Ordinary Differential Equations (noc 24 ma 78)

Dr Kaushik Bal

Department of Mathematics and Statistics

Indian Institute of Technology, Kanpur

Week-04

Lecture 24: Asymptotic behavior of solution to linear system-1

Hello students, in this video we are going to talk about asymptotic behavior of solutions of linear system. What does it mean? So, essentially first of all let us start with a linear system which is a variable coefficient. Consider the system, the system what is it? x prime equals to a of t times x plus b of t.

here and let us assume that x at the point 0 is x1. Now, for this system, let us say that if there exists a solution, we want to see that what is the behavior of the solution. So, let us say let y of t Let us call this system as 1, okay? So, let y of t solves 1.

So, b is a solution 1, okay? Now, our question is this, that let us say that can we say something? See, because existence actually guarantees that there exists a solution, yes? And we can actually see from this problem that the maximal interval of existence for this problem is going to be whole of r, right? But the thing is this, so we know that a y t solves one for all t, for all t, t in R, that is there.

Given x, a and b are smooth, okay, and b are smooth. So, basically if a and b are continuous, that is more than enough, are smooth. Let us just put it smooth for now, okay, it is not a problem, smooth. Now, the thing is we want to see that what happens to this system the solution. So, what happens to the solution?

So, the question is this. Existence inheritance gives us a solution. It does not tell you what the solution is or it does not tell you what is the behavior of the solution. So essentially, if we cannot solve the problem, what is the next best thing we can do? We can actually talk about what is the behavior of the solution.

So let us say what happens at t equals to, you know, as t tends to infinity, whether the solution remains bounded or not, whether it tends to 0 or not. So that sort of questions we are going to answer here. So let us say the question is does the solution so basically let us say all or some solution okay so for now I am just writing it all the solutions. So, in this case, it is a unique solution. So, all the solutions is just the solution.

So, basically, let me put it this way, that the question is, if the solution so obtained, solution so obtained is bounded, is bounded, okay, or the questions like this, let us say, or tends to 0, tends to 0 as t tends to infinity. Okay, so why t tends to infinity? That is the question. First of all, let us understand. See, this problem, since the maximum interval of existence for this problem is r, right?

So what is the maximum interval of existence of one? It is r. Clear? provided a and b are smooth since it is r so you see the thing is it is valid for all t so t as t tends to infinity also there is a solution right now the thing is this we want to see what happens to the solution as t tends to infinity why because you see if t lies between let us say t1 t2 so what is so special about infinity why not in some t1 t2 let us say as some let us say why not 100 1000 1 million yeah the problem thing is it If you are fixing t between t1 and t2, this is a compact set, right? This is a compact set.

And the solution which you have, y. So, let us say y is also 1, right? So, y is a c1 function, c1 from r to rn, right? It is a curve, smooth curve, okay? It is a smooth curve. Now, since it is c1 from r to rn, now if I am restricting my t between t1 and t2, so basically y is restricted to this, you know...

Compact set, T1 and T2. What does that gives you? That gives you that there is a maxima of y, right? So, let us say y naught maxima. set.

So, there is a maxima there is a minima. So, we know that y is bounded in sort of this, but the thing is here if you are looking at the whole interval then we cannot say whether they are bounded or not ok. That is the question which we need to answer in this ah between yeah ok. So, let us look at this see first of all let us look at the see our study will be like this. First of all we are going to start with

constant coefficient equation right once we know how to handle constant coefficient we will go to variable coefficient problem is this okay so let us look at the constant coefficient problem first okay so first of all consider the problem consider x prime equals to x. This question. Clear? And let us say that we are also assuming that x0 equals to x1. Doesn't matter.

Let us say x prime equals to x. This is the thing where a is constant. Where a is a constant n cross n matrix. Is a constant n cross n matrix. Matrix. Now, for this problem, we know that xt is given by the JL solution, okay.

The fundamental matrix times c and fundamental matrix, we know that what is it? It is ta in this case, right, times a constant n cross 1. This is a constant arbitrary matrix. So, the constant matrix. So, you know that any solution looks like this, right?

Okay. Now, the problem thing is this. See, let us say that if it passes through x0 equals to x0. If x0 equals to x0 given to you, then that will imply that x of t is a unique solution given by its reward ta times x0. Okay.

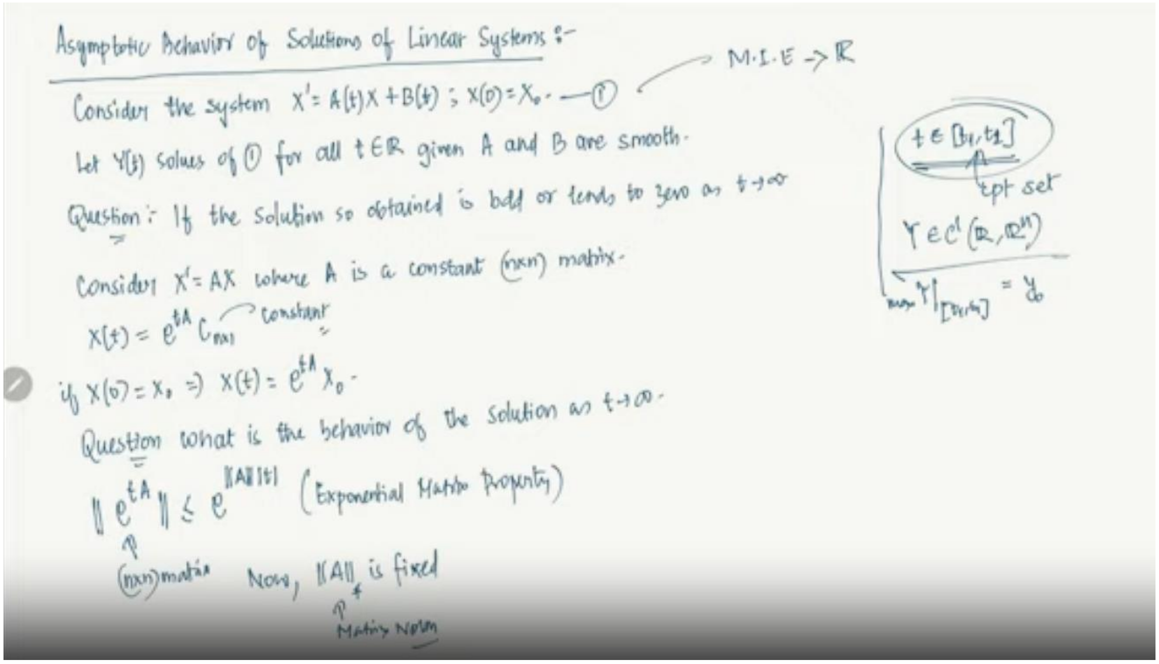
Now, if we want to understand what is the behavior of this solution, okay, as t tends to infinity, now the question is this. What is the behavior of the solution as t tends to infinity? So, what is it? See, first of all, e to the power ta, let us just understand that. This is a n cross n matrix, right?

This is a n cross n matrix. Matrix of course dependent on t that is there but basically it is a n cross n matrix for a fixed t right. Now let us look at the norm of t of a. What is it? So norm of t of a if you remember it is always dominated by e to the power norm of a mod of t right. If you do not remember when we defined exponential, this is the thing which we proved.

So this is done in that exponential video, whatever video that is. So the exponential video is done. Exponential property, matrix property. property okay so this is why we actually showed that this is absolutely convergent the series okay so we have this now if this is the case see the thing is and now now given a is constant right so norm a is fixed norm a is fixed okay you remember what is norm a norm a this is matrix norm Matrix norm or operator norm, whatever you want to call it, does not matter.

Matrix norm. You can take L2 norm also. It does not really matter. So essentially that is basically the same. What I am trying to say is it is always bounded above.

That is the thing. So it is matrix norm. Now you see the thing is and so e to the power norm a mod t. So it is fixed. So we can say that norm a equals to let us say eta. Same.



some number right so it is fixed the if you are fixing the matrix norm let us say operator norm then that norm of a is eta some number real number okay so what happens therefore hence so these two implies that any solution x of t if you take the norm of that what is it it is always bounded by dominated by e to the power ta so you see what is x of t x of t is nothing but e to the power ta times x naught right this is what this is a n cross n matrix this is what this is a n cross 1 matrix so norm of xt is always dominated by norm of e to the power ta times norm of x naught right we prove this this is basically So, kind of Cauchy's Versa that is what we proved. If you remember when we looked at matrix the properties of matrix then we looked at this problem. This is very easy to prove it is not a problem.

So, please do it yourself if you are not convinced. But we did it in one of the videos when we talked about exponential. Anyways, so if this is true, so you see what happens is that now e to the power ta, the norm of e to the power ta is dominated by e to the power eta norm t, right. So, it is dominated by eta mod t, clear, times norm of x naught, you see, okay. And t for now, let us just assume that t is positive, okay.

I mean, it does not have to be, but let us just, okay, does not have, let us just say it is mod t, okay. Now, norm of x naught. norm of x naught yeah if it passes through x naught then it is always dominated by this now you see norm of x naught is fixed right norm of x naught is not changing clear so if norm of x naught is not changing so essentially what happens is the asymptotic behavior of this will depend so on t you see for t so as t tends to infinity let us say as t tends to infinity norm of x t, it is always bounded by norm of x naught, which is let us say some constant c times e to the power eta times mod t, clear. So, what happens to, if you got eta times mod t, eta here is a positive constant.

So, it is always dominated by exponential. That is the only thing which you can say here, at least this is, okay. But this is more than enough. We just have to see that you can, norm of x of t can always be dominated by c times exponential eta t. That is what you need to know, eta times mod t, sorry, eta times mod t, okay. So, t, yeah.

So, that is what you need to know. That is all. So, that is the maximum we can say if you have a solution which is like for the homogeneous problem cost and coefficient. Yes, that is the thing we can say. Now, we consider this differential equation.

So, consider the system. Consider the system. What is the system? x prime of t equals to a. I am just changing this system a little bit. So you see this is a plus bt times a constant matrix x. Here I will again assume that of course a is a constant matrix.

b is continuous. This is the n cross n matrix with continuous entries. with continuous entries. Continuous entries. Entries vij, whatever it is, vij of t. So, now the thing is this, you see this type of system, this is called a part out system.

part of the system, is this okay? We will just call it, for namesake, it is not anything special, but we will just call it like that, okay? So, let us just call it as 2. This is the system 2, part of the system, yeah? And how is it part of?

It is, see, A, the matrix A is part of which A plus B, T, yeah? Constant coefficient matrix, but I am just perturbing it with a variable coefficient. That is the thing. Another thing is, we will have, we want to see what happens to the solution of this problem, yeah? So, the first thing is this we want to find out sufficient conditions on B of t so that

All solutions. You see I am not putting any initial data. So that is why I am writing all solutions. If there is initial data then you will have a unique solution. All solutions of 2 remains bounded.

Remains bounded. Bounded. So, let us look at that theorem. So, theorem 1. Let us call it theorem 1.

Let all solutions of Let's call this a homogeneous problem. This problem. Okay. Maybe I can write it here.

x prime equals to ax. This is the original equation. Right. The constant coefficient homogeneous equation. And I am part of it.

2 is the part of equation. Let's call this a maybe star. Okay. Homogeneous equation. Constant coefficient.

So let all solutions of star be bounded. Okay. So, this is our assumption. We do not know whether they are, let us just assume that in 0, infinity, open 0, infinity. So, basically, now we are not looking at t negative or anything.

It is only t positive. Let us just understand. Because, see, our goal is to find out what happens as t tends to infinity. You can do the exact same sort of, I mean, analysis as t tends to minus infinity. So, it is not a problem.

So, in that case, you just look at minus infinity 0. That is it. So, let all solutions of the differential equation is bounded in 0, infinity. Then all solutions of 2 are bounded in 0, infinity. One second, let me just put it this way.

Okay, maybe I will put it continuous entries in 0, infinity. Let us just put it this way. So, boundaries in 0, infinity provided 0 to infinity norm of B, S, D, S is finite. Is this okay? Fine.

So, we have to prove it. What is the theorem says? It says that if all solutions of the homogeneous problem. See the homogeneous problem has many solutions right. Once you put an initial data then the solution is unique.

But generally speaking it has many solutions. Now what it is saying is this if you have the if you are if you are interested in looking at the solution of the part of equation you look at the you know the homogeneous problem and what we do is this if the homogeneous problem if all solutions are bounded then all solutions of the part of equations are also bounded provided that 0 to infinity b of s ds the norm is finite yeah so let us look at the proof. What is the proof? The proof is let us say that what I will do is see this part of equation. So how do I put it?

Note x prime t if you want to write it properly it is ax plus bt times x. Right. And let us call this term. So let us call this as small b of t. Okay. So let me write it like this.

small b of t is capital Bt times x. See x is also a function of t, right? So capital B of t times capital X of t, I will just write it as B of t, yeah? So if we do that, then by Duhamel's principle, or variation of parameter or whatever you want to call it, by Duhamel's principle, what do you get? You get that x of t, okay? is e power a t times x naught.

Where is this coming from? See, this is the solution of this problem provided x 0 equals to x naught. So, I am just using that. So, x t equals to e power a t times x naught solution of the homogeneous problem plus you remember the next part is 0 to t e to the power a t into the power a s b s, this term x of s d s. I can write it like this.

Now, please understand this. This is what is it? Although I have used this formula, this is nothing but this is actually an integral equation. equation. Why equation?

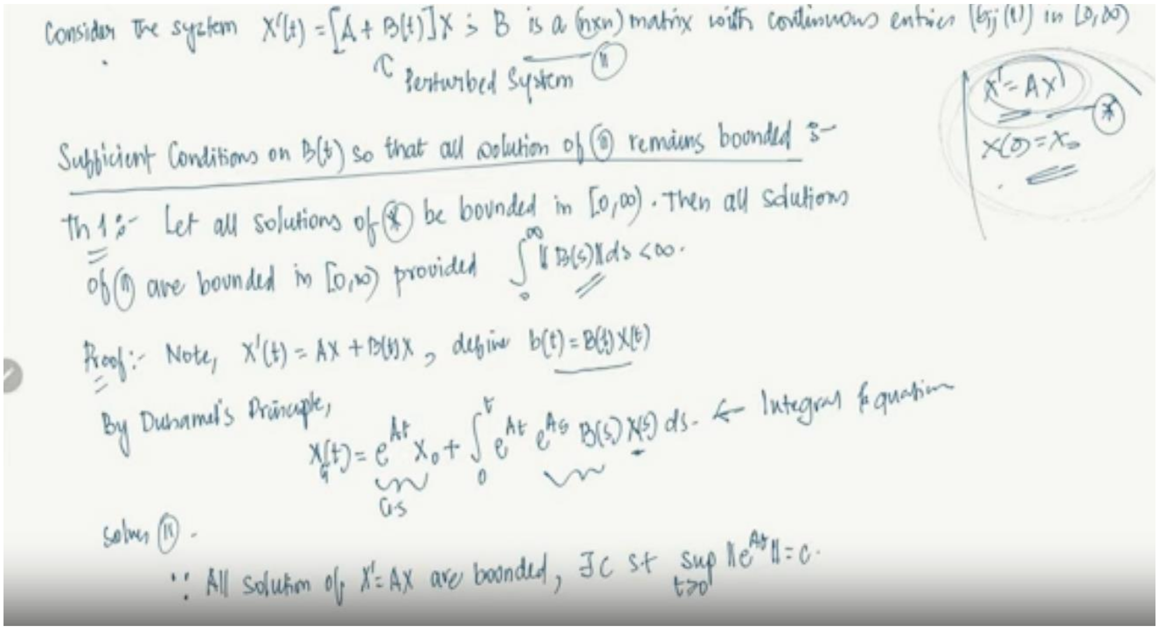
See I have wrote x in terms of x. So it is not a solution. Do not think of this as solution. It is just a equation. It is called an integral equation. Now what we need to do is this.

See what is this solution? This is any given solution of the system. Please understand. What is this solution? This is any even solution.

This is a general solution of the part of system 2. Xt. So let us call that Xg. So you see Xgt by the way of these solves 2. the general solution right this is the general solution we solve to now the thing is this so if you remember this is the general solution of the homogeneous problem see x naught is nothing it is like any if we pass it to x naught it will write it like this and this is the particular solution right ok so it solves to here now you see that since all solutions of this problem the homogeneous problem this problem

So, x prime equals to ax. This is bounded. That is given, right? All solutions are bounded. Since, all solutions of x prime equals to ax are bounded.

Are bounded. Here, there exists C then such that supremum of t positive norm of e to the power a t is c. Is this okay? It has to be right. See, if they are bounded, see x naught does not matter.



x naught is some, it can pass through any point. It is basically an arbitrary number, right? x naught can be, I can change it to x naught plus delta x naught. It does not matter. The only thing which actually, you know, provides the boundedness is basically what happens to the exponential math.

Now definitely if all solutions are bounded exponential the supremum of e power a t as t greater than or equal to 0 that has to be some constant c otherwise they are not going to be bounded right. Hence for all t greater than equal t0, we have norm of x t is less than equal c0, right. So, basically it means that it is, how do I put it, c times norm x0, see. supremum is c so this this particular term x g of t if you take the norm that will be dominated by norm of

this part, x naught e to the power 80 times x naught, x naught times e to the power 80, right, plus norm of integral that term, right, that is triangle inequality. Now, this again is dominated by norm of x naught, norm of e to the power 80. If you remember, we exactly did the same sort of thing when we studied matrix norms. So, this is true. Now, you see, e to the power 80, the supremum of this is c. So, this is dominated by c times norm of x naught.

So, that is what I wrote here, yeah. okay so now now plus this term plus again see here this x of t is there right okay this x of t is there so this x of t if i take the norm on both sides this x of t is again dominated by uh One second, let me just see. Sorry, e to the power a t, the supremum is given by c. So, this e to the power a t, I can dominate it by c, right? So, it becomes c times 0 to t naught, the norm of b of s, norm of x of s, ds.

This is x g, okay, the general solution. You can write it like this. See? x of s, b of s. And this whole thing is e to the power a, t minus s, the norm of s. This is always dominated by c. So, I can just write it as c times bs norm of xs. See, this is what I wrote.

I hope this is clear. Okay. So, let me make it cleaner. Okay. Now,

Now what we have? Once this is true, you see, you have to use, so if you remember properly, we are in what setup? We are in the setup of ground wall. Inequality. So, please understand what is happening.

Once you have this integral kind of equation, what you do is this. This is also x, this is also x, right? So, I have to write it x in terms of x. So, basically if I have to take x out, what is happening? You can actually use ground wall. So, basically if you use ground wall, that will give you x of t.

is dominated by some c naught okay some c naught doesn't matter i mean some some cost and times exponential this c times exponential this c times 0 to t norm of bs once again this is there is a small mistake here ds This I can do. This holds for all t greater than or equal to 0. This is by Grunwald. Again, if you do not remember, first week only when we talked about inequalities, we talked about Grunwald inequality.

So, please go check that video, Grunwald inequality. We are just using Grunwald directly and you get this. Now, we are done, right? See, we assume that all solutions of x prime Ax is bounded. So, this is happening.

Once this is happening, then this also, actually you see, this also says that xt is bounded, right? See, b of t, if you look at the, whatever this assumption is, 0 to infinity norm bsds is finite. So, this part is finite. This is just a constant. So, exponential is some constant to finite, right?

So, norm xt is always bounded. and this is for any solution please remember this okay one thing i only need to clarify this i mean you need to clarify this thing i am not saying that this bound Okay, I am saying that if you fix any solution is always bounded by this particular term. This is what I am saying. Okay, so it does not really matter what solution you get.

You can take any x naught and you get another solution, right? But that solution will again be bounded by this. That is what I am trying to say. Is this clear? Okay, so what did we learn?

Very easy. We learned that if you start with, if you start, we started with this, x prime equals to x. Yes, we assumed. See, we do not know whether such a thing happens or not. Let us just assume that by somehow, somehow we get to this information that all solutions of this problem are bounded.

Once that is there, then you can say that all solutions of this problem, bt times x is also bounded. Yeah, but when is it true? If integral bt is finite. Clear? That is what it basically says.

So, now let us look at the condition. Now we want to find some sufficient condition on the matrix B of x, B of x, sorry B of t, B of t, so that all solutions of what did I call this? 2, right? 2 tends to 0.

So, did you understand the question? See, the first thing is this. We wanted to understand that when are all the solutions are bounded. Now that we know that when the all solutions of the homogenous equations are bounded then all solutions of the part of the equations are bounded provided that the b is basically is integrable. Not of b. Now the question is this.

When can we say that all solutions of this factor Pt they tends to 0. So let me ask you this question. Maybe you can pause this video and think about this. When do you say that all solutions of this part of equations tends to 0 as x tends to infinity. So, the theorem I want you to maybe you can think about it yourself.

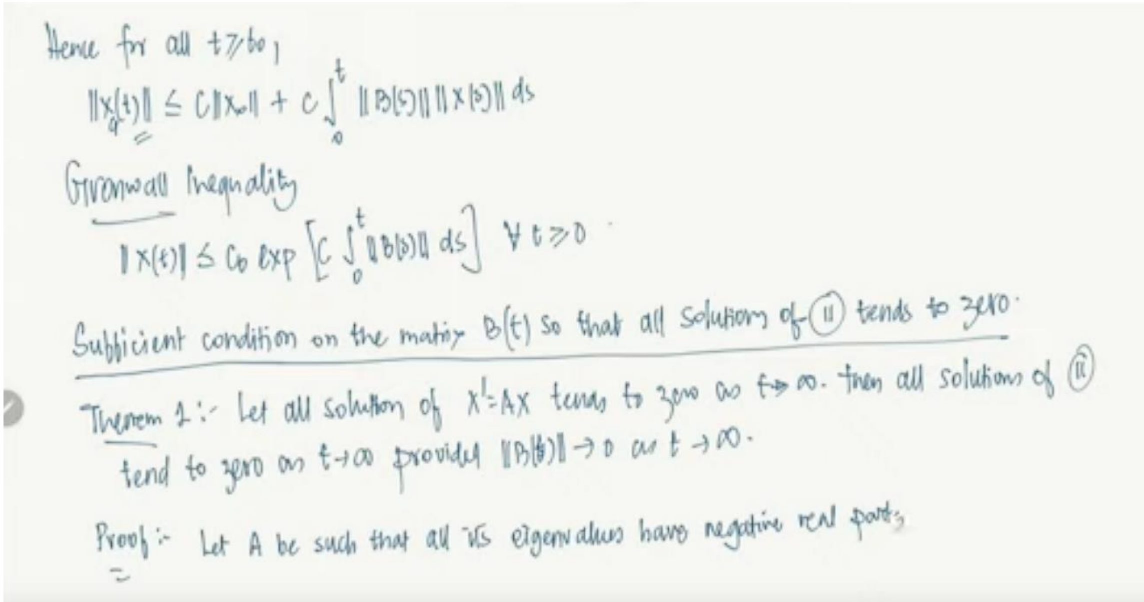
So, pause this do that and then come back. So, see let all solutions again this is exactly the same sort of thing. Let all solutions of x prime equals to ax this system. 0 as t tends to infinity. Then all solutions of 2 tend to 0 as t tends to infinity provided

See, initially, what did we needed? We needed integral of b of s is finite. Now, we need if b of s goes to 0 as, or b of t maybe I can write it, as t goes to infinity. Is this okay? Very, very, I mean, this is very intuitive, right?

It should happen, okay? But the thing is, I mean, intuition is fine. You have to, again, anyways prove it. But the thing is, this is the, I mean, okay? And it is quite intuitive.

From the earlier theorem, it is very intuitive. Okay. So, let us look at the proof. So, first of all, all solutions of x prime equals to ax tends to 0. That is given to us.

Okay. So, if that happens, then one can say that, see, let us say, let a be such that such that all its eigenvalues have negative real part. If that is the case, now you see if that is the case then what can you say? So, if all the iterations are negative real part, then you can see that this is a part of the assignment.



You have to check this part. So, check that the solution, the solutions, any solutions basically tends to 0 to 0 as t tends to infinity. Okay. Right. So, maybe I can give you some ideas how, why this is true.

Okay. Let me give you some ideas. Okay. Again, this is not the whole proof. I am just giving you some ideas how to go along this.

Okay. See, any solution of this, how does it look like? It is basically, let us say lambda 1, A has this eigenvalue. Let us do it for 2 cross 2. Yes.

And lambda 2, two distinct eigenvalues. Yes. I am doing the easy case. I am leaving the hard case for you. So, lambda 1 and lambda 2, two distinct case.

So, you have two eigenvectors, v1 and v2. And we know v and v2, they are linearly independent. v1 and v2 are linearly independent. So, what is the solution? x1 of t, what is it?

It is e power, sorry, one second. It is v1 times e to the power lambda 1 t. And x1 is x2 of t, it is nothing but v2 times e power lambda 2 of t. Now, what is the general solution x of t? It is nothing but c1 v1 times e to the power lambda 1 t plus c2 v2 times e to the power lambda 2 t. Now, if this is the case, so you see that if lambda 1 and lambda 2 are negative. So, if lambda 1 and lambda 2 are negative.

If both are negative, what happens then? What can we say? See, V1 and V2, then, okay, so let me do it properly. Maybe I can do it properly, okay. So, let us assume that, let me do it here.

If A 2 cross 2 matrix has eigenvalues, lambda 1, lambda 2 and v1 and v2 are the corresponding eigenvectors. Okay, then the general solution x of t looks like c1 times e power lambda 1 t v1 plus c2 times e power lambda 2 t v2, right? That we know. This is the, you remember the method of eigenvalues when we studied, we exactly did the same thing, right?

Now you see, then, so if this is the case, then what is norm of x t? norm of ht is always dominated by c1 is sub constant right so let us just put a mod there norm of v1 which is fixed v1 is fixed it is not changing and then it will be 1 lambda 1 t right okay sorry i should not this is a real number okay a positive real number plus c2 norm of v2 E to the power lambda 2 t. Yes. Now you see. As t tends to infinity.

If t tends to infinity. Okay. And lambda 1 and lambda 2. Are both negative. Let us assume that.

What happens to this term? This goes to 0. This goes to 0. And this is always positive. So norm of x t. Must go to 0.

Okay. So if both the eigenvalues are negative. Then this is going to happen. Okay. Now the question is this.

Can you do similar thing. When you have a repeated eigenvalue. Or if you have a. you know complex eigenvalue so basically in that case you have to look at the real part yeah okay so i want you to check this yourself please do that fine this is one of the part fine now see thus so what do what do we have we have that you have the function the matrix a has negative i capital U negative real part that is fine then you can actually say that all solution stays to zero has to stay to infinity that is given to us ok so now what you can say is thus there exists C add eta ok such that

norm of eta e to the world at sorry this is dominated by some costar types e power minus ah let us just call delta type t okay so maybe eta type t is fine yeah i did i use eta somewhere i have used right it should not be confusing i have used eta somewhere okay maybe not I do not yeah I have used eta somewhere okay it is your eta t right so I mean if you want you can do that but anyways is so okay you see the thing is please remember this let us say let us say for this case for this case okay this This is always dominated by, if you take the maximum of lambda 1 and lambda 2, okay. So, you can always dominate it by the maximum of this. So, minus eta times t, right, if you can see.

So, that is what eta is this basically. Okay. Now, this holds for all t greater than or equal to 0. I hope this is clear how this is coming. See x of t is nothing but e power at times x dot.

So essentially if I am taking the norm I can take the norm here and you can see that I can take the I mean maximum of that and then it can be written as c times e to the power minus e to the t. Why minus? Because see both eigenvalues are negative. Yes. At least the real part. If they are negative then only this can happen.

Okay. So c is the all solutions tends to 0. So, basically the real part at least they have to be negative. So, that is why it is minus eta t. Is this clear?

Now, also see p of t this goes to 0 as t tends to infinity. So, what does it means? It means that for t, let us say greater than equal, let us just call it t1. One has norm of bt is bounded by some c1.

I can say that b of t goes to 0, norm of bt. So, it means that for sufficiently large t, this has to be bounded. Otherwise, this is not going to happen. So, this is there. Therefore,

For all t greater than equal to t1, we have norm of xt is bounded by c types e to the power minus delta t minus t0. No, here I am assuming t0 to be 0, so t0 I do not know either. as delta t because this, what is this? Eta t, let us write it as eta t. Eta t top of x naught. See here, this thing.

This is also true, same equation, right? So, this is true. Now, if you take the norm, e to the power a t is dominated by c times e to the power minus eta t. So, a norm of x1, okay? And then I have again this term. So, this term what I can do is if I remember properly, I should do it like this.

0 to t c times e to the power minus delta t minus s, okay? Norm of bs, norm of xs. I hope this is okay. consider maybe I have to break it up I have to break it up because this is for t greater than t1 I see I only have this estimate for t greater than t1 I do not have the I do not know what I have I have to have a different estimate for t less than equal to t1 so what I will do is maybe I can write it as t1 and then I will write it as t1 to t c times e to the power minus delta t minus s c1 times norm of x of s ds.

Is this clear? So, for all t greater than t1, this is dominated by c1. So, I can write it as c1. And for between 0 and t1, bs is just bs. I hope this is clear.

That is why I just broke it up. So, once that is true that is basically nothing but you see if you define e to the power eta t I will take it on the other side. So, define w of t to be e to the power eta t times norm of xs, xt sorry xt. clear. Therefore, what happens is therefore, what do we have?

Let us just call that let us say that this particular thing plus c times so once again let me just look at it. Maybe this whole thing let us call this particular thing as c dot, okay? And, clear? Let me just do that.

So, this and this, okay? So, let us just write it this way. Therefore, W of t, okay? W of t is dominated by this term, C dot I have to norm of x naught.

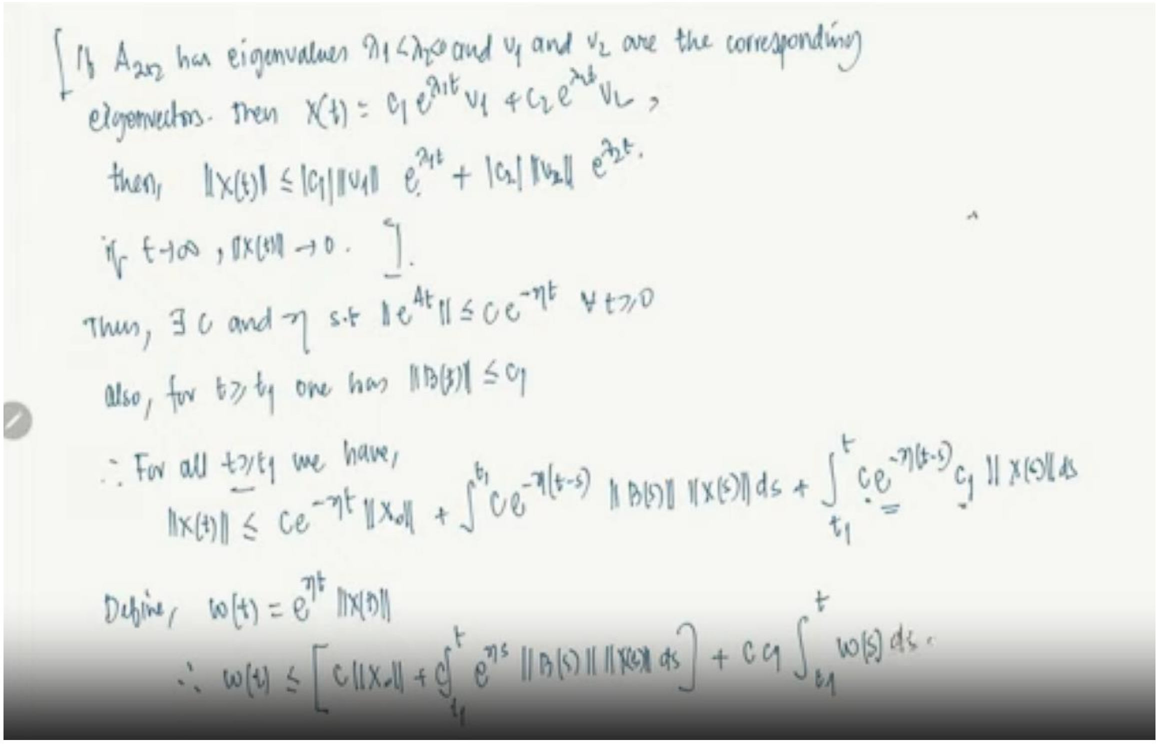
This term, e to the power eta t goes there and this term. So, e to the power eta t, sorry, what is delta? It is eta, no? It is not delta, sorry. This is eta.

That is why I got confused. Okay, it is fine. eta. So, now eta t I can take everywhere. So, this term and this term I will just put it together.

Sorry about the constitution. So, this term plus 0 t1 to t c times e to the power eta s see eta s this term it is a minus eta t goes that way right so if you are minus sorry eta s comes here times v of s i am looking for this term okay so let me write properly v of s x of s So, let us, so these two terms together plus c2, let us just call c2 as c1 times c2 or maybe that is okay, t, t1 to t, e to the power uh eta s xs oh sorry sorry xs yeah hs will be there no i am doing some mistakes somewhere so x of s uh x of s ds oh

If your eta t times, see, if your eta t times x of s is w of s, right? Sorry, sorry, sorry, this is what, yeah, there are lot of terms, right? So, I am getting confused. So, this is x of s, sorry, w of s, w of s, w of s t s. W of s ds.

Yeah, this is okay, right? See, e to the power minus eta t goes that way, right?



So, e to the power minus eta t times x t is W of t, which is this. Now, I have e to the power eta s is there. e to the power eta s times norm x s is nothing but W of s. So, I just wrote W of s, right?

So, sorry for the confusion, okay? Fine. Now what does that imply? So this will imply C. This is nothing but let us just call it this particular term to be maybe C naught and let us just call this term to be C2. C1 is already there.

So once this is there what we can say is W of t. is dominated by c0 plus c2 t1 to t w of s ds, right. I can say, I hope I wrote it correctly, yeah, I can say that. Now, of course, again using Ronval, now again using Ronval inequality. As I told you, Ronval is very, very important.

This is all the reason we have, we have W of t is dominated by c0 times exponential c2 t minus t1. Clear? Yes? This is by Cronwall.

And therefore, what is x of t? It is dominated by c0 times exponential c2 minus eta t1. if you are eta t that if you are eta t is there so minus eta okay t minus c2 times t1 okay now of course you can see that as t tends to infinity c2 minus eta okay see this c2 yeah is c1 c we can always choose always choose c ah I I did some mistake somewhere one second ah this is yeah I think it is ok no c 2 is ok yeah it is ok so I can choose c 1 small such that c2 is cc1, right?

c2 is defined by cc1, see? Okay, c2 is cc1. So, this is sufficiently small, so smaller than eta, right? So, c2 minus eta is negative. If this is negative, then this goes to 0, right?

And this is positive term, norm xt. So, therefore, norm of xt must tends to 0 as t tends to infinity. Is this okay? Right. So we have more or less a very good idea of what is happening right now.

$$w(t) \le c_0 + c_2 \int_{t_1}^{t} w(s)\,ds.$$

Using Gronwall Inequality we have,

$$w(t) \le c_0 \exp\left[c_2 (t - t_1)\right]$$

$$\therefore \|x(t)\| \le c_0 \exp\left[(c_2 - \eta)t - c_2 t_1\right] \to 0$$

We can always c_1 small s.t. $c_2 = c c_1 < \eta$

$$\therefore \|x(t)\| \to 0 \text{ as } t \to \infty.$$

So for a boundedness you need if the original the cost and coefficient problem has all bounded solutions then the part of problem will also have the bounded solution provided the part of part. that integral part is bounded again if it goes to 0 again same sort of thing all solutions of the original equation has to be has to go to 0 and the part of part that should the norm should go to 0 right okay so let us look at one example of where we can use this thing so maybe next page example So, consider this equation. x1 prime x2 prime is nothing but 0 1 minus 1 0 times x1 x2. So, what sort of equation is this?

This is a cost of completion equation. And let us look at the part of equation. x1 prime x2 prime which is 0 1 minus 1 0 x 1 x 2 plus the part of term this is b this is a right see that is your a this is the b and remember b depends on t yeah so you can of course you can do that so basically but here in this example what we are going to do is we are going to take it in such a way that b depends on t so it is very well parted to doesn't matter maybe a let's just see and a t plus b let's just call this x1 x2 okay so this in this case this is your b of t

Okay, so for this problem, for the problem, the constant coefficient problem, right, one can see that the fundamental solution, fundamental solution, okay, let me write it this way. Let me do the problem properly, okay. See here, A is 0, 1, minus 1, 0. clear? So, you do realize that if you, it is very easy to solve, right?

It is not a very difficult problem to solve that, I mean, you can use eigenvalues and eigenvectors, right? And you can solve this problem. So, you have to check that cosine t minus sine t and sine p and cosine p forms the fundamental set, forms the fundamental system, right? Fundamental system of solution.

These two, these two gives you the fundamental matrix, right? Yes. So, I hope this is okay. Now also check this part.

Check that this problem matrix A sine t minus A t plus B times cosine t and A t plus B sine t. This vector and this vector, a cosine t. See, to be very frank with you, this is one of such, you know, how do I put it? System which can be solved, okay? Right. Variable coefficient, but still it can be solved.

a t plus b sine t and then a t plus b This forms the fundamental state of solutions for forms the fundamental system for the part of equation. perturbed equation. I did not do it, but you have to check it yourself, okay. Now, the thing is, you may think that how are you going to check this.

This is actually not a very difficult thing to do, okay. If you break this equation up, you see, the first equation looks like this. x1 prime is nothing but x2, see, 0 times x1 plus x2. So, x1 prime is x2, only this. And what is x2 prime?

x2 prime is nothing but minus x1 okay plus 2a by at plus b times x2 okay this is the system now for this system what you can do is this will imply that x1 double prime clear is x2 prime right so if you replace it here it becomes x1 double prime clear see if you replace it here it becomes x1 double prime is minus x1 plus this term 2a by a t plus b times x2 x2 is nothing but x1 prime now you see what actually happens is this is basically a second order equation you know how to solve the second order equation okay Nah. You can use your, you know, basic, how do I put it, power series expansion.

You can do that. Or you can directly put these two fundamental set of solutions in this equation and see that whether this satisfies the equation, okay. It will be, okay. So, what I suggest you to do is this. You please check this, whether these two forms are fundamental set of solution for this part of equation or not, okay.

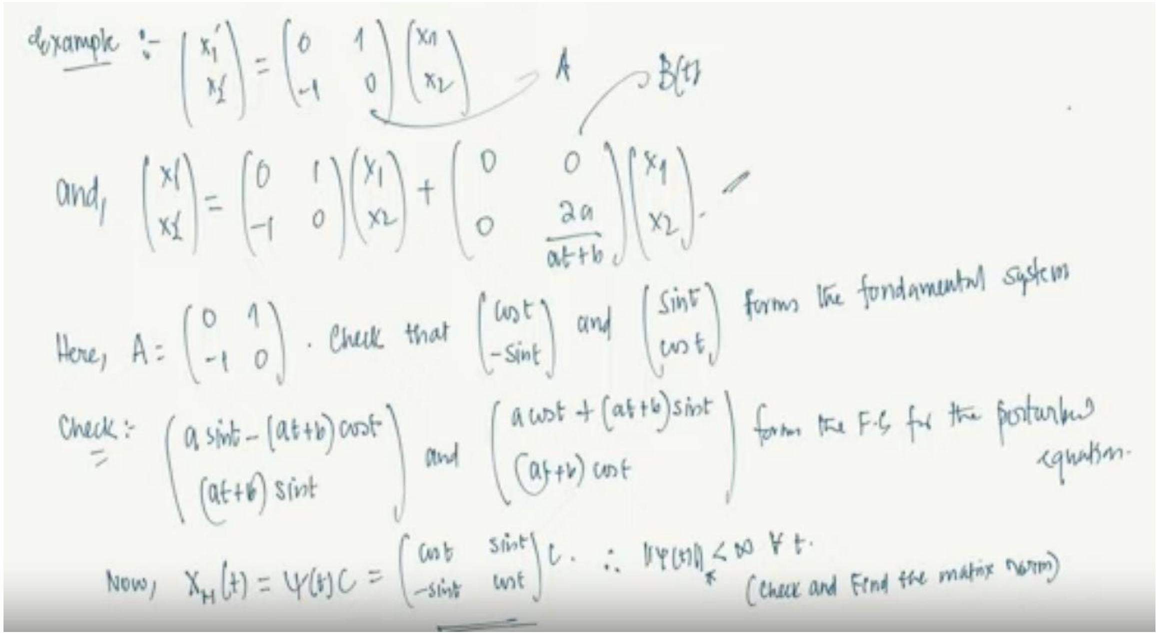
It will be. So, you, I mean, you have to just check it yourself. Just put it directly and calculate it. You can see that these two forms, okay. Okay, now, now, sorry, now, what is this, see, x homogeneous problem, okay, homogeneous problem, xg, let us say, of t, how does it look like?

Any solution will look like phi t times a constant matrix C. What is phi? Phi is nothing but cosine t minus sine t and sine t cosine t, right, times C. Is this okay? Yes. Please look at this. See, this problem, therefore, if you take the norm of phi t,

Yeah. The matrix norm. You can take matrix norm. You can take any other norm. Doesn't really matter.

But you can of course see that this is going to be bounded for all t. Yeah. This is going to be bounded. So these also you have to check. Check and find out the matrix norm. Find the matrix norm.

I hope you can check this part right you just have to check if this is a given matrix you just find out what is the matrix form of this norm of this matrix okay so once this norm is bounded okay then all solutions of this problem how do you show they are bounded tell me you just have to see if the norm of this matrix so check norm of p t this is bounded for all t okay you see what is b t b t is this 0 0 0 2 a by a t plus b okay you just find out what is the matrix stop again calculate it and you show that they are well this is bounded this is very very easy why i am not doing it i just want you to calculate it yourself ok anyways it is part of the assignment whose solution you will get it so it is already there but for now I just want you to do it yourself yeah ok so it is there again again if I am asking you whether all solutions of this problem The putter problem. So basically what did we find out?



All solutions of the original equation, the cost and coefficient equation are bounded, right? So what about the putter problem? Putter problem also since norm of Bt is bounded, clear? Since norm of Bt is bounded, then all solutions of this problem is also bounded. You do not have to calculate this thing.

This is just for your information. But you can directly show that it is going to be bounded. All solutions of this problem. Now what happens if I ask you whether all solutions of this problem goes to 0 or not? How do you find it?

What do you do? What you do is you just find out that all solutions of the original equation must go to 0. at that norm of Pt must go to 0 ok. So, anyways so let us just find out what is it. So, let us say that see was I did I do any mistake or something do not know right I will not be anyway.

So, let us see norm of pd okay this goes to 0 as t tends to infinity right This is infinity. I hope you can of course find it out. And you see, it is 0 to t. This is very easy to, I mean, once you calculate it, you can see that this goes to 0.

So, this is bounded and goes to 0. So, 0 to t, you see, you have to calculate this norm of Bt, Bs, Ds, let us say. Bs, Ds. If this goes to 0, sorry, this goes to infinity. Let us just check that.

See, what is it? It is 0 to t. 2a by a t plus b a s plus b d s right now if you calculate this thing please let me check this part it is nothing but log of a t plus b by b square okay and as t tends to infinity you do realize this is going to infinity right so what you can say is for this problem okay all non-trivial solutions okay are unbounded yeah so therefore all non-trivial solutions solutions are unbounded yes why because you see if they are bounded let us look at the theorem first What is the zero versus?

See, all solutions are bounded. Of course, that is given. All solutions are bounded for this problem, okay? But the thing is, you see, then these two will be bounded provided zero to infinity norm BSDX has to be bounded. But in this case, you see, what is happening is this.

Where is it? You see, for this problem, what is happening is this. All solutions are bounded. For the homogeneous problem, solutions are bounded. No problem.

But, So b of t if you look at the norm you take 0 to t norm of bs ds that is nothing but 0 to t norm of bs is basically 2a by 2s plus b. You can see that this goes to 0 as s goes to 0. So norm of bs goes to 0 as s goes to infinity. But if you take the integral 0 to the 2a by as plus b ds, then that becomes log of this square which goes to infinity. So, basically all non-tubular solutions are going to be unbounded.

Is this okay? So, for this system, this is such a system such that the original equation, all solutions are bounded. But the thing is, for the butter problem, the solutions are going to be unbounded. Is it okay? So, with this I am going to end this video. Bye.