x1 of p, xi of p, xn of p. Okay, sorry, not x but phi, fundamental matrix phi. Now all solutions of this thing, system is bounded, right? So basically, if you take this initial data, right?

So basically, if I am looking at a solution which passes through the ith unit vector at t equals to 0, that solution is also going to be bounded. So each component of this matrix, fundamental matrix phi, which is given by x1, x2, xn, is bounded. If each component is bounded, of course, phi has to be bounded, right? This is trivial, right? This is trivial.

So, please check this part. I gave you everything. You just have to do some calculation. The thing to do is basically every component is bounded. The matrix has to be bounded, okay?

So, please do that part. Okay. So, check it. Now, you see what do we have is this. You remember the Abbas theorem, right?

Yeah. So, by Abbas theorem, Remember what we did using Abel's theorem. We showed that if it is linearly independent at one point, it has to be linearly independent everywhere, right? The set of vectors, right?

We proved that part. So, here also we are going to use Abel's theorem, okay? By Abel's theorem, what you can say is this. If you take the fundamental matrix, the determinant of phi t, okay? It is nothing but determinant of phi at the point 0 times exponential phi.

0 to T grace of ASDS. Is this okay? This is by Abbas theorem, right? And hence, what is phi inverse of T? It is nothing but adjoint of phi T by determinant of phi T.

Yeah, adjoint of phi t by determinant of phi t, right? That is what phi inverse of t is. So, you see, so this will be adjoint of phi t by determinant of phi t is given by determinant of phi 0. Why can we write it like this? See, phi is a fundamental matrix.

Determinant of phi 0 is never going to be 0. Yeah, because phi is invertible, right? So, determinant of phi 0 is non-zero. So, an exponential is always going to be positive. So, essentially this is not a problem.

So, exponential 0 to t raise of a s d s. Fine. Now, So you see this this limf of case of 80 is either greater than minus infinity or it is 0. So if this is the case then exponential of this this particular thing has to be bounded here. This has to be bounded, right?

See, if this is 0, let us say. So it is basically adjoint of phi t by determinant of phi 0. And either or if it is minus infinity also, the trace of it in the integral of that is minus infinity also. So sorry, greater than minus infinity, sorry. Then also this is going to be bounded.

So therefore, one can say that norm of phi inverse of t is bounded. So, yeah. So, you remember. So, this is fine. What we proved is this.

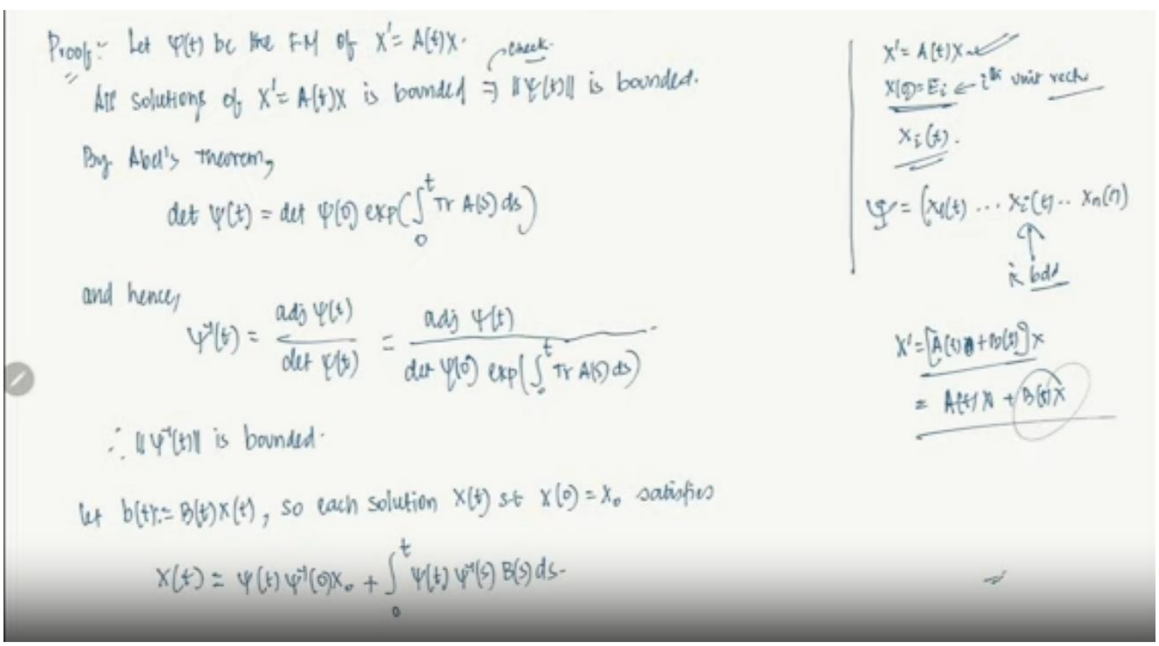
See, if all solutions of the system x1 equals to 80 times x is bounded. Yes. Then all solutions, there is a fundamental matrix is going to be bounded. Yes. Of course, it means there is a norm is bounded essentially.

And norm of P inverse is also bounded. Yes. Okay. Now, you see this equation. X prime equals to A T times X plus B of T.

Yeah, this equation, sorry, times x, right? This is the equation. So, you can write it like this, a t times x plus b t times x, right? This is what we can write it, okay? Now, this part, let us put it as b of, small b of t. You remember we did the exact same thing in that, when we did for the constant coefficient thing, we are also going to do this exact same thing, okay?

So, let, b of t is nothing but capital B of t times x of t. This is what we are defining. This is what we are defining. So, each solution x of t such that x at the point 0 is x naught satisfies this equation, right?

x of t equals to solution of the homogeneous problem, phi of t, phi inverse of 0, x0. Remember, plus Integral 0 to t, phi of t, phi inverse of s, b of s, ds. See, when it is a constant coefficient, small remark, if it is a constant coefficient, what is phi of t? It is nothing but e to the power 80, right?



Then we can write it as e to the power 80. That is why we are writing e to the power 80 there. For the constant coefficient. Here this is variable coefficient. We can't write it as e to the power 80. Right. Okay. So I am not doing that. Okay. I hope this is clear.

Now see. If. If one defines. If. C is nothing but the maximum. Clear. Of. So, supremum of t greater than equal to t naught, norm of c of t and supremum of t greater than equal, sorry, t greater than equal to 0, let us just put it. It does not have to be t naught. It can be t naught also, but 0 greater than equal to 0 to phi inverse of t, right?

So basically I am taking the supine of norm of phi and supine of norm of phi inverse and I am taking the maximum of those two. Let us just call that c. Why? Because if we do that then what do we have is norm of x of t. See here, I have this phi of t, phi of t, phi inverse of s. This part is here. So I can take the maximums outside.

You understand? This can be dominated by c. This can be dominated by c. This can be dominated by c. So I can take that out. That's what we are doing. So by triangle inequality, this is triangle inequality ti. You can actually write it as C naught plus C naught is basically this particular thing, the boundary I am just writing it to be C naught.

So C naught plus C square 0 to T norm of B of S. And norm of x of s ds. That is what we are going to get because small b is b times x, right? Sorry, this has to be, I made a small mistake here. This is not capital B, this is small b. Now, small b is capital B times capital X. So, that is why I just wrote it like this.

Is it okay? So, now, by ground work, What do we have? We have the norm of x of t is dominated by c0 times exponential c square 0 to t norm of bs. ds.

Is this okay? See, here I should write it c0 is c times norm of phi inverse of 0 times x0. This is what I wrote as c0. So, normal will give us this. Is it okay?

Now you see, we have assumed, since we have assumed, that you remember what is the assumption of B of S? It is, the integral of B of S is finite. See? So, if the integral is finite, therefore, x of t is bounded. This is finite, exponential of a finite bounded term, so basically bounded.

So, x of t is bounded. So, the next theorem, which we are going to do, will be in the same line, you know. So, essentially what we will do is this. So, let the fundamental matrix fundamental matrix phi of t of the system x prime equals to a t times x be such that

We search the norm of phi t, phi inverse of t is less than equal some constant c, okay. And this holds for 0 less than equal t, less than equal, sorry, should I change, maybe I should change it to t and s, t and s. Yeah, it is very easy to remember. I will tell you why because this part is nothing but that part, you know, it is part of the, you know, Duhamel's principle. So, that is why we remember this.

So, that's fine. Okay. So, please keep that in mind. Okay. And for now, okay.

So, what did I, this I was writing, right? So, here I have to change it to S less than equal to T less than infinity. Is this okay? This is the condition which I am taking.

Where C where C is a positive constant. Positive constant. Okay? Now, further let further let The condition.

Okay. The other condition. So infinity norm of V. Vs. Ds. Let us say that is finite. That condition which we used last class.

So here this part. Okay. Is satisfied. This is true. For the let this is fine.

Is satisfied. Is satisfied. Okay. Then all solutions. Then all solutions.

Of x prime equals to a t plus b t times x are bounded. Are bounded in 0 infinity. Yes, bounded in 0 infinity. And what is the condition provided? Provided.

No, what am I doing? I wrote the condition. Then all solutions are bounded. This is the condition on the fundamental matrix and of course the integrability of PS is required. Summability.

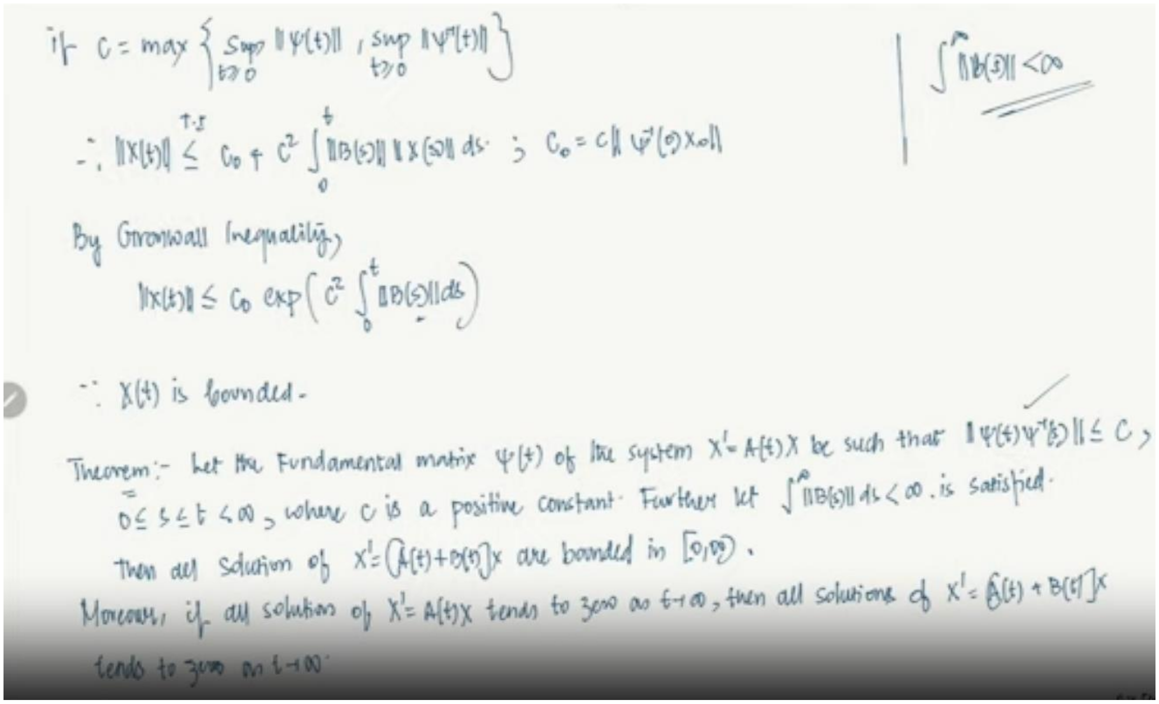
That is it. Moreover, you can say much more here. Moreover, if all solutions if all solutions of x prime equals to 80 times x, 80 times x tends to 0, tends to 0 as t tends to infinity. Clear?

Then all solutions, then all solutions. of x prime. This system, I have to write it all the time, a t times b t times x, the part of system, okay, tends to 0. So you do realize what I am trying to say. Exactly the same sort of theorems which we did in the other videos also.

So here it is saying that you just need that condition on the fundamental matrix that fundamental matrix Pt and P inverse. You take both of them together the norm has to be finite. And of course that the summability. So this is basically this is summability of B office. This is finite.

So, it is summable basically. That is required. Once that is there, then all solutions, so it will mimic the solution. You know, the perturbation is not a big problem. So, it will mimic the solution.

So, essentially, if the solutions of the homogeneous, sorry, yeah, the homogeneous problem, if there is, let us say, bounded, okay, then the uh the solutions of the part of systems are bounded if they go to zero then the solution goes to zero okay which is like a very very straightforward thing it just you know mimics that solution okay let's look at the proof the proof is not very difficult yes what's it once again Okay. So, what do we have? See, this path is already there, right?



That path is already there. I can do the exact same thing. Basically, the same system. So, exactly I can go on doing this. Because I did not use anything else, you see.

This condition, the trace condition, where is it? See. This condition. See. ah so basically what i'm trying to say is this where is that condition what did i write it ah okay oh what is it one second i wrote it in something from no where's the condition ah okay yeah yeah so since you see free t free inverse surface that is given to be bounded right

So, you can use this condition in this part. So, you see, x of t, we know we have x of t is nothing but phi of t, phi inverse of 0, x0 plus 0 to t, phi of t, phi inverse of 0, V of t. x of s, I think I am writing, yeah, it is fine, yeah, it is okay. So, this is all satisfied, right?

You remember, you see, this is the thing which we did earlier. Where is that thing? I wrote somewhere, oh, here, here, okay. So, this is the thing, right? The x will satisfy this equation, yes, okay.

So, you see, what I am going to do is this. Now, since In the earlier proof, what we did is we did it independently. You see, we proved that norm of psi inverse of t and norm of psi t is bounded. That's what we did here.

And then you take the maximum of those two and then you dominate it. That's what we did. Here we are doing it a little bit differently. See, all solutions of the homogeneous problems are bounded. The norm of phi of t is going to be bounded.

That's given. That has to be proved. This is exactly the same thing what we did earlier. And this is given to be bounded. Phi t and phi inverse of 0 is given to be bounded, right?

So, you see that. So, essentially, you see, let's say this bound is given by phi or whatever. So, you see, therefore, what do we have is this norm of x of t. is dominated by c times norm of x naught right so this is c phi t phi inverse of zero see phi t phi inverse of zero where is it phi t phi inverse of this this is for all s right s doesn't have to s can be 0 also you see okay so this is bounded right so i will take it to be uh norm of x naught plus uh c times 0 to t, the norm of this is called norm of Bs, x of s, Ds.

Now, you use your ground wall. So, therefore, ground wall inequality implies that norm of x of t is bounded by c times x naught exponential of c 0 to infinity norm of bs ds okay and which is basically bounded that is given right that is given yeah so it is very very trivial once you write this part in the other part what we did individually we showed that the fundamental matrix and the inverse of the fundamental matrix norms are bounded Here, it is given to be phi t times phi inverse of 0 is given to be bounded. So, we just use that bound.

Nothing doing here. So, it is very trivial. I mean, it is not very difficult actually. Now, again, the second part. What is the second part?

So, this part is also very easy. I am not going to do it in the whole, but I will just give you an idea how to do it. See, it is given this fundamental, sorry, the system, all the solutions are going to 0 as t tends to infinity. And we have to show that the part of system also exactly mimics the same sort of properties. So how do you do it?

So see, now, again, so this is part 2. again x of t yeah is phi of t phi inverse of 0 clear x 0 plus 0 to t phi of t phi inverse of 0 b of s I am making some mistake somewhere, no? One second, I am making some mistake somewhere. Formula, yeah, this is phi inverse of s. I should not write phi inverse of 0.

I am sorry about it. But anyway, it does not change anything, but the formula is correct. This is the formula. Here it is phi inverse of 0. Here it is phi inverse of s. And here also, sorry.

phi inverse of s, b of s, d of s. Now, this is fine. And x of s, d of s. Is it okay? And so, basically what we are going to do with this, I am going to do it from 0 to t 0 plus, I will just break it up, t 0 to infinity, phi of t, phi inverse of s, b of s, x of s ds is this okay this is we can write it now therefore if we take the norm on both sides it is x of t okay if we take norm on both sides that is given by norm of phi of t phi inverse of zero norm of x naught right that's what we're going to get plus norm of phi of t I can take that out, right?

And then 0 to t0 phi inverse of s, b of s, x of s, ds, okay? Plus c times, let us call this as m. This term, this term, let us just call that part, that term as m, as m. You understand? This c times, so basically c times norm x0 exponential, this particular thing, I will just call that as m. Is this okay? So, this is m times t0 to infinity norm of Vs ds.

I will just write that part like this. So, x of t is given like this. Now, let us say let epsilon greater than 0 be given. Small enough. Again, this part of the theorem, I am not doing it explicitly.

I am not doing every single line. What I want you to do is please check it yourself. This is very easy actually, but I am still doing it. But I will leave some gaps. I want you to fill those gaps up.

Now, see, the thing is, if you look at this particular b of s, right? What is the condition on b of s? I wrote somewhere, no? The integral of b of s is finite, I think. That is what I wrote.

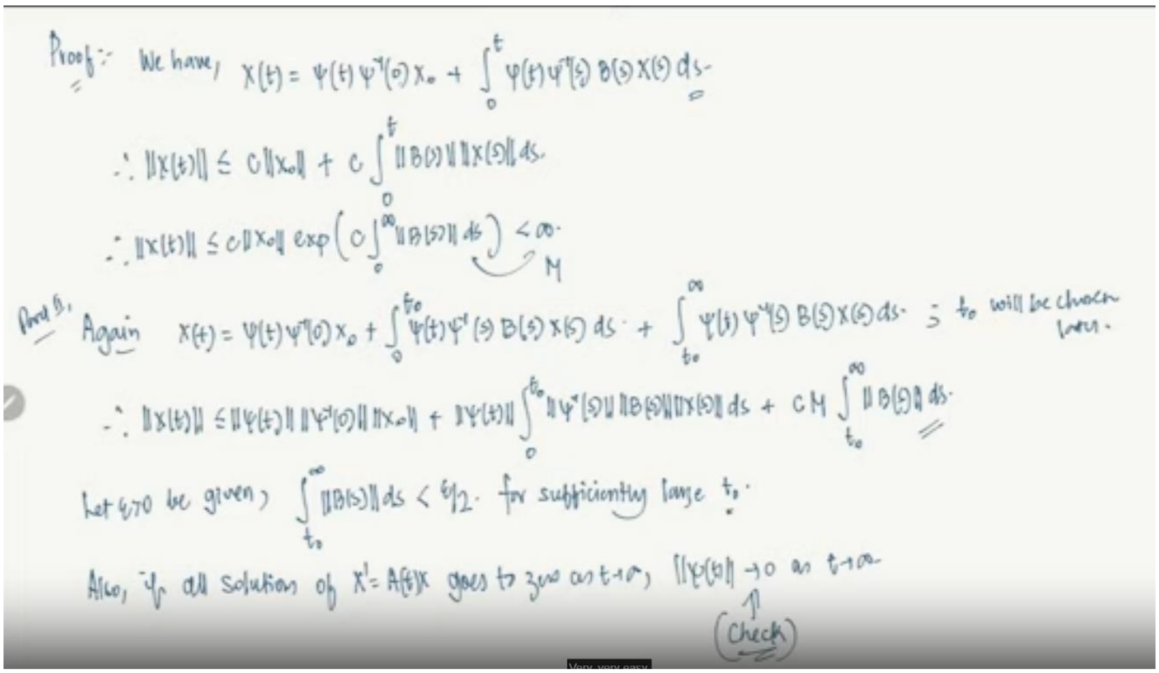
This one is finite, given, right? I hope this is okay, right. See, this part is there, no? Where is it? Yeah, yeah.

This is finite, integral. So, basically Bs is summable. Since Bs is summable, this particular thing, t0 to infinity, norm of Bs ds can be made less than or equal to epsilon, right? So, in that case, t0 to infinity, norm of Bs ds can be made less than or equal to epsilon by 2. for sufficiently large t dot, let us say, for sufficiently large t dot.

Large T naught. You see, this is why T naught here when I wrote, I did not write what T naught is. T naught will be chosen later. So, now we have chosen later. You see, now we chose T naught.

Sufficiently large such that this is that. Clear? Okay? Now, see that also if all solutions if all solutions of x prime equals to 80 times x goes to 0 as t tends to infinity then this is trivial right norm of p t goes to 0 as t tends to infinity

why you remember x1 x2 x3 what is pt consists of x1 x2 xs right and each x1 x2 x it has to go to 0 because all solutions are going to x1 x2 xs these are those are solutions so they have to go to 0 right so if they go to 0 then the norm you can show that the norm will also go to 0 okay so please check this part again Okay. Check this part. Not very difficult. Very, very easy.



But you can do it. Please do that part. Okay. So, if that goes to 0, therefore, see what you can say is the first two terms. Okay. This term. This term and this term. The sum of those two terms. Okay. Can be made arbitrarily small. Right. By choosing T large. Right. See norm of Pt goes to 0, right? So this goes to 0, this goes to 0. Yes? So what we can say is essentially and this is a bounded quantity. This is a bounded quantity. Okay? So what we can say is we can take this part and this part, the sum of that, I can make it less than epsilon by 2. That's what we are saying. So, I will just write it. The sum of first two parts, sum of first two, I am not going to write the whole expression. First two terms can be made less than epsilon by 2 for large n of t.

of t this is okay okay and so therefore what we have is this is less than this two part is less than epsilon by two this is given to be less than epsilon by two two if you add it up then this is going to be therefore norm of x of t is finite for large t large t yeah so hence Hence, that will imply that x of t goes to 0 as t tends to infinity. Done. Very, very simple. It is not very difficult theorem to prove.

So, we are going to end this video. But before we do that, I will just give you one theorem. I want you guys to do this theorem yourself. Very, very important, but extremely easy theorem. So, up till now we were talking about part of system.

What about the inhomogeneous system? Okay. So, the thing is this. Suppose, I will write down the theorems. You have to prove it yourself.

Very, very easy. Again, you can do it. Suppose, every solution of the differential system x prime equals to At times x is bounded in 0, infinity. Again, why 0, infinity?

Because we are looking for as t tends to infinity. So, we do not care about what happens at negative time. Then, every solution of the system I think you can guess what is going to happen. x prime equals to a t times x plus b t. Okay.

Are bounded. Can you think of any condition under which this will be bounded? Okay. Provided. And this is a very simple condition.

At least. Okay. One of its solution is bounded. Solution is bounded. okay i hope this is clear see i will just give you the proof i will not write the proof down but it is trivial see any solution of this inhomogeneous equation what is it it is the solution of the homogeneous problem and then the one solution of the inhomogeneous problem you know that at least one solution of the inhomogeneous problem is bounded right so that part is taken care of

What about the apart for the homogeneous problem? You know that all solutions of the homogeneous problem are bounded. So, the fundamental matrix is bounded and hence everything is bounded. Okay. So, this is trivial.

But anyways, write it down. And there is another theorem which you can say and this is theorem 1, theorem 2. Theorem 2. Again, this I will not prove extremely. It is trivial.

You guys can do it yourself. So, you can do it. So, let us say suppose every solution solution of x prime equals to 80 times x, 80 times x, okay, is bounded. Same sort of thing, in 0, infinity.

Same sort of theorem. You need a different condition. Okay. What is the condition you need? That I wrote the condition somewhere.

Where is it? The trace thing. What is the trace thing? I wrote that condition. One second.

The trace condition I wrote somewhere. This one. So let us just put that condition as star. This is the trace condition. And the star is satisfied.

And Star is satisfied. Satisfied. Okay. Then, all solutions of the homogeneous formula.

Solutions of x prime equals to a t times x plus b of t is bounded. You need an extra condition provided provided norm of BSE summable, yes. What we did up till now, if you have understood all the proofs, this is trivial, okay. So, again theorem 1 check, theorem 2 check.

So, check both the proofs, okay. So, with this I am going to end this video.