Ordinary Differential Equations (noc 24 ma 78)

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Week-04

Lecture-19: Higher Dimensional Matrix Exponential 2

So welcome students to this video and in this video we are going to continue our study of you know exponential of a matrix. So essentially what we looked at in the last video is if you have a matrix A given and with a distinct eigenvalues. What are the eigenvalues? Lambda i's and alpha p plus i beta p. So and where i ranges between 1 to k.

And you know d1, d2, dp. So, p lies between 1, 2. Let us just write it as that is n cross n matrix. So, n minus p. Now, what we are trying to suggest here is this. See that if

A has, A may be a complicated matrix, right? But if it has distinct diagonal values, then you can have a change of variable, yes? So, this actually determines the change of variable T, such that T inverse A, T will look like, like an, you know, very much like a diagonal matrix. May not be fully diagonal, because you see this D1, D2, DP, these are like block matrices, okay? But more or less, very much like a diagonal matrix, okay?

Now this form, so let me make a small remark here. The form which we got, this form, the form, the above form, form is called, is called the canonical form of A. Canonical form Of A. What does it mean? It means that you know you can always have a change of variable which actually reduces a very complicated looking matrix to a very like a diagonal matrix. Okay.

Right. Now the question is this. Why are you suddenly interested in canonical form? Why are we interested in canonical form? Right.

Interested in the canonical form. Now, we already know that form. You see, we know that x prime equals to a, x, and x0 equals to x0. Let us just look at this equation. Yes, this equation admits a solution, right?

Admits an unique solution. An unique solution. Where unique solution? Unique solution. And what is the unique solution?

It is given by x t equals to e to the power a t times x naught, right? That is the unique solution. So, this holds for all t in R, right? Yes. Now, you see, if we want to explicitly calculate what the unique solution is, so basically e power a t, if we can calculate that part, then you just multiply it by the vector x naught and we are done, right?

Theorem :- Let A has distinct eigenvalues. Then $\exists$ a matrix $T_{n\times n}$ s.t

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_k & & & \\ & & & D_1 & & \\ & & & & \ddots & \\ & & & & & D_p \end{pmatrix} \quad \text{where } D_p = \begin{pmatrix} \alpha_p & \beta_p \\ -\beta_p & \alpha_p \end{pmatrix}.$$

where λ_i, $\alpha_p \pm i\beta_p$ are the eigenvalues of A. $(i=1,2,\dots k,\ p=1,2,\dots,(n-p))$

Remark:- the above form is called the canonical form of A.

Question:- Why are we interested in the canonical form.

We know that, $x' = Ax$; $x(0)=x_0$ admits an unique solution $x(t) = e^{At}x_0 \ \forall t \in \mathbb{R}$

Okay. So, how do you calculate e power a t? So, how to calculate e power a t? That is the question, right? How to calculate the exponential?

Calculate e power a t, right? That is the question, right? Now, you see, if a has distinct eigenvalues, okay? So, case 1, so case 1, case 1, if a has distinct eigenvalues, has distinct eigenvalues, then what happens?

So, a admits distinct eigenvalue, distinct eigenvalue. eigenvalues. So, you see, A can be written as, so there is a T such that T inverse A T can be written as this diagonal matrix. So, you see what happens is this. Therefore, and let us just call this matrix as B for now, the diagonal matrix.

Let us just call it to be B. So, B is a nice matrix. Let us just put it this way. So, you see, therefore, E power A T, how does it look like? It is nothing but, you see, If T inverse A T is B, so you see, if T inverse A T, so there is a T such that T inverse A T is B, right?

B is like a very much like a diagonal, almost diagonal, let us just put it this way. So, you see what is A? A is T, B T inverse, right? A is B T and then what is e power A T? It is nothing but e power T B T inverse times small t right so and we know that you know when we were doing matrix exponential we talked about this property of matrix exponential it is nothing but t e power b okay t and t inverse this is what we are going to get right so you see t and t inverse once you If you can, if one can, you know, find out what t is, what t inverse is, okay. So, you just have to calculate what e power bt is, yeah. So, once you know what e power bt is, then you can just put everything together and you can calculate e power at, okay. Now, the main question is this, how do you, how does one find e power bt?

That is the main question, okay. And then we will solve one problem and I will show you how to get the answers. Okay. So, now let us look at this. See, the thing is, what, how to find, see, why this canonical form is important.

What is e power bt? See, b looks like what? For example, without going into like a general case, let me give you an example that will make everything clear. Okay. So, next example.

See, if everything is real, real distinct, lambda 1, lambda 2, lambda k, then you can of course understand what is going to happen. So, let me do it for lambda 1, one real distinct and two complex conjugate. So, basically for a 3 cross 3 matrix, let A, which is a 3 cross 3 matrix, admits the eigenvalues, eigenvalues. alpha, okay, and beta plus minus i gamma, clear, okay. So, three eigenvalues are there, one is alpha and one is beta plus minus i gamma, those are the three eigenvalues, okay.

Of course, alpha is in R, yeah, of course, that is assumed. Now, you see what is happening is this, in this case, what is the, you know, the T inverse H, that is B. So, in this case, therefore, B will look like this, see, it will look like Alpha, 0, 0. So, the first corresponding to alpha, it will be alpha, 0, 0, right? Now, corresponding to beta, you see, it will be alpha, beta, minus beta, alpha, okay?

So, sorry, this is beta, gamma. So, beta, gamma, minus gamma, beta. So, beta, beta, clear? And then, what is it? This is gamma, minus gamma.

gamma minus gamma. This is going to happen. Is this clear? I hope this is clear. So, in this case, this is the matrix which you are going to get after the, you know, this is almost diagonal matrix which we are talking about.

Now, you see, what exactly does it mean? If this is the matrix, let us just say this is the matrix. Now, one way of looking at it is this. Let us say, That you are looking for this problem, right?

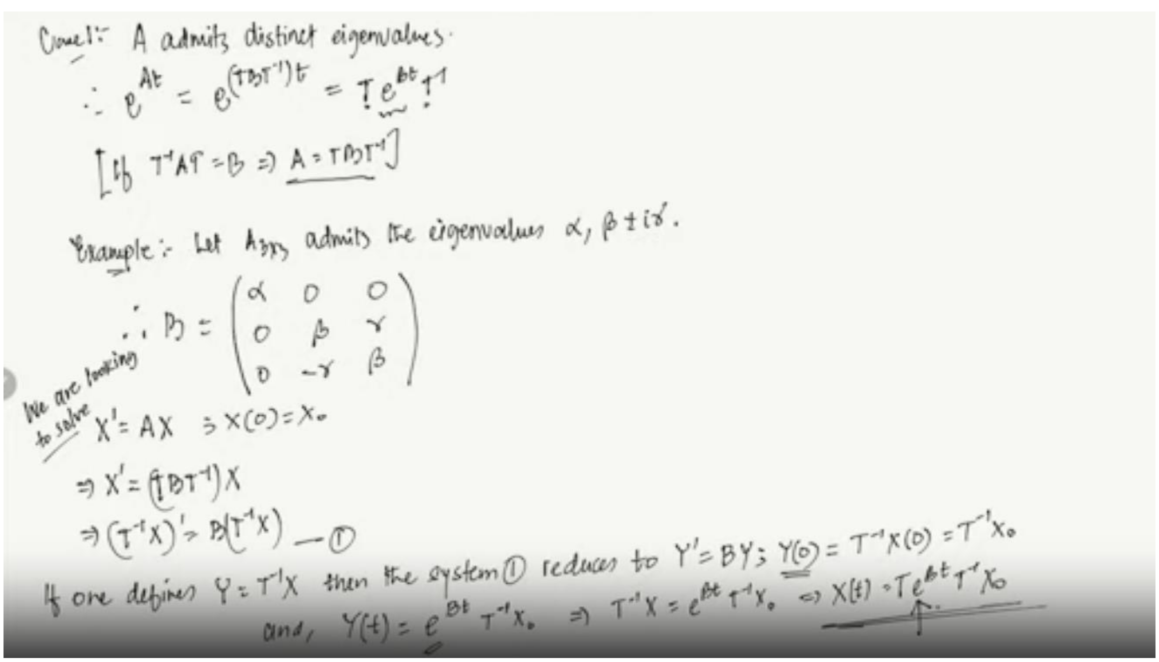
x prime, we are looking to solve this problem. x prime equals to x. We are looking to solve. To solve. Clear? Now, you see, what does it mean?

x prime equals to a. What is a? a is nothing but, you see, a is nothing but t b t inverse. If we can do something like this, then it is nothing but t b t inverse times x. That will imply that t inverse times x whole prime is nothing but be T inverse of x, right? See, if we take this T on the other side, it is T inverse.

And since T is basically a constant matrix, I can take it inside the derivative, okay? Now, if one defines y to be nothing but T inverse of x, then let us call it 1. Then the equation, then the system 1 reduces to reduces to what is it reduced to y prime equals to by yeah and you see the initial data is x 0 equals to x naught okay so in this case y 0 y 0 will be t inverse of x naught so t inverse acting at x0.

Clear? Now, any solution of this problem, you see, and yt, what is yt? yt is nothing but e power bt times y at the point 0, which is t inverse of x0. Clear? Okay, so you see if we can calculate what is e power bt and of course if we know what is t then we can of course know that what is y and then once we know what is y then we know what is x because y is nothing but t inverse of x. So in this case then you see if y of t is this then that will imply

that T inverse of x is nothing but e power bt times T inverse of x naught, right? So, that will imply x of T is nothing but T e to the power bt T inverse acting at x naught. So, you understand. So, it is very important to calculate e power bt. You see, this is the same thing which we did here, okay?



So, that is the idea. Now, Another beautiful thing about this canonical form is this. See, this, if we can reduce this form, what happens is solving, so let us say we want to solve, we want to solve this, we have, or we want to solve, we want to solve, solve y prime equals to by, right? So let us just call, let us just call y to be y1,

y2, y3, 3 cross 3, right? Okay, so it means y1 prime, y2 prime, y3 prime is nothing but that matrix, right? What is the matrix? You see, it is alpha 0, 0, alpha 0, 0, 0, beta, gamma, 0, minus gamma, beta, right? And then y1, y2, y3.

This is what we are getting. Now, if you write down properly, it means y1 prime. That will imply that these are the three equations which you are going to get. y1 prime equals to nothing but alpha times y1. y2 prime is nothing but beta times y2 plus gamma times y3.

And y3 prime is nothing but minus gamma times y2 plus beta times y3. Now, the first equation is very easy to solve, right? So, let's call it 2, 3, 4. So, from 2, what do we get? From 2, one gets y1 of t is nothing but e to the power alpha t, right? Of course, our constant is there.

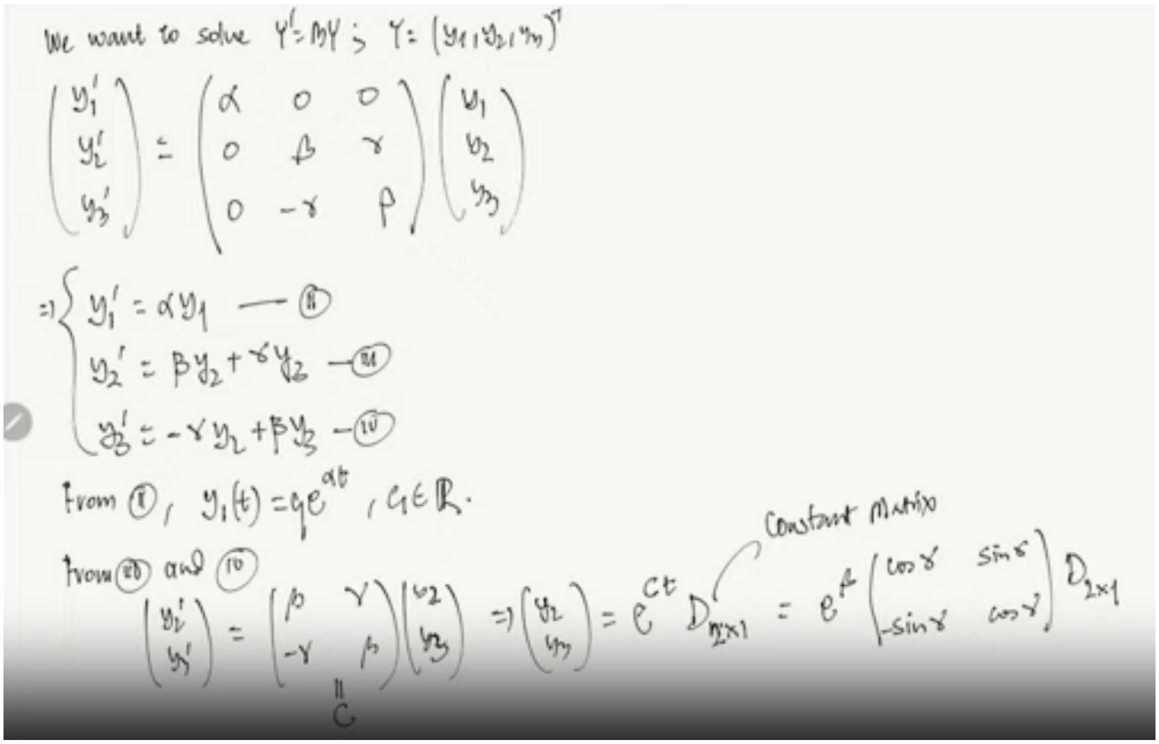
So, let us just write it as c1 times e to the power alpha t as c1 is in R. And from 2, sorry, 3 and 4, from 3 and 4 and 4, You see if you just think about it this way, this is a system of 2 cross 2 equation. There is no y1 involved here. So you have this system y2 prime y3 prime equals to nothing but beta gamma minus gamma beta y1 y2. So what is the solution here?

sorry, y2, y3, y2, y3, okay. So, the solution y2, y3, this will be nothing but e power, this matrix, right, let us call this matrix as c, e power c times, okay, e power c, one second, c times t, right, that will be the solution, yes. So, if we Times, of course, that constant matrix, let us just call it d, which is 2 cross 2 constant matrix. Constant matrix.

Constant matrix. Sorry, it is not 2 cross 2, 2 cross 1. It is just a constant part that I am writing. Now, you see e power ct. Now, if you remember, we talked about this sort of equation, right?

Beta, this sort of exponential, beta gamma minus gamma beta. And we saw how to calculate that exponential, okay? If you remember how to calculate that exponential, what we found out is this, from that, you know, that when we calculated that exponential essentially, is this is nothing but e to the power... beta, cosine gamma, sine gamma. sine minus sine gamma and cosine gamma.

This is what we got times a matrix D, D, 2 cross 1 matrix constant, okay? So this is what we are getting. If you remember, we did it in the, I mean, maybe another two videos back, we just calculated this thing, okay? Again, if you are not convinced how we got from here to here, let us just do it in another way.



That will be a more easy way. But I am not going to calculate the exact same thing. What I am going to do is I am going to show you, I mean, how to do this part and then you can do it yourself. This is another method of looking at the exponential of this particular thing. So, you see, let us say that let a matrix P is nothing but alpha, alpha,

Okay, and gamma or beta, let us just call it beta, beta, beta, gamma, minus gamma, right? That is the matrix given to us, you see, this matrix. I want to calculate the exponential, yes? Now, how do you calculate? You see, I can always write it as beta 0, 0, beta 0.

Plus 0, gamma, minus gamma, 0. Right? That will be beta times identity plus gamma times this matrix. 0, 1, minus 1, 0. Right?

So, you see, therefore, if we want to calculate e to the power p, let's say. Okay? What is it? It is e to the power beta times e to the power 0. 0, gamma, minus gamma, 0.

This matrix, right? Okay? Beta i, sorry, e to the power beta i times this. Okay? Now you see, what is this matrix?

So e to the power beta i is very easy to find out, right? What is e to the power beta i? It is nothing but identity plus beta times identity square i plus beta times beta square by 2 factorial. Okay? i squared and it goes on like this times e to the power 0 gamma minus gamma 0.

Now, this is nothing but exponential beta. This is exponential beta i. How do I put it? So, let me write it this way. See, this is nothing but 1 0 0 1 plus beta 0 0 beta i. Plus, you can write it like this.

So, you do understand this 1, beta, beta square, those things will come, okay? Times e to the power 0, gamma, minus gamma, 0. So, you see, this is nothing but e to the power beta, right? Times e to the power 0, gamma, minus gamma, 0. You see, e to the power beta.

That is why this e to the power beta is there, see? Okay. Now, the thing is, this matrix, we are just saying that this is nothing but this matrix. e to the power 0, gamma, minus gamma, 0. Okay.

Let us just, you know... Calculate it. Let us see what happens. Now, e to the power 0 gamma minus gamma 0. What does it mean?

It means, first of all, it is identity plus identity. 0 gamma minus gamma 0 times, yeah, there is nothing times. So, this is plus 0. So, let us calculate if you are this matrix times t, yeah, because all of this is like if you are p t. So, this is t, this is t, okay. So, that is always there.

Once again, I did some mistake here, I think. Yeah, so I forgot to write the t part here. Okay, so it is beta t, gamma t, this t part is there, this I forgot to write. Okay, so this t part is there. So please keep that in mind.

I forgot to write it. But anyways, you do understand what is happening, right? So sorry, here there will not be a t. plus this t plus goes on like this. So, this is times t, this times t. So, t is that part I thought, but anyways, you do understand what I am trying to say.

Now, you see, let us calculate this thing. See, 0 gamma minus gamma 0, 0 gamma minus gamma 0. If we calculate the whole thing down, what is happening is this. This will actually give you minus gamma square 0, 0 minus gamma square. right.

And then this is square. What about the cube? The cube is minus gamma square 0, 0 minus gamma square times 0 gamma minus gamma 0, right. So, what will happen if we are doing the third, the second part that is, okay, the third part cube. This is minus 0 minus b cube and then b cube 0, okay.

Let us do another one, the last one, okay. Now, let us say 0 minus b cube, b cube 0 times 0, sorry, what is, this is gamma, gamma, gamma cube. And then the last part, the last part is 0 minus gamma cube, gamma cube 0 times 0. 0 gamma minus gamma 0. If you multiply, you see you are going to get gamma power 4, 0 gamma power 4.

This is what we are getting. See, if you put it together, then what happens is r square is minus r, sorry, r square is minus gamma square 0 0 minus gamma square times t square by 2 factorial plus q part 0 minus gamma q gamma q 0 t cube by 3 factorial plus gamma power 4 0 0 gamma power 4 q power 4 by 4 factorial and it goes on like this. Now, you see what is happening is what is the first term? It is 1

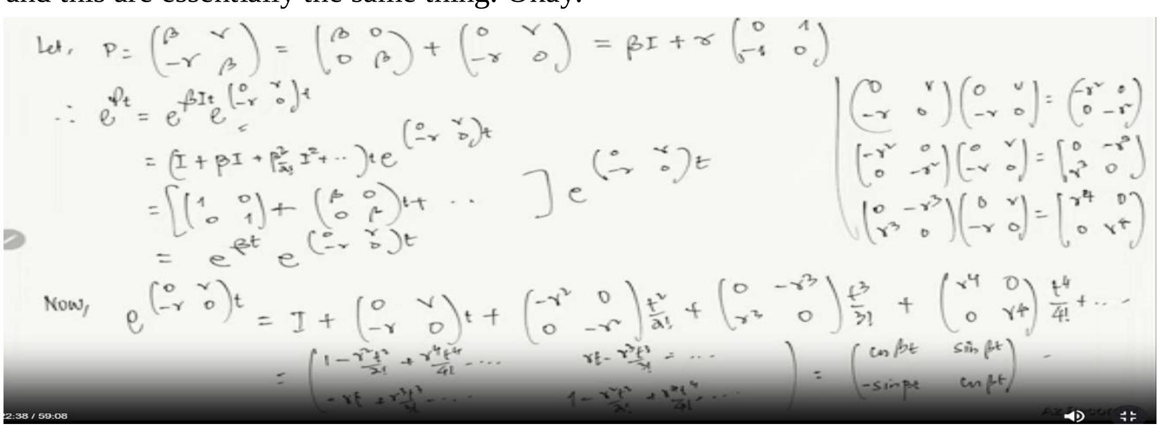
The second term is not there. Minus gamma square t square by 2 factorial. The third term is not there. The fourth term is plus gamma power 4 to the power 4 by 4 factorial. Minus it goes on like this.

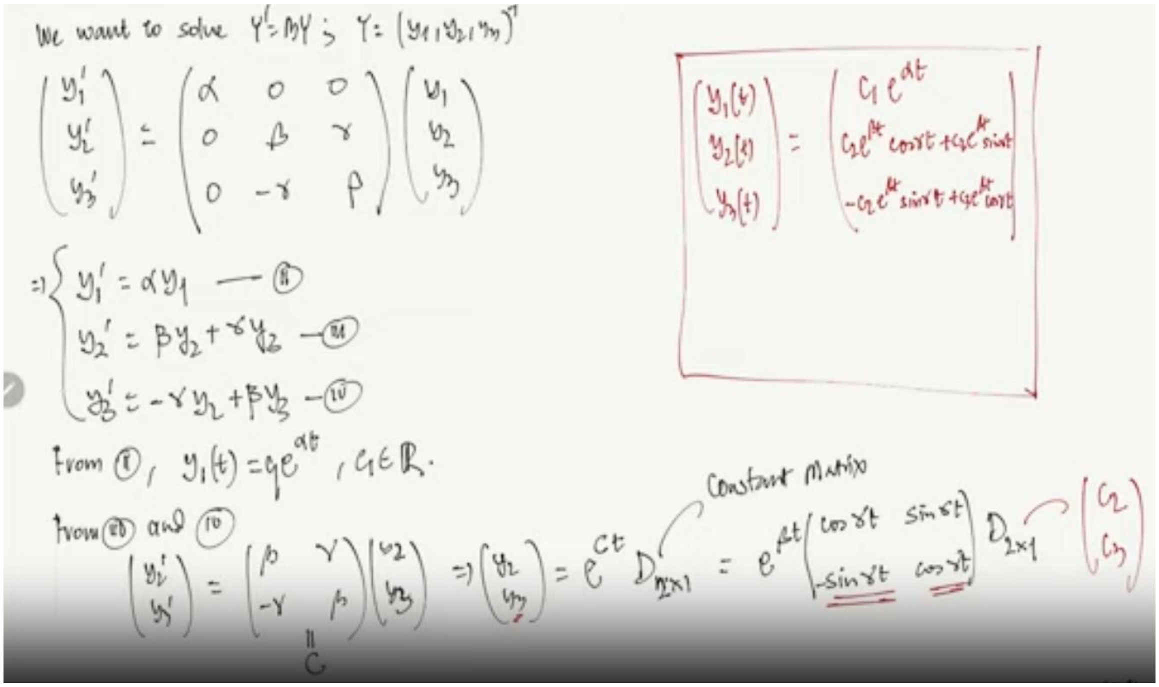
That is the first entry. What is the first row second entry? The first row second entry is 0. So the second is gamma. And from here gamma t gamma cube d cube by 3 factorial.

And it goes on like this. And what about this term 2 1? 2 1 is minus gamma t. okay plus gamma cube t cube by 3 factorial plus gamma cube t cube by 3 factorial okay minus it goes on like this okay and what is the this term here the 2 2 term 2 2 term is 1 minus gamma square t square by 2 factorial plus gamma to the power 4, p to the power 4 by 4 factorial and it goes on like this, right?

Now, you know this is nothing but sine and cosine. So, essentially if you write it down, it is nothing but cosine beta t, sine beta t, okay? And minus, this is minus sine beta t, okay? And this is cosine beta t. Yes.

Okay. So, you see this and this is essentially the same thing which I wrote here. Okay. This and this are essentially the same thing. Okay.





So, you see what I want you to do now is this. So, there is nothing to check. I calculated everything. Okay. So, anyway.

So, you understand how this is going to be calculated. So, this is done. So, this is how we calculate this part. Now, you see. What happens is, if you want to find out what is y1, y2, y3, we already got it, right?

y1 is this. So, therefore, we can write it like this. Let me put it this way, as in a red. What did we get? We get y1 of t, y2 of t, y3 of t. How does it look like?

It looks like this. What is y1 of t? It is c1 times e to the power alpha t, right? What is y2 of t? See, this d is nothing but, let us say, c2, c3, some matrix, right?

So, it will look like this. e to the power beta t is always there, right? So, c2 times e to the power beta t, cosine gamma t, cosine gamma t. plus c3 e to the power beta t sine gamma t. And what is the third expression? Minus c2 e to the power beta t sine gamma t, because you see what is y3.

y3 is nothing but c2 times this plus c3 times this. plus c3 times e to the power beta t cosine gamma t. So, you see, we have the whole expression. So, this is the solution. So, this is y of t. So, y of t can be written like this. See, y of t is nothing but this vector.

Now, once we get y of t, now the only thing which is required to show that what is t, how do you get such as t. So, basically once you find that t, see x of t is nothing but, see here x of t is nothing but this. Okay, e to the power beta t, you know what is y of t. What is y of t? y of t, see y is t inverse x, right. So, y of t we already got. Now, if we just can calculate what t inverse is we are done.

So, this we will learn how to do it, but let us do the you know the other part that is the when there is a repeated truth once we do it and then I will do one example where all of this will be put together. I hope this is clear why we are actually working with the canonical form because it makes our life easier to deal with. Now, as for the repeated roots, what we are going to do is, this is a little complicated issue, right? And we are going to do it for 3 cross 3, okay? And for this course, we are only going to assume, I mean, more or less, we are going to do 3 cross 3 system, yes?

So, do not worry about it. Now, for a 3 cross 3 system and having repeated roots, so let me just put it as a theorem, okay? Theorem. Let us look at what is the canonical form. So, say, let A

be a 3x3 matrix, okay, 3x3 matrix for which lambda is the only eigenvalue, right, eigenvalue. Okay, then what happens is you can find, so there exists a transformation T, change of variable coordinates that is such that T inverse A T assumes one of the following forms. forms. So, what are the forms? The first form is it may look like this 0 lambda 0 0 0 lambda 0 0 0 lambda.

So, when is this case essentially you can just think I mean this is very much like intuition. The first case will be achieved when you know there is one at least there is just one eigenvalue, but the thing is for that eigenvalue you have three linearly independent eigenvectors. Okay. So, this actually for n cross n matrix also exactly the same sort of thing will happen. So, you do not have to worry about it.

So, for n cross n matrix, if you have like for one eigenvalue, if you have like, you know, n linearly independent eigenvectors, then the corresponding matrix will look like this. Okay. So, what is it? Lambda times i, essentially. Okay.

Now, if, I mean, the other case is this, it should look like this, lambda 1 0 0. 0 lambda 0 0 0 and lambda yeah and the third case will look like this lambda okay so you see in both the cases in all the cases they are going to be triangular matrices just like a different kind of triangular matrices okay the first thing is of course the easy one the diagonal matrix and this is like the this sort of matrix okay so 0 0 0 okay right so let's look at the proof of this yes Now, if you see, it all depends on the basically the kernel, right? So, let us say define the kernel of A minus lambda I, let us say, let us just call it as K, clear?

If we call that as K and the dimension of K is 3, let us say, okay? So, case 1, case 1, the dimension, okay, of K is 3, right? Clear? So what does it mean? It means A minus lambda I. Okay.

See A minus lambda is from R3 to R3. Right? As a linear operator it is from R3 to R3. Okay. So this acting on V is going to be 0 for all V in R3.

Right? So, you take any vector, does not really matter what. Since the kernel dimension is k, so basically what will be your k? k is the whole of R3. So, essentially for any v in R3, you have a minus lambda i v equals to 0.

What does that imply? Since this relation holds for all v, that will imply that a equals to lambda i, of course it has to be. there is no other option right a equals to lambda i yeah and if a equals to lambda you see this is what lambda is okay so the first case is trivial and from this all actually holds for n cross n also it's not a problem okay it's the same thing okay now comes the difficult uh two cases okay case two now let's say that the dimension of the kernel okay dimension of the kernel is let's say two yeah two now what happens is let's call uh that I mean, we define r to be the range of a minus lambda i. Clear? We just call it as r. Then, since dimension of k is 2, you do realize that the dimension of r will be 1.

Why? Because a minus lambda is from rc to rc, right? So, the Ragnarok theorem says that the dimension of r has to be 1. That is quite easy to see, right? It has to be.

Okay. So, now, you see, we will claim, we claim that 1 is contained in another. Which one is contained in another? r is contained in k. Okay. Bold claim that is.

But the thing is, this is true. Why this is true? Let us see. That if It is not the case, is not the case, okay.

Then, you see, there exists a element in R which is non-zero, okay. So, there exists V, let us say non-zero, in R, in the range, clear, okay. Okay. Now, you see, what did I want to show? r is contained in k, right?

r is contained in k. So, I start with a non-zero element in r. Let us call it a v. Now, you see, a minus lambda i acting at v. Yes, this is in r. Yes. See, V, the vector V is in the range. So, what does it mean? A minus lambda I times V should be in that set R. And R is one dimensional.

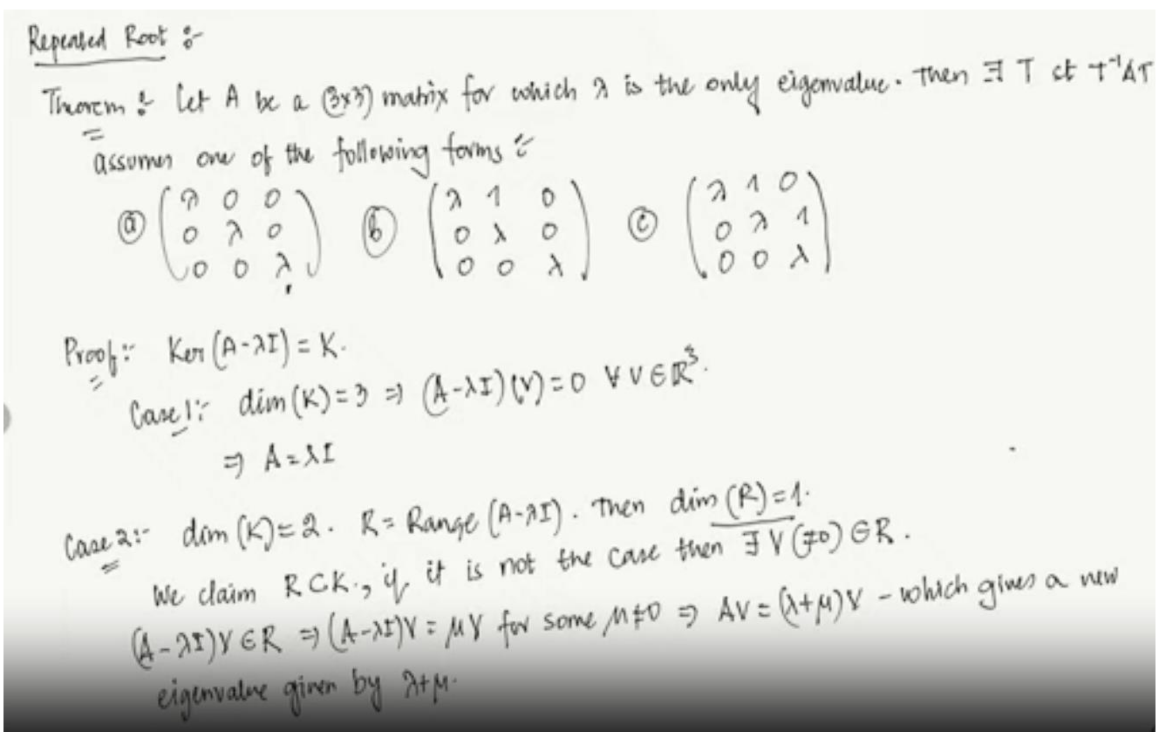
That is given. You see, dimension of R is one. So, that will imply A minus lambda I times V will look like some mu times V. Mu times V, right? See, V is not 0.

It depends, right? In R. And since the dimension of R is 1, we can think of the basis of R to be just containing V, right? Because V is non-zero. Okay. So, I can just write it as mu times V. For some mu.

For some mu non-zero. Clear? Now, if that happens, then AV will look like what? Lambda plus mu times V. Lambda times mu, lambda plus mu times v. So, you see what happened is, then we actually, and this is nothing but the eigenvalue problem.

So, we found out a new eigenvalue which is given by lambda plus mu, which gives us a new eigenvalue, eigenvalue given by Lambda plus mu. Okay. But we assume that lambda is the only eigenvalue. Okay.

So lambda plus mu has to be equals to lambda. That will imply mu equals to 0. But here we have assumed mu not equals to 0. So therefore, so r has to be continuity. Yeah.


Repeated Root :-

Theorem :- Let A be a (3×3) matrix for which λ is the only eigenvalue. Then $\exists\, T$ st $T^{-1}AT$ assumes one of the following forms :-

(a) $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ (b) $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ (c) $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$

Proof :- $\mathrm{Ker}(A-\lambda I) = K$.

Case 1 :- $\dim(K) = 3 \Rightarrow (A-\lambda I)(V) = 0 \ \forall V \in \mathbb{R}^3$.

$\Rightarrow A = \lambda I$

Case 2 :- $\dim(K) = 2$. $R = \mathrm{Range}(A-\lambda I)$. Then $\dim(R) = 1$.

We claim $R \subset K$., if it is not the case then $\exists\, V (\neq 0) \in R$.

$(A-\lambda I)V \in R \Rightarrow (A-\lambda I)V = \mu V$ for some $\mu \neq 0 \Rightarrow AV = (\lambda+\mu)V$ – which gives a new eigenvalue given by $\lambda+\mu$.

Very, very simple. It is not very difficult to see this. Okay. So let us look at the next part. So r is continuity. That is fine. Now what happens? So you see, let So, V1, I will start with another element V1, be non-zero in an R. Non-zero. Now you see, since V1 is non-zero in R, so it is in K, because R is contained in K, we just showed.

And V1 is an eigenvector. Eigenvector. Why? Because if v1 is in k, then a minus lambda times v1, you see v1 is in k, a minus lambda i times v1 is going to be 0. So, v1 has to be eigenvector.

So, that implies, I should write it here, a minus lambda i times v1 is 0. If v1 is in k, this happens. So, that is fine. Now, Since again V1 is also in R, right, that is our assumption, V1 is in R, okay, that will imply there exists a V2 in R3 minus K, okay, with A minus lambda I times V2 is V1.

Yes, so you see K, what is the dimension of K? It is 2. right, that is what our assumption is, see, dimension of k is 2. So, essentially, you can, of course, find a non-zero element v2, let us say, yes, which lies in r3 minus k, and since v1 is in r, you can always find a minus lambda a times v2 to be v1, yeah, you can always have that element there, that is always true, okay, right. Now, you see, k, the

where is it? Yeah. So, now, what am I doing? So, since v1 is in R, you can of course find the v2 which is in the R3 minus k such that this is happening. So, this is fine.

Now, you see, okay. So, since the dimension of k is 2 now, okay, then we can choose another element v3. We can choose v3, okay, in the kernel k such that V1 and V3 are linearly independent. Are linearly independent.

Clear? See, V1 is in R, right? So, V1 is in K also. I am choosing, since the dimension is 2, of course, I can find another element in K such that V1 and V3 are linearly independent. Since the dimension is 2.

Now, you see, V3 is also an eigenvector. Also, V3 is an eigenvector. He is an eigenvector. Right? Okay.

So, now what happens is you choose a change of coordinate. What is the change of coordinate? Now, choose this change of coordinate. P of E j is nothing but V j. Okay. For j equals to 1, 2, 3.

Clear? If we do that, then you see what is happening to T inverse A T. Let us just see. Therefore, what is T inverse A T, this matrix? It is nothing but T inverse A P acting at E1, right? And then T inverse A P acting at E2 and then T inverse A P acting at E3.

That is the matrix. So, you see T of E1 is nothing but V1. A of V1 is nothing but what is A of V1? A of V1 is nothing but minus lambda V1. Now, T inverse of minus lambda V1 is lambda times you see this is lambda times T inverse of V1.

T inverse of V1 is nothing but E1. So, E1. So, E1 is 1 0 0. 0, right? So, we can write it as lambda 0, 0, clear?

Okay, again, T of E2, what is T of E2? It is V2, okay? T of E2 is V2 and V2 is what, you see? Now, A of T E2 means A of T E2 is A of V2, A of V2, A of V2, What is A of V2?

A of V2 is nothing but V1 plus lambda V2. Lambda V2. Now, you see T inverse of V1 plus lambda V2 is T inverse of V1 plus lambda times T inverse of V2. T inverse of V1 is E1 plus lambda times T inverse of V2 is E2. So, it is 1, 0, 0 plus lambda times 0, 1, 0.

So, it is 1, lambda and 0. Yeah. And similarly, last one, if you just calculate it, it becomes 0, 0, lambda. Why? Because you see, the last one is V3, which is again an eigenvector.

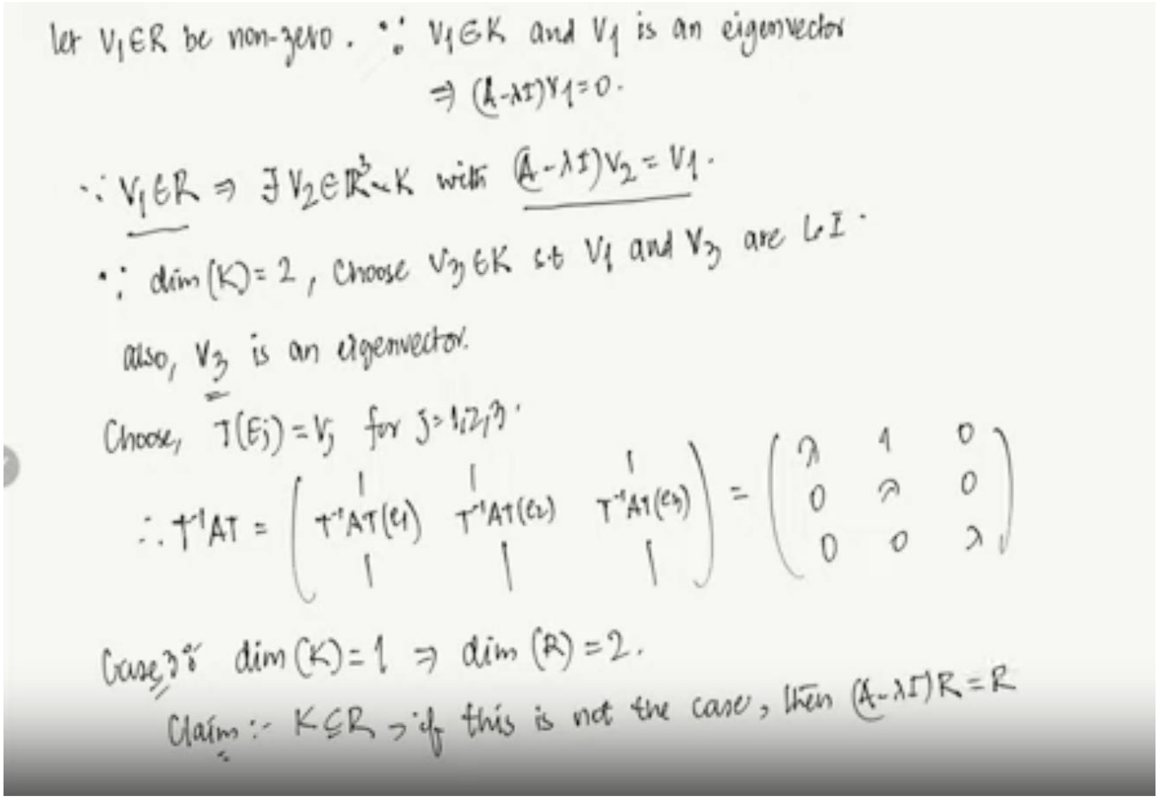
So, essentially, the same sort of, this thing will happen with E3, essentially. Yeah. Okay. So, this is the thing. So, you see, this is the second part.

Clearly, this is B part. Okay. I hope this is clear. Now, let us look at the third part. Okay.

So, now, the case 3. Case 3. is the dimension of K is 1. Dimension of K is 1. And then what happens is, what is the dimension of R then?

Dimension R is 2 then, rank-narrative theorem again. And then, as earlier, we will claim that 1 is contained in another. You see, since the dimension of R is 2, K is contained in R. This is what we will, I mean, you know, claim. We have to prove that. We did not prove it, but this is our claim.

So, if this is not the case, so let us say, if this is not the case not the case then a minus lambda i times r equals to r right yes if k is not contained in r then a minus lambda i times r has to be equals to r okay and a minus lambda i r is invertible. Double on r, right? On that range, it is invertible.



Okay, it has to be, right? So, thus, if you have a vector v, which is in the range, okay, there is a unique w in the range, such that a minus lambda i times w has to be equals to v. Clear? So, therefore, what is a v? a v can be written as a a minus lambda i times w. Clear?

v is a minus lambda i times w. So, I am just replacing it. Now, this is nothing but a square minus lambda a times w. Okay, that is nothing but A minus lambda I times AW. So, I wrote it like this. Clear?

Okay. See, what this shows is if V is in R, right, if V is in the range, then AV is also in the range. That's what it is showing. See, if V is in the range, AV is also in the range, right? So, what it shows and that implies, therefore, it implies that A preserves the subspace R. Now,

a if it preserves subspace r and it does this so you see definitely a has to be an eigenvector in r right so therefore also that will imply that a has to be be an eigenvector in r so has to have an eigenvector, let me put it, rephrase it like this, has to have an eigenvector, has to have an eigenvector in R, clear? Then, that will imply that K has to be containing R and which is a contradiction, which is a contradiction. Because we have assumed that K is not contained in R, right? So that's a contradiction.

And this follows, okay? So now what we are going to do is we are going to claim the second claim. This is a complicated situation here for this, okay? The second claim is this. A minus lambda I of R is nothing but K, okay?

Now, why is it? This is not a very difficult thing to see. See, A minus lambda I of R is one-dimensional, is 1D, okay, one-dimensional. Why one-dimensional?

Because, you see, K is containing R, right? See, here, this is the claim which we proved earlier, right? K is containing R, K is containing R, okay? So, of course, A minus lambda I times R is one-dimensional. Since K is

This K is containing R. Now, let us say if A minus lambda I times R is not the same as K, then there exists a non-zero vector V, non-zero, which is contained in K. For which A minus lambda I times R has to be look like T times V. Because it is one dimensional, right? T times V. T is in R. See, here we have assumed the dimension of case 1, right? So, that is why A minus lambda I times R has to look like T times V. Okay. So,

If this is the case and P is not 0, then that will imply that, you see, one second, Tb, clear. Now, one second, yes, a minus lambda i times r should be Tb, right? Yes, that is what we are going to get. But, you see, a minus lambda i, times V should be equals to T times V, right?

For some T non-zero, it should be equals to T times V. That and therefore, what happens is A times V should be equals to T plus lambda times V. Yes, you take this thing here, then it works, right? And then what happens is you are going to get a new eigenvector. You see, t plus lambda, getting a new eigenvector, which is a contradiction. New eigenvector. Eigenvalue, sorry, eigenvalue.

Now, this is a contradiction. So, therefore, a minus lambda i times r has to be equals to k. Now, we are more or less at the end of the proof. Now, what we are going to do is this. So, this is fine.

and $(A-\lambda I)$ is invertible on K.

Thus if $V \in R$, $\exists!\, W \in R$ s.t. $(A-\lambda I)W = V$.

$\therefore AV = A(A-\lambda I)W = (A^2-\lambda A)W = (A-\lambda I)AW$.

If $V \in R \Rightarrow AV \in R$.

$\therefore$ A preserves the subspace R, A has to have an eigenvector in R.

$\Rightarrow K \subset R$ – contradiction.

Claim: $(A-\lambda I)R = K$. $(A-\lambda I)R$ is 1-Dim, since $K \subset R$.

If $(A-\lambda I)R \neq K$, $\exists\, V(\neq 0) \in K$, $(A-\lambda I)R = \{tV\}$; $t \in \mathbb{R}$

But, $(A-\lambda I)V = tV$ for $t \neq 0 \Rightarrow AV = (t+\lambda)V$ – getting a new eigenvalue

$\therefore (A-\lambda I)R = K$.

So, see, let V1, we will start with V1, which is in the kernel K. So, this is an eigenvector. So, V, an eigenvector for A. Okay. So, what will happen is this. So, we can say there exists V2 in R, right, such that A minus lambda I times V2 is V1.

Yes. see k is one dimensional r is two dimensional right so essentially what is happening is this you can always get and r is the range right so you can always get a v2 in r such that a minus lambda i times v2 has to be v1 here and one is contained in another k is contained in r right so that's happening now since v2 is in r okay there exists a v3 then such that a minus lambda i times v3 is v2 Okay, so that will imply that A minus lambda I square times V3 is V1. Clear? Yes, and you can see, it is very easy to see, so please check this part.

Check that V1, V2, V3 are linearly independent. Okay, that is just by the construction it is like that only. Yes. So, you define your T as T of E1 to be EI to EVI for I equals to 1 to N, 1, 2, 3. And therefore, T inverse AT.

This again, I want you to check T inverse AT. What will this be? It is the first thing is T inverse AT. acting at E1, right? The first column.

Second column is T inverse FD acting at E2 and the third column is T inverse FD acting at E3, okay? Just do exactly the same sort of thing what we did here, okay? If you do it, you see V1, this is, if you calculate it, you know what V1, V2 and V3 are, okay? So, if you calculate the whole thing, you will actually end up getting this, lambda 0 0 1 lambda 0 0 1 lambda. Clear? So, you have to check this part. Check. Okay?

Let $v_1 \in K$ be an eigenvector for A.

$\exists\, v_2 \in R$ s.t $(A-\lambda I)v_2 = v_1$.

$\because v_2 \in R$, $\exists\, v_3$ s.t $(A-\lambda I)v_3 = v_2 \Rightarrow (A-\lambda I)^2 v_3 = v_1$

Check :- v_1, v_2, v_3 are L.I

Defining, $T(e_i) = v_i$ for $i = 1,2,3$.

Check

$$\therefore T^{-1}AT = \begin{pmatrix} | & | & | \\ T^{-1}AT(e_1) & T^{-1}AT(e_2) & T^{-1}AT(e_3) \\ | & | & | \end{pmatrix} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

So, this is what happens when if you have a 3 cross 3 matrix with a repeated eigenvalue, this is actually something which is called the canonical form.

This is the canonical form. So, canonical form is you can reduce it either in one of those three forms and you are good to go. So, any one of those two three forms will work. So, what I am going to do is since this particular example is little difficult, let us just do a small exercise. How to calculate this?

That will make life easier. So, first of all, let us just assume a matrix A which is 2, 0, minus 1. 0, 2, 1, minus 1, minus 1, 2. Okay. It is very easy to check that, you know, lambda equals to 2 is a repeated eigenvalue.

Is a repeated eigenvalue. Eigenvalue. Okay. And, of course, you can always get one eigenvector out of it. That is not a problem.

Okay. So, for lambda equals to 2, V1 let us call it 1 minus 1 0 is an eigenvector. Now the thing is this. I have 2 is repeated twice, thrice, right? 2, 2, 2.

But the thing is, I only have one eigenvector. Yes? And since it is just one eigenvector, it is independent. It is not a problem. Okay?

And since this is non-zero, it is independent. Now, the thing is this. I need to know, I need to get another two eigenvectors. Okay? How am I going to get it?

Okay? So, this is the procedure to do it. Please remember this. Okay? So, first of all, we compute this.

Compute. We start with a, right? So, we compute a minus 2i square. Clear? Let us just compute it.

If you compute it, it becomes 1, 1, 0, minus 1, minus 1, 0, 0, 0, 0. clear. Now, the thing is this, I am going to find out a vector v3. Now, the thing is this, how do you get, you know, I want to get v1, v1 I already got, I want to get v2 and v3. So, see, I want to, so we will choose

Choose v3 such that a minus 2i square v3 is nothing but v1. So we are targeting v1. We want to achieve v1. And then a minus 2i square v3 is v1. This is what we are trying to do here.

Now, see, if you calculate this thing, what happens is essentially 1, 1, 0, minus 1, minus 1, 0, 0, 0, 0, okay, v3, 1, v3, 2, v3, 3. So, these are the three components of v3, let us just assume that. And then that will give you v1. What is v1? This is our target.

I want to reach v1 using square, the a minus 2y square, okay. So, this particular thing, what we are going to get, these are called generalized eigenvectors, okay, right. So, what you get is this V31, V32 and V33, we are going to get it as 1, 0, 0, right, okay. And we also choose, okay, so we also choose a minus 2i times v3, okay, equals to v2.

Clear? So, essentially, using this v3, okay, using this v3, we want to get v2, right? Okay. So, how do we get it? You see,

So essentially what we are doing is this. See, first of all, I have a minus lambda i times v1 is 0, right? Yeah, lambda 1 is 2 here. So basically a minus 2i, 2i times v1 is 0. That's how we found out, basically we tried to find out a v non-zero such that this happens, right?

So solution was v1. That is what we got. Now we use v1. Yes. And we try to find it this way that a minus lambda i square.

We try to find a v such that this is equals to v1. Right. That amounts to getting v3. Now we are going to start with cube. Clear?

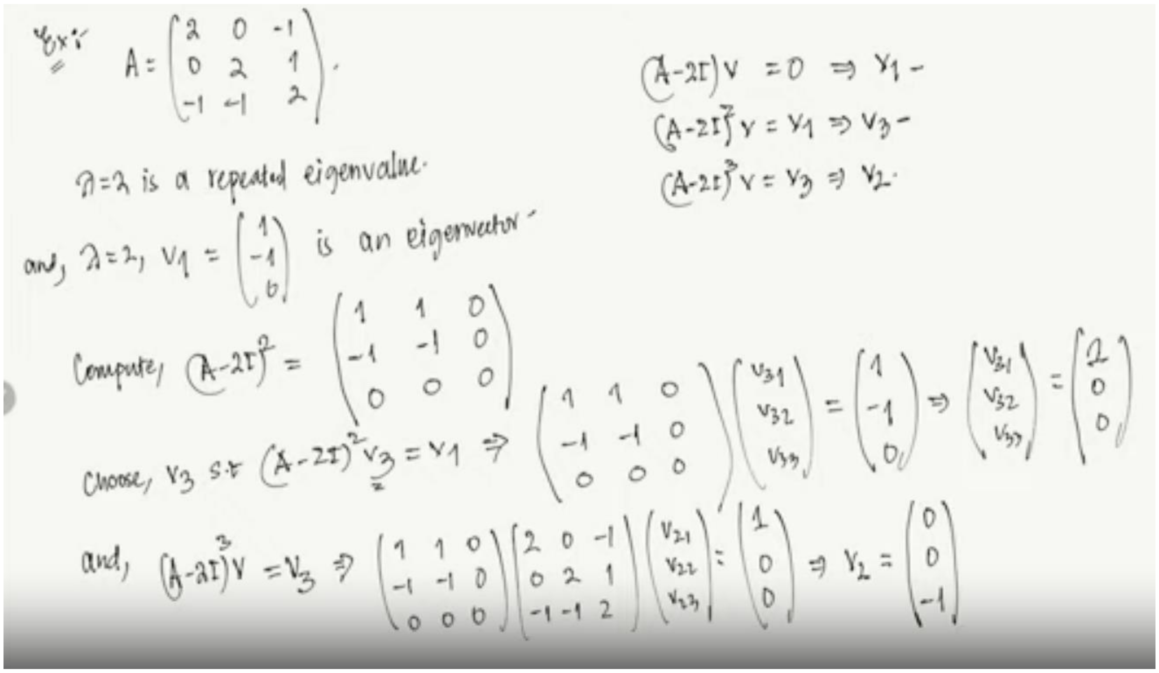
Of V, I want to target now V3. V3 we already have. I want to target V3 here. And I need to find this. If we can find it, we will call it a V2.

Okay? And then we are done. It does not have to be V1, V2. You can just change the numbers also. It is not a problem.

But anyways, we are doing it like this. Okay? Right. So, a minus 2i times v3 is v2, right? So, what does it mean?

It means that, sorry, I should write it as, one second, I should write it as a minus lambda i cube v2. what is it? V, yeah, so I have to find a V2, right? So, I will target, I have V3, so I will target V3, okay? Now, this is what?

So, you see, this is nothing but 1, 1, 0, minus 1, minus 1, 0, 0, 0, 0, times this matrix again, 2, 0, minus 1, 0, 2, 1, minus 1, minus 1, 2, times V1, let us say V2, V3, 1, 1, V, okay, let us just call, this is V, V2, right, let us just write it as V21, V22, V23, and the target is this, 1, 0, 0, that is the target of V3, okay, now, if you calculate this, this is, we are going to get V2 here, okay, if you calculate it, you see, it is 2, okay and 0 0 so basically the first thing is you are going to get 0 and then minus 2 minus 2 yes, times 2, 4, v1, v2, v3, okay, so this is 0, and the last part is minus 1, okay, so this is how, what we are going to get, v1, v2, v3 is according to this, okay, this is how we got it, 1, now, what happens is, we need to know, we need to write down the, this thing, the diagonal element, how is it supposed to write down, okay, so basically, we let, T of E1 to be V1.



T of EI to be VI for I equals to 1, 2, 3. So, what is T? T is nothing but 1, minus 1, 0. See, T, if you remember, T is essentially nothing but this matrix.

T acting at E1, T acting at E2, T acting at E3. E1 is what? 1, minus 1, 0. E2 is 0, 0, minus 1. 0, 0, minus 1.

And T acting at E3 is nothing but, what is it? T acting at E3, 1, 0, 0. 1, 0, 0. Okay, so therefore what is T inverse AT? T inverse AT if you calculate it, it is nothing but, so basically T inverse AT acting at E1 which will be 2, 0, 0. T inverse AT acting at E2 which is 1, 2, 0 and T inverse AT acting at E3 is 0, 1, 2.

Yes, so this is the T inverse AT. Yes. Okay, I will give you some exercise for that, you know, just for you to understand what is going on.

let $T(E_i) = v_i$ for $i = 1, 2, 3$

$$T = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\therefore T^{-1}AT = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

So, some calculations, it will be better for you to understand that and then you can do this part. So, it is not very difficult, it is just some calculation, you have to do it yourself and then you can find this eigenvector.

So, this is the idea, the nomenclature here is, this is called generalized eigenvector. Okay, so with this I am going to close this video. Bye.