Ordinary Differential Equations (noc 24 ma 78)

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Week-04

Lecture 23 : Periodic Linear System

Welcome students to this video and in this part we are going to do something called a periodic linear system. So, what does it mean? So, let us consider this problem. Consider x prime equals to a t times x this problem yes and let us assume that it passes through x naught at the point t equals to 0 the curve now you see by because existence and uniqueness you know we have already looked at this problem right that if a is continuous so if a is continuous

What does it mean? It means that all, you know, entries of the matrix A are continuous. Yes, in whatever interval I, some interval I, then you have the existence of a global solution, right. So, there exists a solution which, you know, satisfies the equation for all t in I. So, there exists, then there exists a x here satisfying, satisfying The problem, let us put this problem as P, P for all t in I, of course, which contains 0.

So, that is always there. Now, the thing is this. So, this is fine. But, you know, if A t, now see, for A t equals to t. yes, if A t is identically equals to A, let us say, so that is a constant matrix, yes, constant matrix, then we know, constant matrix, then we know that this solution will be given by x of t is nothing but e power A t times x naught, yes, but this is not true, so please remember, this is not true, not true in case of, in case of,

of variable coefficient but the thing is so Can we say something more if you have a variable coefficient problem? That is the question. So, what we are going to do now is to see, we are going to for this problem, for this video, we are going to consider. So, we are going to consider a much more general problem.

We are going to consider the problem. x prime equals to a t. Clear? So, a is a variable coefficient matrix, n cross n matrix plus b of t. Is this okay? Right? So, this is the problem which we are working with and let us call this problem as p prime.

Clear? Where we will assume that a And B have continuous entries. Have continuous entries. In some interval.

Yes. Continuous entries in some interval I. So, all the entries are continuous. That is what we are assuming here. Now, the thing is this. The question is this.

We know that there is a solution, right? We know that there is a solution for all T in I. That is fine. Yes? But the problem is this. If you put some more conditions on A, T. So, if one imposes, one imposes more conditions, conditions on A and B, on A and B.

Can we derive, can we derive something more? Something more. Is this okay? So, what I am trying to say is this. Can we say something more about the solution if we put, you know, more conditions on the coefficients?

Of course, we, I mean, that is just, I mean, our hope, right? That if we put some more conditions on A t, some nicer conditions on A t and B t. Then, of course, we hope that the solution will also reflect that generosity, right? So, what we are going to do here is we are going to look at linear systems which are periodic. I am going to explain to you what all of this means.

So, essentially, first of all, we start with the definition. What is the definition? So, this we will put it this way. A function y of x, clear? Some function y of x is called periodic, called periodic, yes, of period, of period, let us say w positive, okay.

if for all x in the domain, whatever the domain of y is, yes, for all x, y of x plus w is equals to y of x. Clear? Okay. So, geometrically what it means is the graph of yx repeats itself in successive intervals of length w. That is what it means. Clear?

Okay. So, it is basically repeating itself. And as an example, of course, you guys know that the sines, cosines, there are like periodic functions. And what is the period? 2 pi.

Okay. So, for example, we know that sine of x cosine of x yes see i am writing here x don't get confused or maybe i think it is better to put it this way so let me just change the definition as little bit let me write it as y of t yeah so you shouldn't don't get confused that's why otherwise it's not a problem everything is same so sine of t let's say or cosine of t they are all 2 pi periodic periodic right So what does it mean? It means the sine of x plus 2 pi is nothing but sine of x. Again cosine of 2 pi plus x is also going to be cosine of x. Now you see the thing is this.

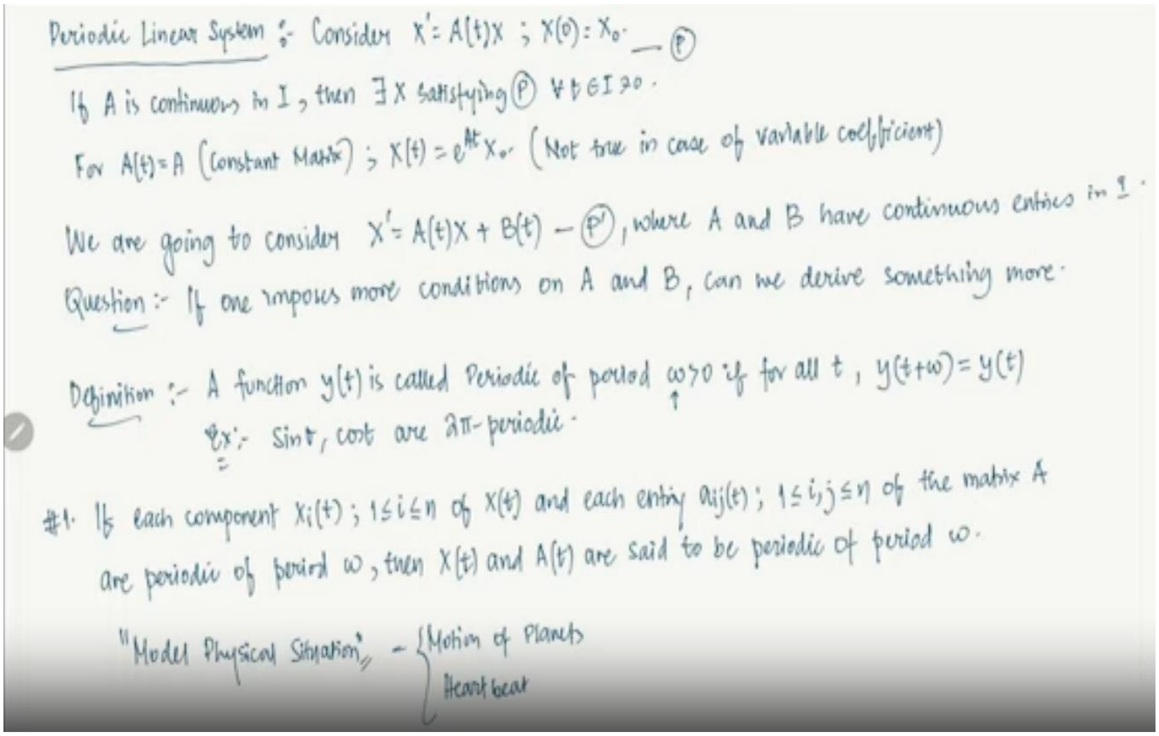
And of course, we are going to assume, you do realize that I can, this W is not unique, right? Okay. So, what we are going to do is for now, for our convenience, we are going to choose the W as the smallest possible such number for which this holds. Yeah. Okay.

So, that is what we are going to call that, this thing, the period. Now, the thing is this, let us say, if each component, component, So, this is kind of a remark 1 let us just put it. If each component xi of t, 1 less than i less than n here of x of t and each entry aij Okay, of T for 1 less than IJ between N, clear, of the matrix A, matrix A, clear, RPDAT, RPDAT.

of period 2 pi, oh sorry, of period w. Is this okay? Then, then x of t and a of x, sorry, a of t, clear, are said to be, are said to be periodic periodic of period W. Period W. Is this okay? So, essentially, what I mean by this is, you see, let us say A is a matrix, right?

And it depends on T, essentially. You see, A matrix has all these entries and all these entries depends on T. So, for every entry, it is a function of T. So, as a function of T, if that is basically periodic of some same period, the period has to be same, W. And again, the solution, let us say, the function which we were talking about, the unknown function X of T. That is also like all the components are like periodic function. Then we say that basically those are periodic of period W. That is just what you understand by period. Now you see periodicity is a very very important aspect of study.

yes why because if you if you think about it properly you see what does ode do when we study a system or if we study an equation essentially what we are trying to do is we are trying to you know model a situation right so that's the thing model physical situations physical situations now if you think about it Most physical situations which we deal with are generally periodic. So for example, let's say the motion of planets or sun, moon, whatever, right? The motion of planets. or let's say in biology our heartbeat yes heartbeat or let's say the growth of our hair let's say yes so as you understand most biological features or you know the let's say space science also motions of planets okay or pendulums and all all these motions are generally periodic motions clear okay so it is actually makes sense to study



the periodic systems okay and that's the idea of studying this okay so first of all we are going to do this theorem it's called the necessary necessary and sufficient condition and sufficient conditions for P prime, clear? What is P prime? X prime equals to A T times X plus B of T, clear? P prime to admit periodic solution, okay?

And I will put it like this, W periodic solution, W periodic solution. What does this mean? It means that it has a periodic solution of period W, right? W periodic solution. So let us give you the necessary and sufficient condition for this thing to happen.

So first of all this theorem. What is the theorem says? It says that let the matrix A of T and B of T are continuous. And W periodic. W periodic.

Okay. I hope you understand what this means. It means that let us say A is a matrix. N cross N matrix. Right.

So all the entries are essentially a function of B. They should be continuous and W periodic. And the same holds for B also. It is a N cross 1 matrix. And for that every entry has to be continuous and W periodic. Yeah.

In R. In R. Yeah. Then. p prime here, this p prime, this problem, p prime x prime equals to 80 times x plus bt has a periodic solution, has a periodic solution x of t of, has a W periodic solution, let us put it this way, has a W periodic solution x of t if and only if x of 0 equals to x of w. Is this okay? That is the theorem.

So, essentially what it is saying is this. Very, very expensive. See, think about it this way. Let us say p prime is this equation, right? x prime equals to a t times x plus b of t. Yes?

If a is periodic, B is periodic. You expect your solution to be periodic, right? Kind of, yes? So, that is what is happening here.

So, it is saying that if you have a solution, W periodic solution. So, basically what it is saying is this. If you have a W periodic solution, of course, this is happening. You see, this is the definition of periodicity. X of 0 has to be X of W.

Right. So there is nothing to prove here. But the other part is also true. So what it is saying is this. If x of 0 equals to x of w, if you can show just for two points, just the two points.

Yes. The curve is closed essentially. So, basically what you are showing is this. You are basically looking at the curve, right? And you are showing that the curve is closed.

So, basically at the point t equals to 0 and at the point t equals to w, the curve comes back to itself. If that happens, so if you have a closed curve, then the solution is going to be period. That is just the, you know, that is what you expect, right? So, if you have a solution which is like a closed curve, so what happens? It will again go on doing the same thing, right?

So, basically, it is going to be periodic. So, that is the idea essentially, okay? If you have the system which is, the system is periodic. So, basically, A and B are W periodic. Then, the solution which you are going to get, yeah, will be W periodic if and only if x is equal to x w, okay?

So, let us look at the proof. Of course, if x0 equals to, you see, if it is a W periodic solution, you see, if x is W periodic, let us say, yes, periodic, then x of t plus w equals to x of t for all t. So, therefore, for t equals to 0 also it is true and hence x of w equals to x of 0. So, that is just trivial. So, conversely, now the important part, conversely, what happens is this.

Let us say, let x of t be the solution of p prime. Is this okay? yes, and satisfying, satisfying, satisfying x 0 equals to x w, is this okay? See, now we are assuming that x is a closed, so essentially at the point 0 and at the point w, it is basically the same, yes, then we are going to show that if it is a solution, it has to be w period, right, yes, okay, so to do that, what happens is, If we define v of t is nothing but y of t, sorry, let me put it like y of t, it will be much better to put it this way, x of t plus w. If we define it like this, then that will imply, you see, since x is a solution of this problem, then y also solves this problem.

See, y prime of t is nothing but x prime of t plus w. Clear? This is just a chain rule, right? Chain rule. Now, if this is the case and what is x prime of t plus w?

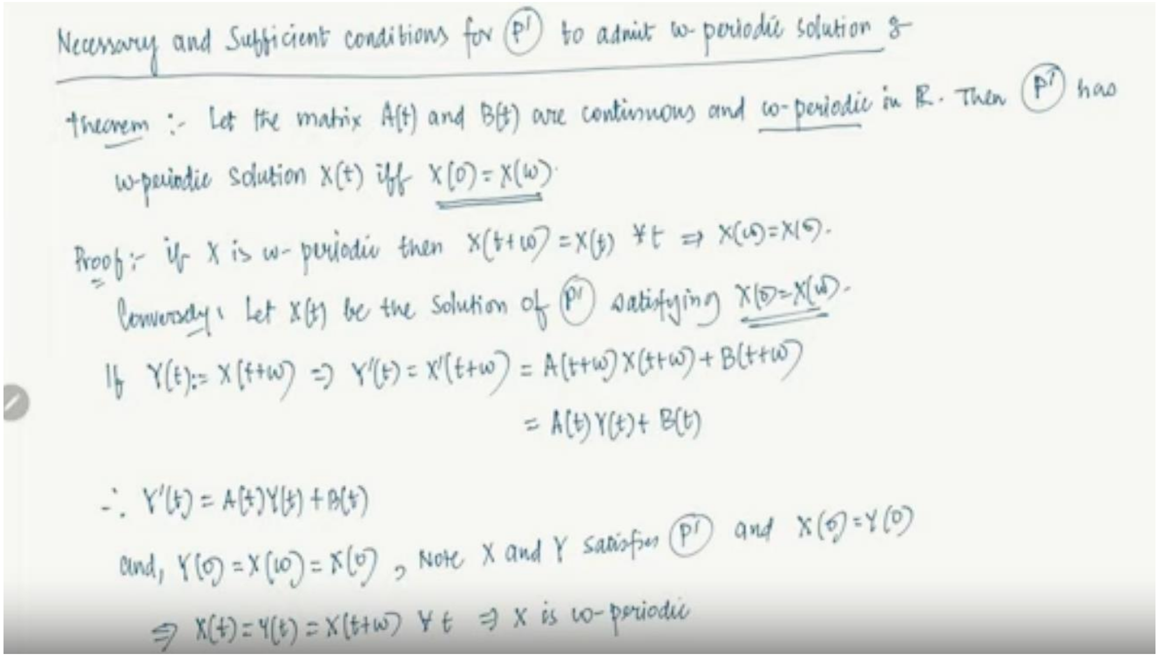
It is nothing but a of t plus w and then x of t plus w plus b of t plus w. I hope this is clear. Yes. See, if x is a solution of the problem, so it satisfies the equation for all points, right, in the interval. So, basically, this is t plus w is some point, right.

So, we can just substitute it by t plus w. Now, you see, this is what this implies is, since a is a and b, they are, you know, they w periodic it is given you see a and b are w periodic so a of t plus w is nothing but a of t and what is x plus t x of t plus w it is nothing but by definition it is y of t so it is y of t plus what is b of t plus w again it is b of t that is just the given okay since a and b are w periodic this happens so you see you have a problem such that so therefore what does y do y prime of t equals to a t times y of t plus b t. Clear? And moreover, and moreover, y at the point 0, what is it? It is x at the point w, right?

And x at the point w is given to be x at the point 0. So, that is what we have started out with, right? So, it is x at the point 0. So, you see, y and x both satisfies the exact same equation, right? Yes.

See, note, note, x and y satisfies the exact same equation. Satisfies the, satisfies p prime, that is. Exact same equation means p prime. Satisfies p prime. And at the point 0, x equals to x and y are essentially same.

What does that imply? Then, of course, since the solution is going to be unique, then it implies x of t has to be equals to y of t. There is no other way. And that will imply what is y of t? It is nothing but x of t plus w. This should hold for all t, right? in whatever interval t is, this should hold for all t. So, that is implies that x is w to t. I hope this is clear.



Okay. So, you see, you do not have to worry about periodicity. So, once you have the coefficients of periodic, you just have to check that there are two points, 0 and w. Yes. If x, you know, are same, x basically achieves the same value at the point 0 and w, then you have a periodic solution of exactly that w, w periodic solution. Yes.

Okay. So, now, we are going to look at a corollary of this theorem. So, let A of t be continuous and W period. Continuous and W period. W period in R. Here, further, let, let,

phi of T be the fundamental matrix FMR. So, basically it means it is a fundamental matrix, okay, of X prime equals to AT times X. Clear? Then, P prime has, sorry, not P prime, this problem. x prime equals to 80 times x okay has a non-trivial periodic solution has a non-trivial periodic solution okay uh actually w periodic i should write it as a non-trivial w periodic solution uh let me put it this way W periodic solution, x of t, if and only if determinant of x of 0 minus x of w is 0.

Is this okay? Yes. So, basically what it is saying is this, if you have a periodic system, so for now, if a and b are periodic, let us just call those systems periodic system. This is a corollary and this corollary is not about the inhomogeneous problem, but for a homogeneous problem. So, basically, let us say x prime equals to 80 times x is the equation given to us, right?

The homogeneous problem that is. And you have a fundamental solution psi of t corresponding to that. Now, you see what it is saying is this equation, of course, 0 is always a solution of this homogeneous equation that is given. But the thing is, you can also have a non-trivial solution. Yes.

And the solution will be periodic also. If and only if the determinant of x0 minus xw, the fundamental solution, evaluated at 0 and evaluated at w, you look at that matrix, if you take the determinant of that matrix, that has to be 0. Yes. Now, this looks a little complicated, but the proof is extremely easy. Let us look at the proof of that.

Proof. See, the thing is, for this problem, The general solution of x prime equals to a t times x is given by x g. Let us call it the general solution is nothing but if you remember it is psi of t times Psi is a n cross n matrix, right? Fundamental matrix times a constant C. And this C is a n cross 1 constant matrix.

Constant matrix. Constant matrix. Now, you see, what do we need to do? If this, you know, solution, if one of this solution is a W periodic solution, yes? So, see, if let us say x of t, g of t is W periodic for some particular constant, yes?

For some constant essentially, okay, is W periodic. Of course, that will give you if and only if condition, right? What is if and only if condition? If phi of 0 times c equals to phi of w times c, okay? This is always true.

Why it is true? Because of the earliest problem, you see. if this in this problem if you take b to be 0 essentially it's saying there is a w periodic solution problem has w periodic substitution if and only if x0 equals to xw right so this problem has a w periodic solution okay so let's say x is a w periodic solution this is that's a general solution but there's a w periodic solution if and only x of t equals to x of w x of t has to be x of w And that will imply phi 0 times c is phi w times c. What does that imply? That will imply the system phi 0 minus phi w times c is equal to 0 has a non-trivial solution.

non-trivial solution is this okay yeah why is this true because you see this w periodic solution which you have this is non-trivial we are assuming non-trivial okay so this system must have a p zero c minus p w c that system has to have a non-trivial solution clear Otherwise, this is going to be 0, right? So, it has to have a non-trivial solution. But for this system to have a non-trivial solution, what is the requirement? That will imply that, so it is both if and only if actually, that will imply the determinant of psi of

0 minus psi of w has to be 0. Otherwise, this system cannot have a non-tubule solution. That is just the basic, you know, system of linear equation stuff, right? If this system, the

determinant is non-zero, then this system has a unique solution, right? This system will have a unique solution.

And the unique solution will be given by c equals to 0, yeah? But since c the system has a non-trivial solution, yes, c naught equals to 0 essentially. So, basically, you know, there is one entry which is non-zero. So, hence, this has to be

This is okay. So, what did we prove? We proved that the homogeneous system, please remember this thing. This is called a homogeneous problem. The homogeneous system admits a non-trivial W periodic solution, Xt, if and only if the determinant of X0 minus Xw is going to be 0.

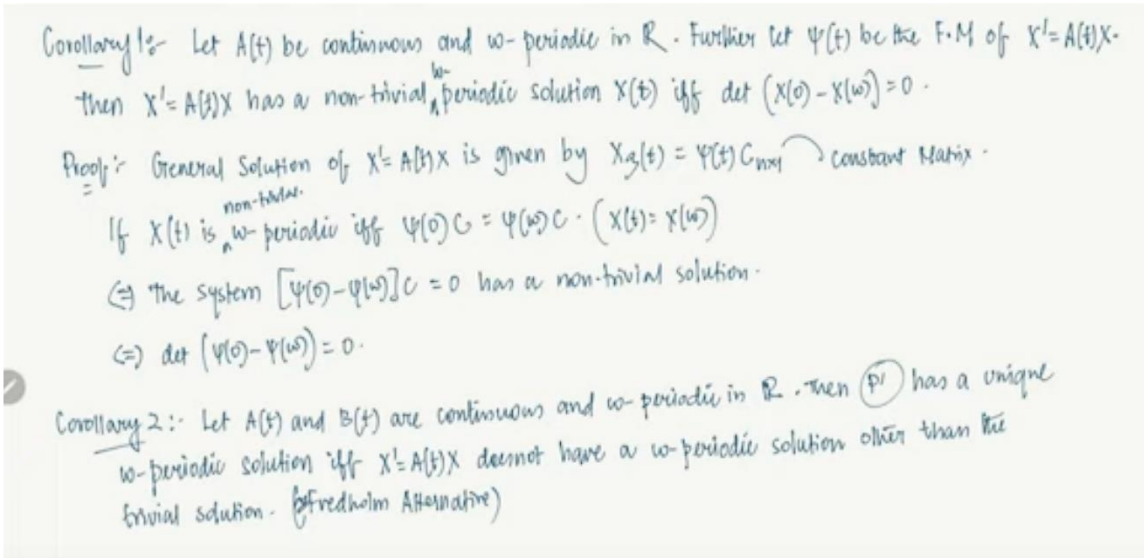
I hope this is clear to you. Now, let us look at another corollary. So, let us say this is corollary 1. Let us look at the corollary 2. So, what is it?

It says that let us say that A and B are continuous. So, let A of T and B of T are continuous and W periodic continuous and W periodic periodic. in R. This is what we are assuming. Yes, then P prime, the problem P prime, this problem, what is it? P prime is, where is the P prime?

P prime, x prime equals to 80 times x plus bt. Yes, this problem has a unique periodic solution, unique periodic solution solution, okay, W PDDX solution, W PDDX solution, if and only if, if and only if x prime equals to 80 times x, okay, does not have a PDDX solution other than the figure solution, okay. So, if this, if x prime equals to 80 times x does not have a periodic solution W periodic solution ok, does not have a W periodic solution other than the previous solution, the previous solution.

So it is a beautiful corollary actually. See what it is saying is this. We want to know that if you have a W periodic solution which is unique for the system, you have an inhomogeneous system and we want to see whether there is a unique W periodic solution. What you can actually prove is it is an if and only if condition. If the homogeneous system does not have a W periodic, of course, you see, the homogeneous system has the previous solution, which is always W periodic or any periodic for that matter, right?

So, that is not a problem. But the thing is, if it does not have a non-trivial W periodic solution, yes, then the original equation, the inhomogeneous problem has a unique W periodic solution, okay? So, these type of properties are called freedom alternatives. freedom alternatives okay so it's a beautiful theorem right it says that not you see this is kind of freedom alternative not exactly what kind of you can think of it like this see what it says is this please understand this x prime equals to 80 times x has always a trivial solution you don't have to worry about it and that is always created what it is saying is this If you can show that there is no other periodic solution other than the PBR solution for this homogeneous problem, then you can guarantee that the original inhomogeneous problem has a unique W periodic solution.



Corollary 1:- Let $A(t)$ be continuous and ω-periodic in $\mathbb{R}$. Further let $\Psi(t)$ be the F.M of $X' = A(t)X$. then $X' = A(t)X$ has a non-trivial ω-periodic solution $X(t)$ iff $\det(X(0) - X(\omega)) = 0$.

Proof:- General Solution of $X' = A(t)X$ is given by $X_g(t) = \Psi(t) C_{n\times 1}$ (constant Matrix).

If $X(t)$ is non-trivial ω-periodic iff $\Psi(0)C = \Psi(\omega)C$. $(X(t) = X(\omega))$

$\Leftrightarrow$ The system $[\Psi(0) - \Psi(\omega)]C = 0$ has a non-trivial solution.

$\Leftrightarrow \det(\Psi(0) - \Psi(\omega)) = 0$.

Corollary 2:- Let $A(t)$ and $B(t)$ are continuous and ω-periodic in $\mathbb{R}$. Then (P) has a unique ω-periodic solution iff $X' = A(t)X$ doesnot have a ω-periodic solution other than the trivial solution. (Fredholm Alternative)

okay beautiful theorem so you don't have to see if you have to say something about the inhomogeneous problem you don't have to study the inhomogeneous problem you study the homogeneous problem and if you can somehow guarantee that there is no w periodic solution you can guarantee that there is a unique w periodic solution for the inhomogeneous problem okay so the proof maybe i can do it in the next page so proof so let's say that let phi of t Yes, be the fundamental matrix, be the fundamental matrix of the equation x prime equals to 80 times x. Is this okay? Now, you see, you remember, if this is the fundamental matrix, then by Duhamel's principle, we have proved this, yes, Duhamel, Duhamel's principle. What do we have? We have that x of t for the inhomogeneous problem.

So, x satisfies the inhomogeneous problem now. x i is a solution of homogeneous problem is given by the solution of the homogeneous problem, which is psi t times a constant matrix C, n cross 1 matrix, constant, plus this part, t naught. So, I am taking it from, so let us say 0 to t, psi of t, psi inverse of s, b of s, ds. Yes, that is Duhammer's principle or we, I mean, I am not quite sure whether I told you that this is Duhammer's principle or not.

Essentially, we, I think we called it a variation of parameter. Okay, this is basically variation of parameter, but the principle is called Duhammer's principle. Okay, variation of parameter formula. So, we have this particular function is a solution of P prime, right? Yes, in terms of the fundamental matrix.

Now, see that if this xi of P is W periodic, is W periodic, PDD. When is this W PDD? That is the question. See, xi of t is W PDD if and only if you remember x at the point 0, xi at the point 0 equals to, sorry, xi at the point W. Is this okay?

So, what does that imply? That will imply that phi of c. See, what is xi at the point 0? It is phi of 0 times c. And here, if you put 0, this term is not there. So, this is equals to phi at the point w times c plus 0 to w phi at the point w phi inverse at the point s b of s ds.

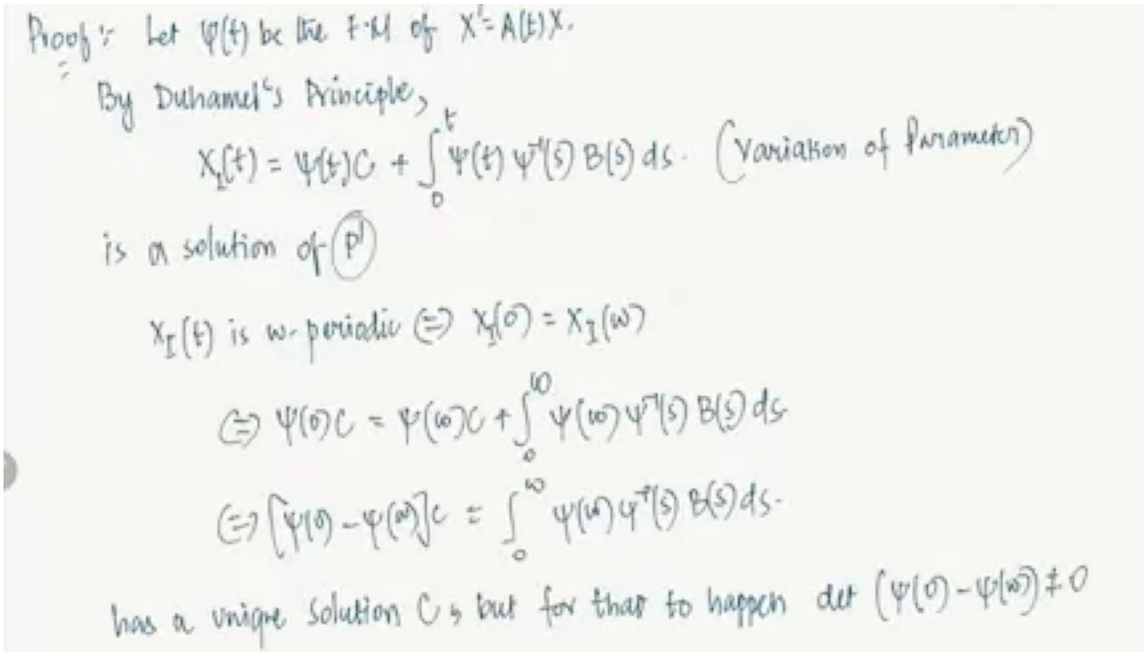
That is what we are going to get, right? Okay. So, that is an if and only condition again. Now, and what does that imply? That will imply if and only phi of 0 minus phi of w. That is the system, right?

That is the system of linear equation. This is nothing but this. Phi of w, phi inverse of s, b of s, ds. This is okay. Now, you see, the thing is, this system has a unique solution c, right?

See, for this thing to happen, this has a unique solution C. Yes? Has a unique solution. Unique solution C. Okay? But for this system to have a unique solution, you need the determinant to be non-zero. Right?

But for that to happen... to happen, determinant of phi 0 minus phi w, that has to be non-zero. Is this okay? So, now, of course, the theorem holds, see, the theorem says that if this has, does not have a W periodic solution, then P prime has a unique solution, unique solution, right? So, you see, the thing is, we use this, we use this.

See, determinant of phi 0 minus phi w is 0. If this happens, then there is a non-trivial W periodic solution of this system. Is this okay? What we showed is determinant in this, for this problem, determinant of psi 0 minus psi w is not 0. Is this okay?



Yes. And the thing is, since this is non-zero, that will imply that the homogeneous equation does not have a non-trivial W periodic solution. Yes. So. you see, so you see, this is an informal information.

So, x prime equals to 80 times x does not have a W periodic solution. So, basically, if it has a unique W, this thing, if the P prime has a unique solution, W periodic solution, yes, then determinant of psi 0 minus psi of W has to be non-zero. Is this okay? I think this is quite clear here. Fine.

So now what we are going to do is very, very important. This theorem, yes, will actually, we can use a lot this theorem and for our purposes, you know, for studying various differential equations, this theorem is called a Flocke theorem. Flocke theorem. Okay. I will give you an application of Flocke theorem also.

We will see why Flocke theorem is so much important. Okay. So, let us say that let us consider again. Let A of T and B of T. B of T are continuous and W periodic. That is given.

Continuous and W periodic in R. W periodic. In R. Is this okay? Okay. Then the following holds. Then the following holds.

Holds. Okay. What is it? Number 1. The matrix.

Matrix. Let us call it pi of t. Okay. Which is defined by psi of t plus w. Okay. here is also a fundamental matrix, is also a fundamental matrix of the system x prime equals to 80 times x. Is this okay? And to

They are exist. This is important. So, first of all, what am I saying? I am saying that if phi of t is a fundamental matrix, phi of t plus w, that is also going to be a fundamental matrix of the same system. Also, you will have a periodic non-singular matrix P, non-singular matrix

P of T. Okay. Of period W. Of period W. Clear. And the constant matrix R. And a constant matrix. R. Such that. You see.

This. Phi of T. Right. The fundamental matrix phi of T. Can be written as. P of T. Times e power rt.

Is this okay? So, what we are saying is this. See, that you can actually break down the fundamental matrix. Okay. See, the first part actually we need it for the second part.

That is why we just wrote the first part. It is not very special. The second part is the special part. It says that any fundamental matrix P of t. Clear? So, you understand?

What is your system given? x prime equals to at times x plus bt. This is the system given to you, clear? And what it is saying is this, if this is continuous and W periodic, this is also continuous and W periodic, clear? If that happens, then the fundamental matrix of which system, the homogeneous system can be written as phi of t looks like p of t plus, okay, one thing, you do not need p here, sorry, you do not need p here, yes, why am I writing p here?

It does not need to have any p, okay, let us Let me put it this way. We do not need to have B continuous. What it is saying is you can actually write the fundamental matrix P of t as P of t times exponential RT. Is it okay?

Exponential RT. So, you see what is the proof? And what is the speciality of P of T? P of T will be a periodic matrix, W periodic matrix and R will be a constant matrix. So you can write it as E power RT which is like a constant matrix times T. So basically you can calculate E power RT and then P of T will be a non-singular matrix which is W periodic.

So what is the proof? So since P of T okay, is a fundamental matrix, is a fundamental matrix of the differential equation x prime equals to a t times x, right? That is what we are assuming. Then that will imply that phi prime of t, it will satisfy the matrix differential equation, you remember that, right?

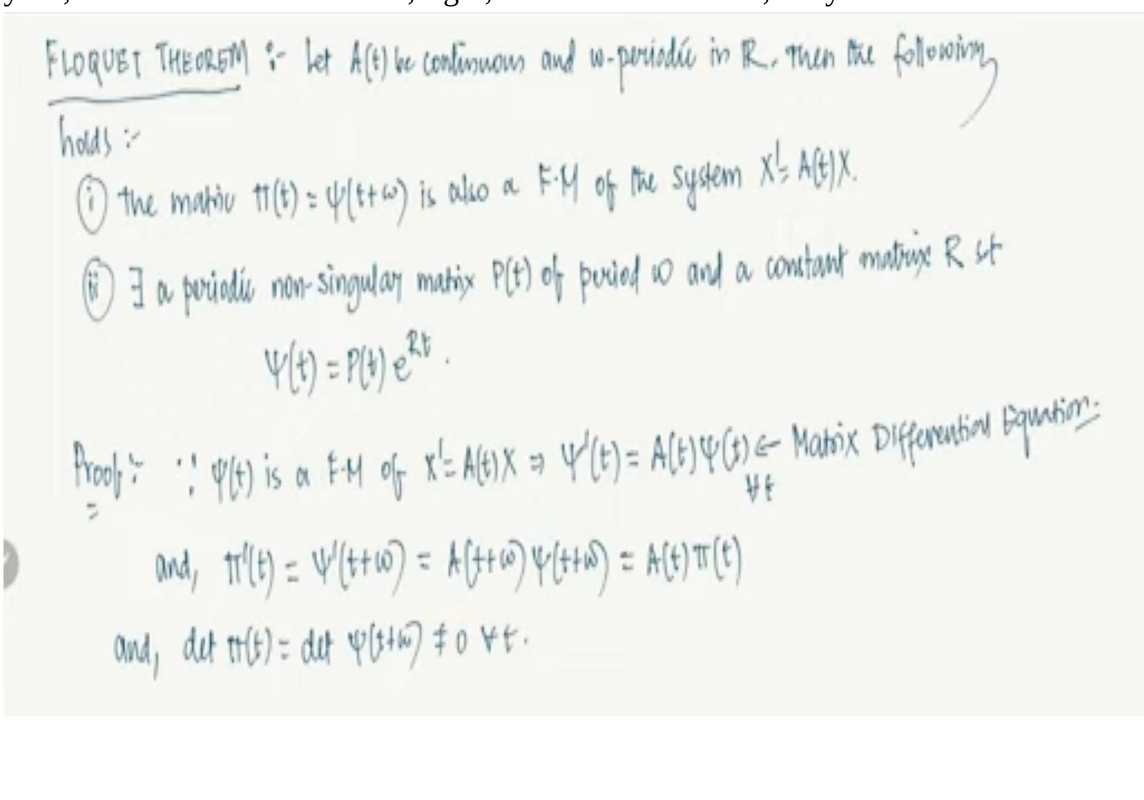
So, one second, sorry, yeah, phi of t will look like a t times phi of t, Please remember this is matrix differential equation, not your usual one. Matrix differential equation. Differential equation. okay see phi prime t is also a bunch of matrix entries okay which is a n cross n matrix and a is a n cross n phi is n cross n so basically both sides is a n cross n matrix this is a matrix differential equation not your easier one please remember that clear okay now you see what happens to p and let's look at what is pi prime of t see if it is a fundamental matrix

Yes, then pi prime, pi will satisfy the equation, right? So, what is pi prime of t? It is nothing but phi prime of t plus w, right? Chain root and which is nothing but a of t plus w. You see, phi is a fundamental matrix, right? So, it satisfies the fundamental matrix differential equation.

So, this works for all t. So, it works for t plus w also. So, we just replace it by t plus w. Clear? Now, you see, if A is W periodic, then A of t plus W is nothing but A of t. And what is phi of t plus W? It is nothing but pi of t. Okay?

So, pi is another matrix which satisfies the matrix differential equation. Yes? The homogeneous equation, that is. Yeah? And, of course, the determinant of pi t is nothing but the determinant of

this matrix, right, psi of t plus w, yeah, and that is of course going to be non-zero for all t, yeah, it is a fundamental matrix, right, it has to be non-zero, okay.



FLOQUET THEOREM :- let $A(t)$ be continuous and ω-periodic in $\mathbb{R}$. Then the following holds :-

(i) the matrix $\pi(t) = \psi(t+\omega)$ is also a F.M of the system $X' = A(t)X$.

(ii) $\exists$ a periodic non-singular matrix $P(t)$ of period ω and a constant matrix R s.t

$$\psi(t) = P(t)e^{Rt}.$$

Proof :- $\because$ $\psi(t)$ is a F.M of $X' = A(t)X \Rightarrow \psi'(t) = A(t)\psi(t)$ $\forall t$ $\leftarrow$ Matrix Differential Equation.

and, $\pi'(t) = \psi'(t+\omega) = A(t+\omega)\psi(t+\omega) = A(t)\pi(t)$

and, $\det \pi(t) = \det \psi(t+\omega) \neq 0 \ \forall t.$

So, what we have is therefore pi is a fundamental matrix, is a fundamental matrix of the homogeneous problem x prime equals to a t times x i hope this is clear now okay now the second part and this is the important part yes how do you find you see what the second part is saying is this is important in this way we can actually break the fundamental matrix in terms of a non-singular periodic matrix and then an exponential part yes okay how do you do something like this so let's look at it okay so for the second part part two See, the thing is, since phi of t and phi of t plus w, this is what? This is pi of t, right?

So, they are both fundamental matrices. Both fundamental matrices. I hope you remember the properties of fundamental matrices. Then, phi of t plus w can be written as phi of t. times a c for some constant c c is a non-singular constant matrix constant matrix matrix

Is this okay? See, again, please remember this thing. We proved, when we talked about fundamental matrices, we told you, we said that any fundamental matrix, see, if phi is a fundamental matrix, pt times c is also a fundamental matrix, where c is a non-zero, non-singular constant matrix, right? And we also showed that any fundamental matrix doesn't really matter, but any fundamental matrix will look like pt times c. There is no other way. So, since phi of t plus w is a fundamental matrix,

So, P of T plus W can be written as P of T times C. Is it okay? Now, you see, what we are going to do is this. What, this C is a non-singular constant matrix, right? So, now, we will do this. Then, there exists a constant matrix, constant matrix, matrix R such that

such that c is non-single, right? So, c can be written as e power rw. What is w? w is the period, okay? c is a constant matrix.

So, we can write it as a exponential of rw. This can always be done, yes? If you are not convinced, you just please do it yourself, yes? So, probably I am going to put it as a part of our assignment. So, that is it.

So, what we are going to say is this C. C is a non-singular matrix. So, what we can do is if you have a non-singular constant matrix, then you can always have a constant matrix R such that C can be written as e power R. And here that W is some number. So, it is not a problem. It is just a real number. So, we can just put W there.

So, this theorem says that if you have a non-singular matrix then there exist another matrix R such that the non-singular matrix C can be written as the exponential of that this given matrix. So, now once this is true therefore, what do we have is this C phi of t plus w is nothing but phi of t times C. What is c? It is e power r w. Is this okay? Now, we define a new function.

Define a new function which is p of t. This is a matrix, matrix function that is. And this is defined like this. This psi of t, fundamental matrix psi of t, then e to the power minus r t. This is okay. This is how we define it.

This is just a trick, nothing special here. We just want to put it in that form, that is why. This is just a trick. So we define a new function p of t, of course it is a matrix, which is given by p of t times exponential minus rt. You can do that.

Therefore, you see, what is p of t plus w? Let us look at it. What is p of t plus w? It is nothing but psi of t plus w e to the power minus r t plus w. We can of course write it and what happens is this, see this is nothing but psi of t plus w, psi of t times c. So it is this is psi of t times c is nothing but e power rw and this is nothing but e power minus rt times e to the power minus rw.

Clear? We can write it like this. Why? Because you can actually see that they are going to commute. So, we can write it like this.

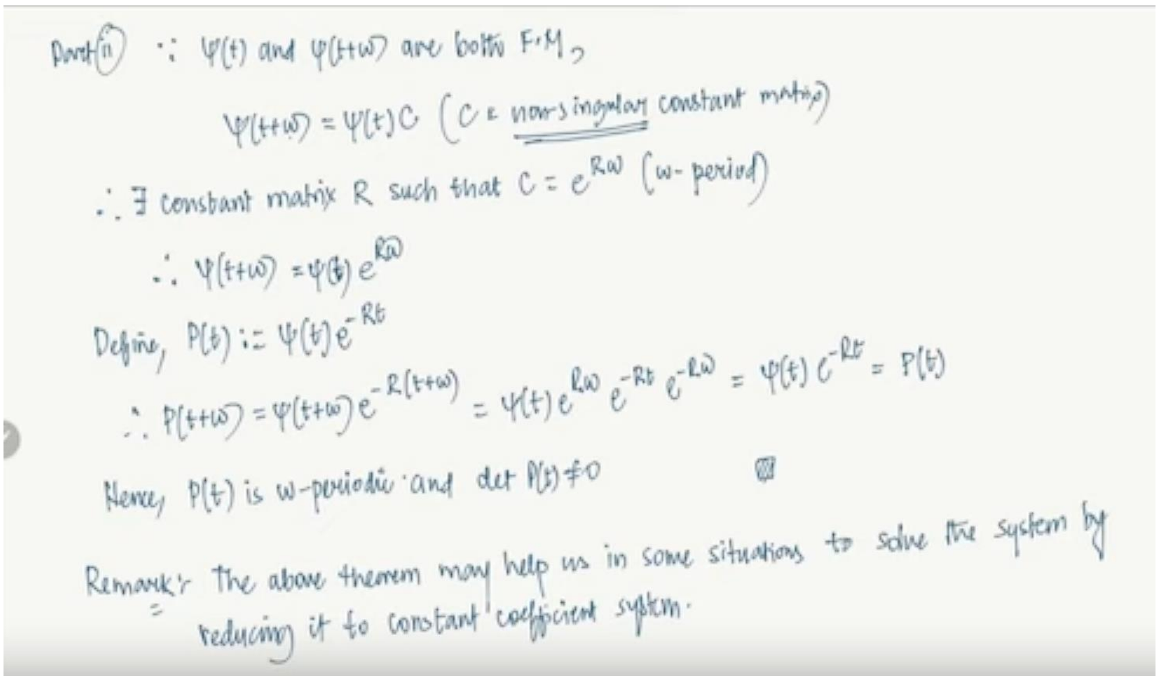
And then, this is nothing but phi of t times e to the power minus r t. Is this okay? And then, this is nothing but p of Clear? So, you see this matrix, the matrix which we define P of t, which is P of t times e power minus r t, that is actually W period. Therefore, hence P of t is W period.

Clear? Now, of course, phi is non-zero. Exponential that is non-zero. So, determinant and determinant of P of t that is also non-zero. It has to be because phi is a fundamental matrix.

The determinant is non-zero. Exponential determinant is of course going to be non-zero. So, determinant of a b is determinant a times determinant of p so it's non-zero right so we can just write it like this here okay so the beautiful part about this is once this is done you see then phi of t can be written as e power phi of t can be written as p of t times e power r t okay so hence we proved out see phi of t can be written as p of t times e power r t hence we proved our theorem here now the thing this is the important part okay Why are we suddenly interested in this?

And that's the beauty here. See, the thing is, sometimes what happens is this. You can actually reduce. So, this theorem, the above theorem, remarked. Let me put it as a small remark.

The above theorem may help us. may help us in some circumstances in some situations that is to solve the system to solve the system okay by reducing reducing it to a constant coefficient system see If you remember, constant coefficient systems are relatively easy. We can easily solve them, right? So, if some, there is, let us say, if you can find out that there is some transformation which actually reduces, you know, this variable coefficient system to a constant coefficient system, then that is going to be very helpful, right? That is what this theorem, actually, this decomposition does. Flocke theorem, this decomposition does, okay? We will see. Okay. So, what is that?



Let us look at the theorem there. Theorem. let p of t and r okay so what is p of t and r p is w periodic r is just constant non-singular matrix okay you remember we just do it delete here you see it's a constant matrix r and p is w ready okay so we are just assuming that so let's say that those are the are the matrix are the matrix such that we can write it the fundamental matrix P of t is nothing but P of t times e power rt. Then the transformation, this is very important.

So, you can have a transformation. What is the transformation? So, let us say you are given a fundamental matrix. You can of course decompose it into Pt times e power rt. That is your property theorem, right?

Once that is true, then the transformation, this transformation, what is it? Y of t given by P of t times W of t reduces the system. What is the system? X prime equals to 80 times x to the system. You see, this homogeneous, see, once we know how to solve the homogeneous problem, we can of course go and use Duhamel's principle to solve the homogeneous problem, right?

That is always done. So, we just have to somehow work out the homogeneous problem. What happens is this, it is saying that you have a transformation. You can have this transformation. Y is given by, what is it?

What did I write? But, anyways. I should write it as x maybe. It will be better. So, sorry, what is y?

I have to write y in another case. So, let us just write it as x of t. Let us just do it this way. Otherwise, I have to again go through that y is a solution of that problem. Anyways, reduces the system this to the system. What is the system?

w prime equals to r of w. You see that the in the variable coefficient system actually gets reduced to a constant coefficient. So how do you prove it? Let us look at the proof. So since see phi is a fundamental matrix.

So what is it? Then phi prime of t we know that this is nothing but a of t times phi of t. And you see If you look at that, this theorem, this relation, phi of t, and since phi of t is nothing but p of t times e power r t, e power r t, then you see, we can take the derivative, right? So, you see, therefore, p of t times e power r t, the derivative of that is nothing but a of t Right.

And what is p of t? It is pt times e power rt. So, it is pt times e power r times t. Clear. What does that imply? That will imply that first of all it is p prime of t e power rt.

This is from the first assignment we are using here. Plus p of t e power rt. r e power rt. Equals to at times pt e power rt. Is this fine?

Now you see what happens is this. I can write it like this. It is P prime of T plus P of T times R minus A T times P T. Clear? Times e power R T is going to be 0. Is this okay?

Now, you see, if you use this transformation, so basically what is this transformation? x is given by Pt times Wt, okay? So, using the transformation now as x of t is nothing but Pt times Wt, okay? So, see x, why I wrote y there, basically x solves this problem, right? Okay, so you take that solution x and then you write it as x equals to pt times w, that is what the idea is, okay?

So, see x solves this problem, right? So, what happens, what I am trying to say is this, p, so okay, let me do it this way. x of t is pt times wt, right? wt. So, what does that give us?

x prime of t is nothing but p prime of t wt and then p of t w prime of t. Now, what does that imply? What is x prime of t? It is nothing but at times x of t. at times x of t. x of t is nothing but pt times wt, clear? That is nothing but p prime of t wt plus p of t w prime of t. That is what we are going to get, right?

So, if you put everything together, see what we are going to get is this. That will imply that p of t p of t and then w of t okay plus one second let me just check this part p of t w of t w prime no p of t, w prime of t, sorry, plus p prime of t, w of t. Okay, once again, I did some mistake somewhere, I think. So, this expression is there.

Okay, this is p prime is there, right? So, if p prime is there, okay, yeah, fine, fine, fine, okay, fine, fine, yeah. So, this will imply that this is basically nothing but it says that p of t okay, and w prime plus, so, w prime of t, that is, w prime of t, sorry, w prime of t, so, p of t, w prime of t, this particular thing plus p prime of t, okay, minus a of t, p of t, okay, and times w of t, This is going to be 0.

That is what we are going to get, right? If you just write it, if you just put it together, that is what you are going to get. This part we are just taking on the other side here. So, it is P prime minus A times P times WT equals to 0. Sorry, I just got confused a little bit.

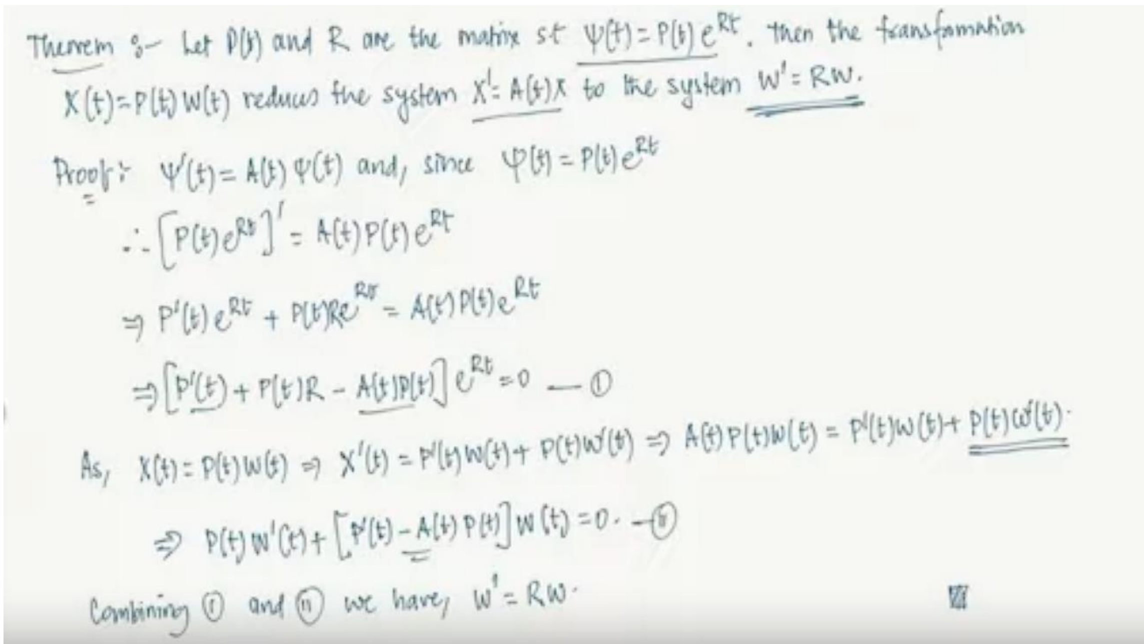
So, now you see, let us call this as 1 and let us call this as 2. If you combine 1 and 2, you are going to get your result. So therefore combining 1 and 2, what you are going to get is your result which is we have. We have, you see, this is P prime minus AP, that is 0, right? So, what happens is, we have that W prime, W prime.

See, P prime minus AP is minus P times R, right? So, you just replace here minus P times R, okay? So, and P is non-zero. P is a non-zero matrix. So, basically, it becomes W prime minus RW equals to 0.

That's what, okay? So, then that will imply W prime equals to RW. I hope this is clear, okay? So, hence proved. So, what did we show?

We showed that if you have sufficient you know if there is like So, if A is continuous and it is W periodic, then Flocke's theorem says that you can just take this, you know, break it up into a exponential matrix times a periodic matrix, a periodic matrix times a exponential matrix. And then using that, you take a transformation which is x equals to P times W and you reduce the original equation to a, which is a variable coefficient equation. This is variable coefficient, you reduce it to a constant coefficient equation. Yes, once you do it, now you see what happens is this.

Let us say if I give you to solve this variable coefficient equation where a is nice, let us say all these properties are satisfied, this is W quadratic and everything. Then what happens is if you can just break it up into this p times e power rt, then you can actually know what r is. you solve w times because of rw, which is easy to solve because this is a constant coefficient equation, this is e power rw times some constant, right? So, you just calculate that. Once you get it, you get what the solution w is, you just put it with the p. So, p times that solution will be your x of t, which satisfies the equation and you are done, yes?



So, with this, I am going to end this video. Bye.