Ordinary Differential Equations (noc 24 ma 78)

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Week-04

Lecture 25: Asymptotic behavior of solution to linear system-2

So, welcome students. In this video, we are going to talk about asymptotic behavior and of solutions of a solutions of the linear system okay and this is the second part linear system so in the first part we have started out with the you know constant coefficient system and we looked at a part of system when can we say that there exists a solution which is all the solutions are bounded and or they tend to zero those are the conditions which you have seen in this video What we are going to see is basically we are looking at a inhomogeneous problem. So consider this problem.

Consider x prime equals to a t times x plus b of, sorry, not a t times x, a x plus b, a x plus b of t, right, where A is a n cross n constant matrix. Constant matrix. Clear? Constant matrix.

See, this, you have to understand, this system and the earlier system was different. The earlier system which we studied was this. The perturbation of the original system. See, the original system is this. Constant coefficient, homogeneous.

Now, the perturbed system is this. X prime equals to, what are you perturbing? You are perturbing the initial matrix, right? So, A is perturbed with a little bit And the perturbation is a function of t. Yeah, that was the part of system.

But this is an inhomogeneous system, different, right? So, it is x prime plus ax. And then we are doing, taking a small, you know, perturbation, which is like given by b of t. Is this okay? And here we will assume b, okay, is a n cross 1 matrix, of course, n cross 1 matrix. with continuous entries.

So, this is valued. Now, the thing is for this assumption if x at the point 0 let us say if x naught then we know then we know From Duhamel, from Duhamel, I hope you remember Duhamel, right? Duhamel's principle or the, you know, calculation of parameter, right?

What we know is any solution x of t, okay, the, sorry, the unique solution which passes through this point, x of t, that will look like this.

It is the fundamental matrix times c. In this case, the fundamental matrix is e to the power a t times x naught, right? Plus this is the solution of the homogeneous equation. So, the solution of the homogeneous equation and then the particular solution of the inhomogeneous equation, which is given by e to the power a t. Integral 0 to t e to the power minus as b of s ds. That you do have a principle.

I wrote it in terms of fundamental matrix but in this case the fundamental matrix is e to the power 80. So basically it is just that. Now these if you want you can write it like this. e to the power 80 times x naught plus integral 0 to t e to the power a t minus s b of s ds. Now, let us look at something what happens.

If we take the norm on both sides, what you can say is that will be dominated by the norm of e to the power a t times x naught s plus it will be again plus 0 to t e to the power a t minus s b of s ds. We can write it like this right. Why we can write it? This is basically triangle inequality using triangle inequality.

equality okay now see this term again you remember in the first lecture when we talked about it norm of e to the power a t we wrote it as less than equal e to the power eta t right you remember for some eta for some eta we can just do that okay so what i am going to do is i am going to use that fact here okay so using this maybe i can write it here otherwise you make it confused huh So using the fact that there exists a eta positive such that it doesn't have to be positive it can be negative also but anyway so it's okay positive such that or maybe eta there exists eta such that Norm of e to the power at is dominated by e to the power eta t. Is this okay? This holds for all t, right? This holds for all t greater than or equal to 0.

Let us just put it this way. Then, we can use that inequality here. And therefore, norm of xt can be less than or equal. See, it is, if you remember, e to the power at times x0. can be written as norm of e to the power 80 times norm of x naught, okay?

This norm is a matrix norm. This is Rn norm. Is this okay? So, this is dominated by e to the power 80 is dominated by e to the power eta t, right? So, e to the power eta t and this is, let us say, norm of x naught is c. So, c naught times e to the power eta t.

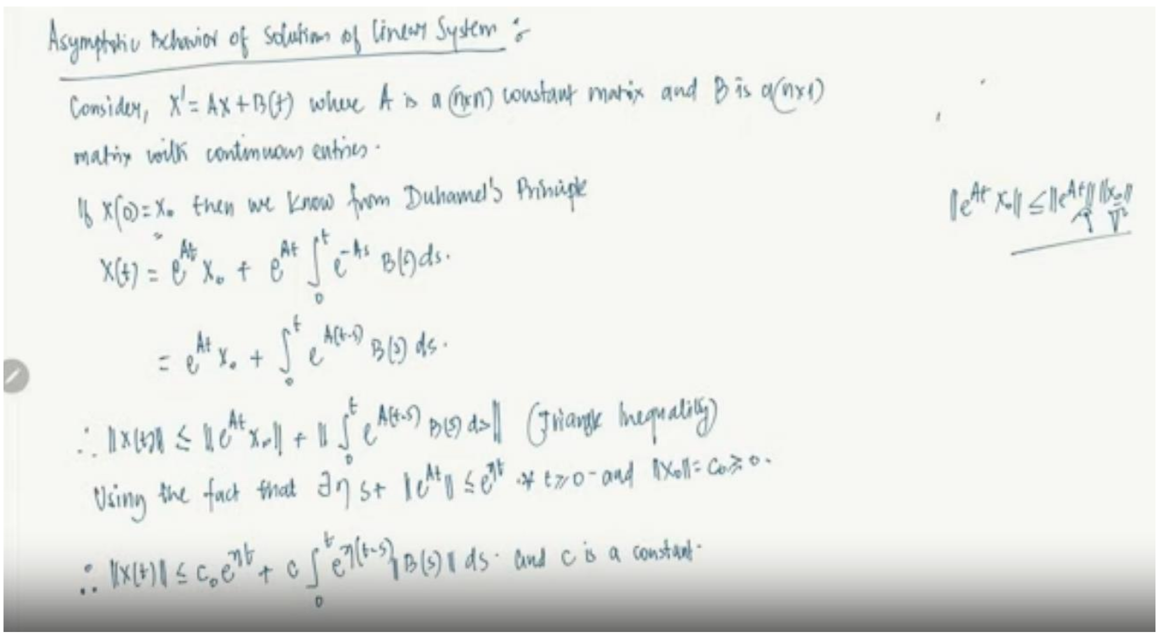
Okay, and let us say that norm of x naught is given by some c naught, clear, which is of course greater than or equal to 0. Yes, that is given. So, this is there plus we can say that c times, okay, norm of, I will write what c is here, 0 to t e to the power eta, clear, that eta, this eta, yes. eta and what do I have here? t minus s, yes, t minus s, b of s, ds, norm of b of s, ds.

This is okay. I am just taking norm of both sides and I am just writing it. And what is c here? This we need to write where c naught is nothing but we will write c naught as c times c and c naught c naught as c times e to the power sorry norm of x naught.

It doesn't have to, I mean, see, it's just basically constant. So, you don't have to worry about it. It's just some constant. Let's just put it that way. Okay.

So, maybe I can put it this way. It's okay. Where C naught and C is a constant. Let's just put it this way. It will be better.

Okay. Now, what do we have? See, this is fine. Let us call this 2. So, we are going to deduce a theorem using 2.



What is that theorem? So, basically the theorem says, it says that, let us say that suppose the inhomogeneous part B of t is such that, so the theorem says that if you have a bound on B, the norm of b of course yeah what does it mean that if you know that the norm of b grows okay no more faster than an exponential so basically it is bounded by an exponential map then the solution will also be bounded by an exponential that's what it is saying essentially okay so if bt is such that norm of bt this is okay norm of bt is bounded let's call that c three times exponential gamma t okay for large t for large t if it holds for all t then no problem but anyways for large t we are just assuming okay And where C3 and gamma are constants.

Are constants. With C3. Greater than or equal to 0. When is C3 0? C3 if it is 0, then norm of Bt is dominated by 0, which is again dominated 0.

So, basically norm of Bt has to be 0. So, all entries has to be 0. So, basically we are in the case of homogeneous equation then. So, C3 is greater than or equal to 0. This we will assume.

Then what we can say is every solution, every solution, x of t, of 1 okay satisfies satisfies norm of x t is dominated by some constant c 4 times exponential xi t is this okay for all all t greater than equal how do i put it maybe So, large enough that is greater than or equal to T naught, which is of course greater than 0. Where, what is C4? C4 and xi, these are all some constants, are constants with C4.

greater than or equal to 0. Anyways, again c4 if it is greater than or equal to 0, xt has to be 0, right? Similar thing. See, if c4 equals to 0, let us say, then what happens to norm of xt is dominated by 0 and again this dominates 0 since it is norm. So, norm of xt has to be equals to 0.

So, if norm of xt is 0, that implies xt has to be 0, right? There is no other way. So, 0 is the only solution. So, basically we are in a homogeneous system case, just like what we talked about in case of C3. So, that is assumed, right?

So, how do I show something like this? See, proof. Proof is actually not very difficult. Yeah, it is some calculation. See, since norm of bt

Norm of Bt is dominated by C3 times exponential gamma t for large t. For large t, that will imply there exists, let us say, t1, let us say, greater than or equal to 0, such that norm of Bt is dominated by C3 times exponential gamma t. okay for all t greater than equal to t1 i hope this is clear okay see for a large t this is true that so i will just choose on t and such that for all t greater than so that particular threshold bt is dominated by e to the power gamma t that's fine now let's say if gamma is not equals to eta what is eta eta you remember this is the eta this is the eta okay so if gamma is not equals to eta what happens then norm of x t norm of x t okay you remember we are starting from here yes i am not doing this again norm of x t is dominated by this right c naught times e to the power gamma t so basically if i am taking e to the power gamma t common because e to the power gamma t is here also okay so let's take it common this is dominated by e to the power gamma t is this okay then c naught plus c some constant c let's see let's see okay uh e to the power gamma t we already took out so e to the power minus gamma s norm of bs ds okay so e to the power 0 to t e to the power minus gamma s norm of bs ds this is what we are left out with okay now see this is again dominated by e to the power eta t okay c naught plus c 0 to c for t greater than equal t1 i have a bound not b so i have to use it okay so i will divide this part into 0 to t1 and t1 to t right it is minus e to the power minus bs sorry minus eta bs b of s ds

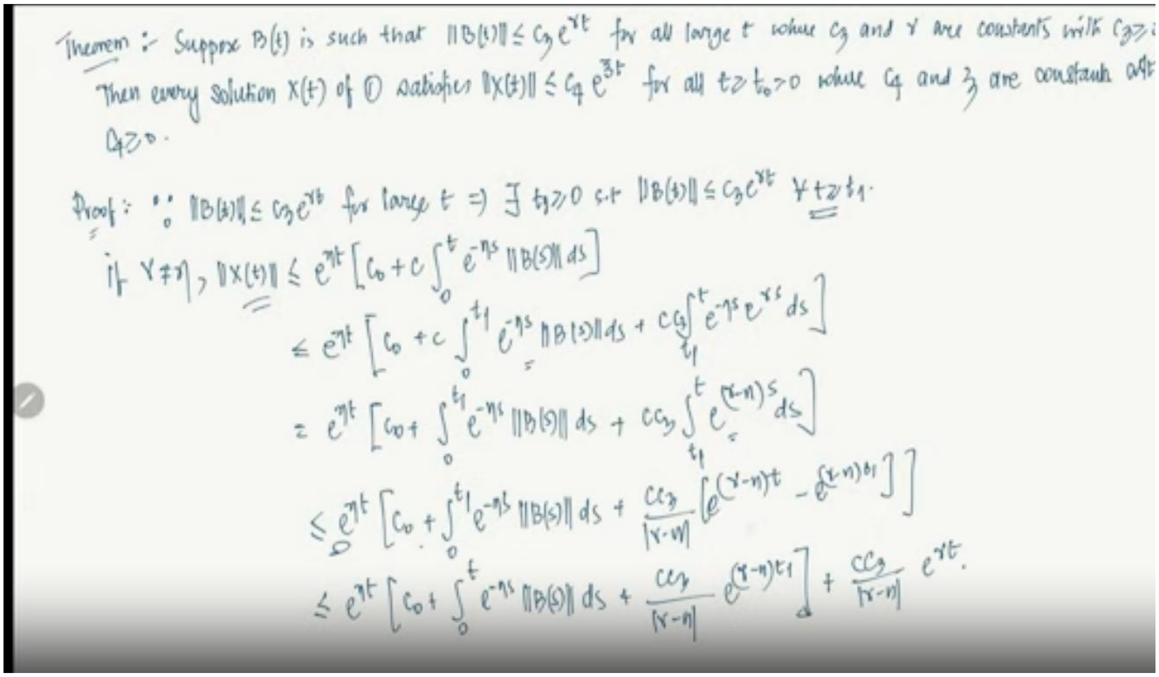
plus c times t1 to t e to the power minus eta s okay norm of bs is nothing but c3 times e to the power gamma t gamma s sorry gamma s bs right i'm just replacing it yes so if that is true now you see ah then we can write it like this eta t c naught plus ah okay this will just leave it as it is okay so 0 to t1 e to the power minus gamma s norm of bs ds okay plus you see c c3 this is t1 to t now this is nothing but if you want gamma minus eta times s ds right that's what we are going to get now so we can write it like this see we can write it like dominated by your gamma t c naught this integral we can do that so we just break it up plus 0 to t1 e to the power minus eta s norm of ps ds plus this this integral we just break it up so c c3 by and i will use the modulus here so it's not a problem i can use the modulus e to the power e to the power gamma minus eta okay uh t1 minus e to the power gamma minus eta, sorry, t and t1, t and then t1, t1, clear?

This is what I am going to get, okay? Now, this can again be dominated by e to the power gamma t c0 plus 0 to t e to the power minus eta s norm of bs ds, right? Plus see here what I am going to do is I am going to write it like this cc3 by gamma minus eta into the power gamma minus eta times t1. This term I am writing like this. Why I can change it to plus?

See this is a positive term. So basically this x of t is dominated by this particular term minus a positive term. So, this will be dominated by this term plus that term also, okay. So, I can just change the sign, it is not a problem, yeah. Since this is positive, the exponential gamma minus eta times t1 is positive, right, exponential map is positive.

So, we are using that fact, clear. So, that part plus you have cc3, cc3 by gamma, mod gamma minus eta e to the power gamma t. I hope this is clear, okay. Sorry, I should write it like this bracket. I have to change the bracket here. Yeah, bracket I should write it here.

You see, e to the power gamma t and this e to the power gamma, sorry, e to the power gamma t, this gets, I mean, cancel out, right? So, e to the power gamma t.



sorry e to the power eta t and e to the power minus eta t gets cancelled out so e to the power gamma t is here right that is what i wrote okay fine so now you see this we are more or less done actually yes so you see if you take maximum of eta and gamma okay so let let xi is maximum of eta and gamma and we define c4 okay to be this particular term the whole term this is c4 the term in the bracket i am not writing it anymore so what happens is if we do that then one can see that we have therefore we have norm of xt is dominated by C4 times e power psi t, right?

This holds for all t greater than or equal to 0, right? And hence it is proved, okay? So, we learnt a beautiful theorem here that if the, you know, if the inhomogene as a source term has at most exponential growth, then the solution will also have an at most exponential growth. It cannot grow more than that, yes? I hope this is clear.

So with this I am going to end this video.