Ordinary Differential Equations (noc 24 ma 78)

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Week-04

Lecture-18: Higher Dimensional Matrix Exponential

Hello students, in this video, we are going to start with some, you know, linear algebra. So, basically we learnt about some linear algebra in the previous video itself, but now we are also, we are going to discuss a little more, okay. So, first of all, we are going to start with something called the eigenvalues and eigenvectors, okay. eigenvalues and eigenvectors. So, what is it?

So, let us say a vector v is an eigenvector of a n cross n system. So, basically given a n cross n system, the eigenvector will be given like matrix A. If V is a non-zero solution, is a non-zero solution to the system of linear equations, of linear equations what are the equations a minus lambda i times v is equals to 0. see v if v is equals to 0 of course a minus lambda is equal to 0 for all lambda right but the thing is basically we are looking for a non-trivial solution of this system here and this quantity Lambda will be called, lambda is called the eigenvector, right?

It's called the eigenvalue, sorry. It's called the eigenvalue of A. Value of A. Clear? Okay. So, this eigenvalue and eigenvector definitions we already know. So, it is not a very, I mean, I am not going to discuss it further.

Now, the thing is, you see, we are going to start with this. So, essentially, what is our goal? Why are we suddenly started with eigenvalues and eigenvectors out of nowhere? essentially we want to calculate exponential of a matrix right and let's say for a big matrix let's say 3 cross 3 or 4 cross 4 matrix 5 cross 5 matrices okay this is the best way that i know of to calculate a matrix exponential there are other ways there are many different ways of doing it but this is the most you know intuitive and modern way of doing this okay so first of all let's look at this property or maybe let me write it as a theorem.

Okay, so before that let us put down a small remark. Now, of course, I am assuming that you guys already know what eigenvalues and eigenvectors are given a matrix and you know what are the properties. So, I am not discussing it further. But anyways, as a remark, I am just writing that let lambda 1, lambda 2, lambda n, okay, are real and distinct and distinct. eigenvalues of A, eigenvalues of A with associated eigenvectors, with associated eigenvectors, vectors given by v1, v2, vn.

okay then what is the relation between these vi's then vi's are linearly independent okay li so i am not writing this thing it is linearly dependent okay right so with these we are coming to the important theorem okay so theorem theorem so try to understand this theorem and you will understand what's happening so essentially let's So let A n cross n matrix essentially has real and distinct eigenvalues. eigenvalue. Then, what can you say? Basically, it means that the matrix is diagonalizable.

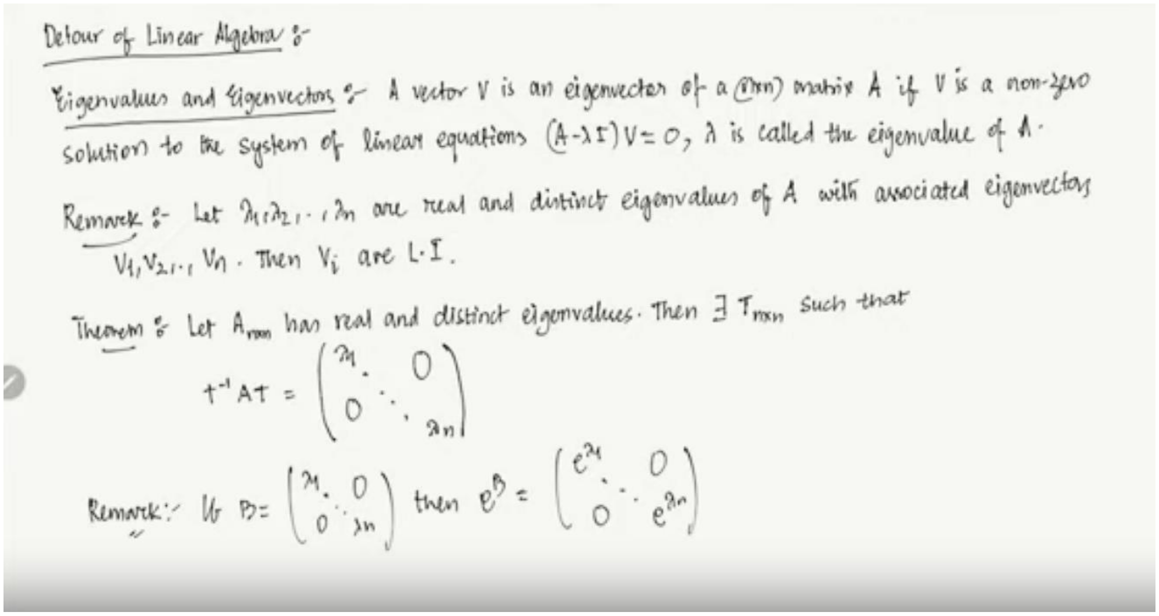
So, basically, then there exists a matrix T, which is a n cross n matrix again, such that such that Okay, T inverse A of T. Okay, that will be even by nothing but this very simple matrix even by lambda. So basically, lambda n. So everything, all other, you know, of diagonal entries are basically 0. And this is a diagonal matrix with entries lambda 1, lambda 2, lambda n. Okay. So, this is very simple.

Now, why is it, let us say, if you are in this situation, why is this important? Because we know that if we can do something like this, exponential of t inverse a t, if we have to calculate, let us say, if we can calculate something like this, if you want to calculate something like this, it will be nothing but t inverse exponential a times t, right? So, an exponential a, so you see, remark, as a remark, if you If your matrix, let us say a matrix B is nothing but lambda 1, lambda n, 0, 0. Then it is easy to say that exponential B is nothing but e to the power lambda 1, e to the power lambda n, 0, 0.

So you do realize that calculating the exponential is going to be very simple. If you can show that the math, you know, if it is real and distinct, if you are starting with a matrix which is real and distinct, then what happens is this is diagonalizable. Then you can write it in this form. And once you can write it in this form, essentially what happens is calculating the exponential gets very easy. That is the idea.

So, let us look at this theorem. So, what does the theorem says? It says that if A has real and distinct eigenvalues, then you will have a matrix here such that T inverse A T, when you calculate T inverse A T, that is going to be a diagonal matrix. We can also write this thing as diagonal lambda 1 lambda A c.lear? How do you prove something like this?

Proof is actually easy. See, here what we are doing is basically we are taking a change of variable here. So, let v i be an eigenvector associated with lambda i. okay so vi is a eigenvector associated with lambda this is what we are assuming right now okay and um what happens is now see uh we want to okay uh write down that that i mean we have to find that transformation t see the thing is you are given a real distinct uh this matrix right okay with the sorry you are given a matrix with the real distinct eigenvalues okay Now, you have to find a transformation which actually reduces this to a diagonal matrix, right.



So, essentially what we are trying to do is find, you know, a change of variable which actually reduces the matrix to a diagonal matrix. Think of it like that, okay. So, what we are doing is this. Now, you define, define P, clear. such that T of ei is nothing but vi, clear.

So, you see since lambda 1, lambda 2, lambda This is distinct, okay? Lambda n, they are distinct. The corresponding eigenvectors which you are going to get, let us say v1, v2, vn, they are linearly independent, right? Okay, so let us say if I map v, e1, e1, the first basis element that is, okay?

to v1, e2 to v2, en to vn, okay? Then what happens is I have a one-to-one map, right? So basically, this is a very, very, this is a valid change of variable. That's what we're trying to do here. So when you define t of ei to be vi, okay?

And what is ei? Ei is nothing but 0, 0, 1 on the i f cordon entry and then this is i f, the standard basis that is a, okay. So, this is there. Now, you see that T, so essentially if you write it like this, so what is the matrix T?

Therefore, the matrix T, if you want to write it as n cross n matrix, it is nothing but T of E1, okay, T of E1. e to p of en, right? That is what the matrix T is. If the linear transformation is given like this, what is the corresponding matrix? It is of course this, that basically it is determined by how the columns of this matrix, so basically the columns of the matrix

of the matrix is determined by its action on the ith basis vector. is this fine okay so this is what we are basically trying to do so basically you see you have a matrix now which is tn cross n such that the first column is t of e1 the second column is t of e2 and then the nth column is t of e n yes so see this matrix nothing but v1 the first column is v1 the second column is v2 and the nth column is vn because that's our definition right these are the columns okay Now, you see the thing is this v1, v2, vn, they are linearly independent. So, since vi's are linearly independent, linearly independent, why they are linearly independent? Because lambda i's are distinct, real distinct.

So, vi's are linearly independent. That will imply that t is invertible. T is invertible. If you do not, so this is very easy to check. If you do not know that, please check this.

T is invertible. And, and, Let us look at what happens to T inverse AT. You see, what we need to do? We need to write down T inverse AT as a diagonal matrix, right?

So, what happens to T inverse AT? So, this is a matrix, right? If we want to determine the matrix, we have to determine what it does on the basis elements, right? So, T inverse AT acting at, let us say, EI. What happens to this?

This is nothing but T inverse A. Now, T acting at EI, that is the first thing. What is T acting at EI? So, T inverse A. Now, T acting at EI is VJ. So, this is nothing but VJ.

So now you see T inverse A acting at VJ. What is A acting at VJ? Now it is A acting at VJ. Now you see VI's are sorry this is J here. Let us do it for J. It does not matter I or J but anyway same.

So, you see T of e i is v i and v i is the associated eigenvalue corresponding is the eigenvalue associated with lambda i. So, this is nothing but T inverse of lambda j v j because since this is v i is the eigenvector associated with lambda i, then a of v i is nothing but lambda i v i. So, this holds for all i. Clear? So, this is there. Now, T is a linear transformation that implies T inverse is also going to be a linear transformation. So, I can take this lambda J, that is a constant, right?

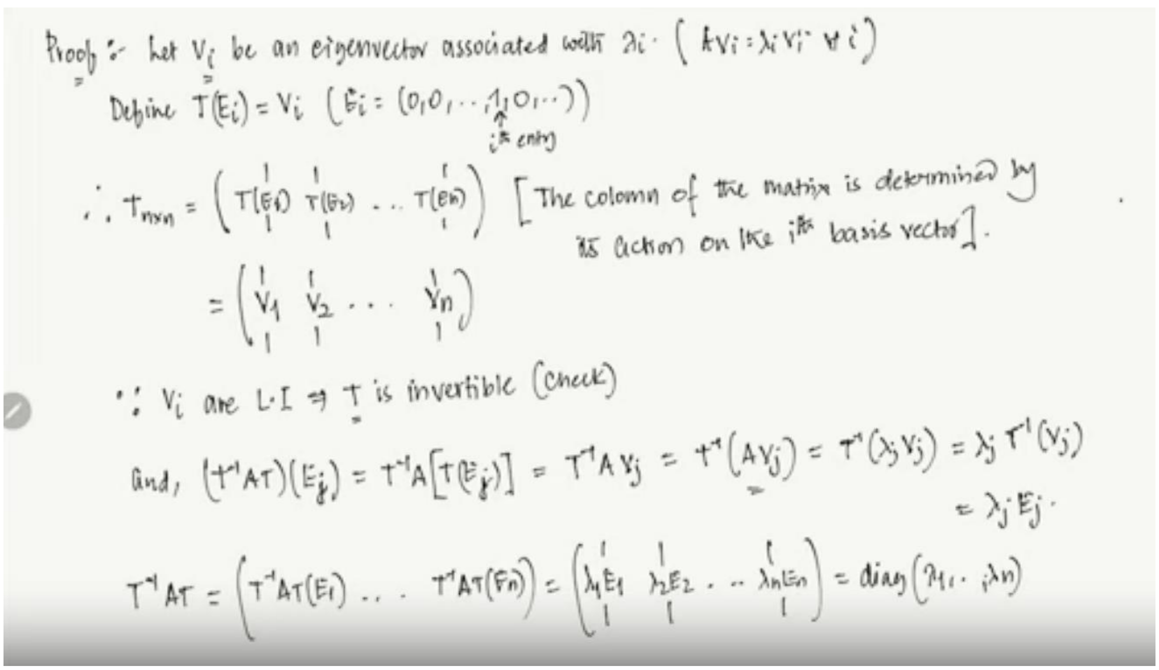
So, scalar. So, I can take that out and then I have this lambda J T inverse of VJ. Clear? Okay. Now, what is lambda J T inverse of VJ?

This is nothing but, you see, T is invertible. So, T inverse of VJ is nothing but EI, EJ, sorry, EJ. Clear? Okay. So, what is T inverse of AT?

See. So, what is the matrix T inverse of AT? Let us look at this matrix. This matrix is nothing but T inverse of AT acting at E1. T inverse of A T acting at E N, right?

That is what it is. So, essentially, you see, the first thing is this. This matrix is nothing but lambda 1 E 1, okay? And then the second one is lambda 2 E 2. And the nth thing is lambda N E N.

Now, if you, E1 is what? 1, 0, 0, 1. E2 is 0, 1, 0, 0, 0. So, if you write it down, this is nothing but the diagonal of lambda 1, lambda n. Is this okay?



So, what did we learn? We learned that if A is a matrix with a real and distinct eigenvalue, this is very important, real and distinct eigenvalue.

Then, of course, I mean, there are other, I mean, situations also, but in this situation, what happens is, you can always find a transformation, which actually reduces A. So, basically, T inverse A T is going to be the diurnal element, okay. Right. So let us look at the next situation. Okay. So you can take some example.

So in the assignment what we are going to do is we are going to give you some assignments, some problems which actually asks you to find out this transformation for some, you know, given specific examples. Now the second case. So this is first case. Real and distinct. Okay.

Theorem 1, let us just call it. Now, before we go on to theorem 2, so basically theorem 2 is about the complex eigenvalues. Complex eigenvalues. Eigenvalues. here okay so now you see that a has a non so we'll assume that it has a complex eigenvalue okay so we know that if alpha plus i will so let a has a complex eigenvalue eigenvalue okay so

Hence, if alpha plus i beta is an eigenvalue, of course, alpha minus i beta is going, also going to be an eigenvalue. So alpha plus minus i beta is other eigenvalues, other eigenvalues. Clear? So, that is always there. Now, the thing is this, you see that we have to, I mean, we are assuming that, of course, if alpha plus i beta is an eigenvalue, alpha minus i beta is also an eigenvalue, okay.

Now, let us say that there can be many different cases, right. So, in this case, suppose also we are assuming that A is a 2n cross 2n matrix with distinct non-zero eigenvalues. Also, A is a 2n cross 2n matrix with non-real, so basically complex matrix. I gain values.

Complex I gain values. What are the I gain values? Alpha I plus minus I beta I. I eqauals to, 1, 2, n. Now you do realize let us say you may have this situation that you have a 3 cross matrix. There is this eigenvalue alpha plus minus i beta and then another real eigenvalue.

It may happen. So we are not dealing with that case right now. What we are doing right now is we are assuming that you have like a square matrix of 2n cross 2n size. And what happens is all the eigenvalues for now, they are non-real. So, essentially they are complex eigenvalues.

And we want to see what happens then. So, what happens now is this. See that first of all. These are complex eigenvalues. So, can we, we also have associated eigenvectors.

So, what we will do is, we start, let vj and vj bar denotes the associated eigenvectors. Here I wrote i, so let us write down i. VI and VI denotes the associated eigenvectors. Now, the thing is this, if let us say summation c i v i plus d i v i bar let's say is 0 i equals to 1 to n okay and c i d i's are in complex number c okay then that will imply that c i equals to d i equals to 0 right that is easy to see

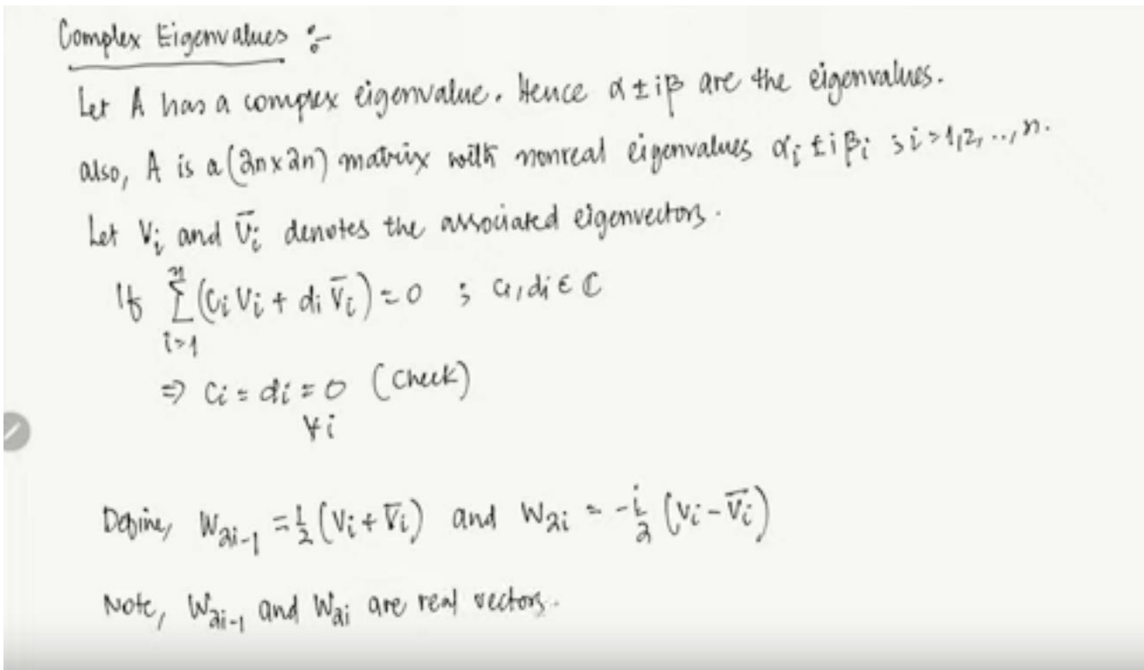
Okay, so I am not doing this thing. If you, I mean it is better you check this part. Okay, so what I am saying is this v i is alpha plus i beta, v i bar is alpha minus i beta. So you just make the real and you know many part separate and then you can see that this is true. Okay, so c i and d i they are going to be 0.

Okay, so for all i, I am not writing so for all i. Okay, now the thing is this, see we want to put, you know, we want to find a change of coordinate. So, here what we did is, you see, we did a change of coordinate, okay, and after that put it into this form, diagonal form. So, we also want to do the similar sort of thing here, okay. So, can we do that?

Now, what do we do? We change the coordinate of A, okay. So, to do that, what we do is this. So, now define W. 2i minus 1.

This we are defining as half of, so the real part of v i, that is v i plus v i bar. And w 2i to be half, sorry, minus i by 2. So that is the imaginary part, that is minus i by 2 v i minus v i bar. So, these are all real vectors. So, note w2i minus i1 and w2i are real vectors.

vectors, clear? These are real vectors, okay?



So, since this is the real part of vj, vi and this is the part of vim, so real vectors, okay? So, now we have this thing, the theorem or let us just put it as a proposition. What is it?

It says that the vectors, the vectors w1, w2, w2n are linearly independent. So, how do we show something like this? Let us look at the proof. So let's say that they are not, right?

So then, then one can find, one can find, can find C i d i, okay, i equals to 1 to n such that, such that summation c i w 2 i minus 1 plus d i w 2 i, i equals to 1 to n, this is 0, okay. See, if they are linearly independent, if they are not linearly independent, then this has to be 0 for some c i d i non-zero okay so essentially what i mean by this is you can actually find a non-zero c i or d i doesn't matter i mean at least something has to be non-zero such that okay this relation holds okay now if this relation holds you see what is w 2 i minus i 1 that is nothing but half times v i plus v i bar right and what is w2i it is the imaginary part so you see minus i by 2 vi minus vi bar okay so if you put it together what happens is you can write it as i equals to 1 to n c from here ci will be there and the other part is minus i di and v you see vi okay vi plus

ci plus i di vi bar, right? That is what we are going to get if we are writing it in terms of vi, 0, clear? Now, you see vi and vi bars are linearly independent, okay? They are linearly independent. What does that imply?

So, since vi and vi bar are linearly independent. Clear? Then, of course, C i equals to plus minus i d i. Okay? That is always true.

What will that imply? That a real is equals to a complex. That can only happen if c i and d i are going to be 0. That will imply c i equals to d i equals to 0 for all i. So, that is a contradiction. And hence, what happens is w 1, w 2, w 2 n are linearly independent.

So, very, very simple proposition. What basically it says is this. If you have eigenvectors, this thing vectors, okay, corresponding to eigenvalues lambda, this thing alpha i plus minus i beta i. So, and what happens is, those eigenvalues, so basically if you are forming it and you are looking just the VL and the imaginary part, they are actually going to be linearly independent. Clear?

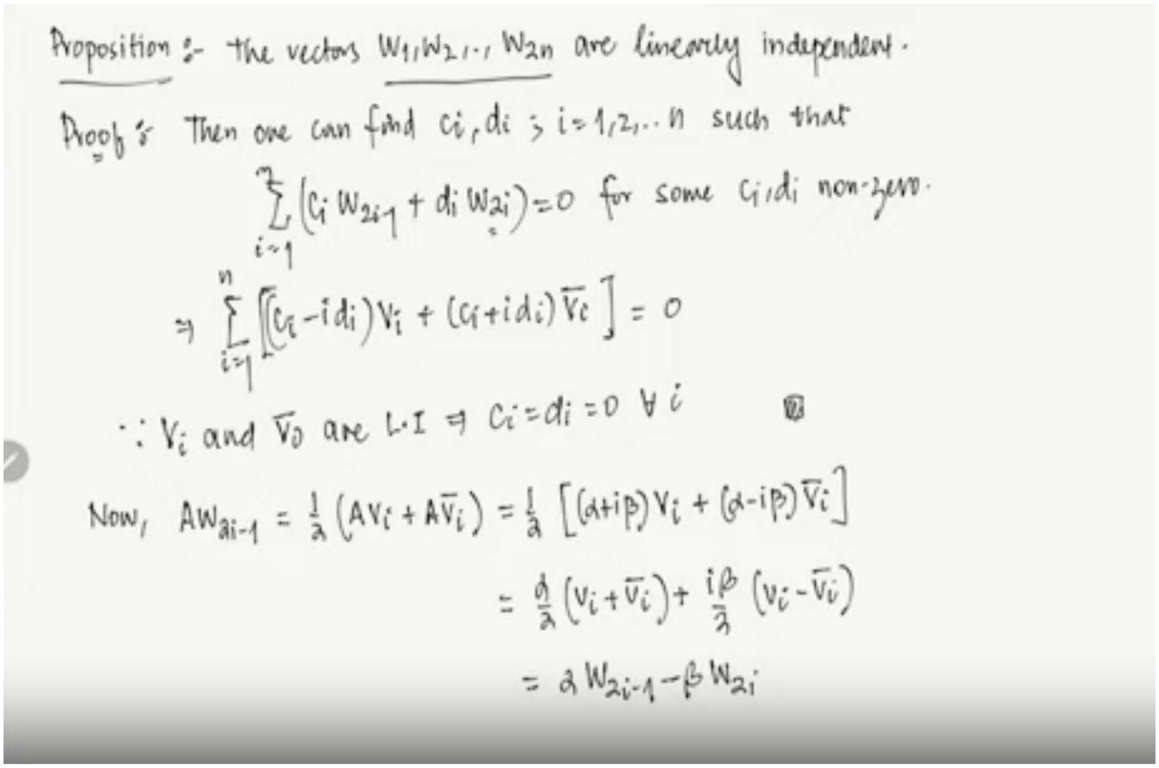
Okay. Now, so now, you see, now, The theorem is done. Now, you remember we last time in this real and distinct case, we just wrote if you are given this thing, a matrix A, T inverse A, T, you can write it as a diagonal element, right? Yes, which actually helps us in calculating the exponential.

So, now we want to do the similar thing here. Now, A of w 2i minus 1. What is it? So, let us look at this.

This is half a of vi plus a of vi bar. Clear? See, w 2i minus 1 is nothing but half of vi plus vi bar. So, a of this, a is linear. So, I can take it inside, right?

So, this happens. So, now, this is equal to half alpha plus i beta vi plus alpha minus i beta vi bar, clear? So, that is equal to alpha by 2 vi plus vi bar plus i beta by 2 vi minus vi bar. Now, if you write it down, this is nothing but alpha times w 2i minus 1 minus beta times w 2i.

Clear? So, see now what we are doing is this. We have a linearly independent set of vectors, right? We want to see what happens when a acts on those vectors. So, a of w2i minus 1 is this.



Again, you have to calculate what happens to a of wi. Okay, so this you have to check it yourself. Please do this part. Now, this is very easy. Nothing special is happening here.

It is beta times w 2j minus, sorry, 2i minus 1, 2i minus 1 plus alpha times w 2i. Okay, so this is what we are, so this you have to check it yourself. Clear? Now, you see, we define a map, define T, okay, such that T of EI is nothing but WI. This holds for I equals to 1, 2, 2N.

Clear? See, W1, W2, WN are linearly independent. So, basically, what I am doing is this, corresponding to each, so, the unit basis element just like what we did earlier okay so essentially what is your matrix t so basically the columns are w1 w2 w2 and that's the thing okay real entries of course okay right and w1 w2 w2 and so basically you see the matrix t the matrix t has columns what are the columns w1 w2, w2n.

Yes, and we just showed that w1, w2, w2n are linearly independent, right? Since they are linearly independent, T matrix is invertible. So, that will imply that T inverse exists. Clear? Okay.

Now, we want to look at what happens to T inverse A T. Just like before. See, I want to make it like a diagonal or as close as possible. Okay. I want to look, I want to make this matrix as a diagonal matrix. Okay.

If it is possible. So, what happens to T inverse A T? Acting at E, let us say 2I minus 1. Let us look at this. It is nothing but T inverse A

T acting at E 2 I minus 1. And what is E 2 I minus 1? This is W 2 I minus 1. So, T inverse A, it is W 2 I minus 1. Now, what is A W 2 I minus 1?

It is nothing but, you see, what is A W 2 I minus 1? It is alpha times W 2 I minus 1 minus beta W 2 I. So, this is nothing but T inverse of alpha times W 2i minus 1 minus beta times W 2i. Clear? Now, T inverse is a linear map.

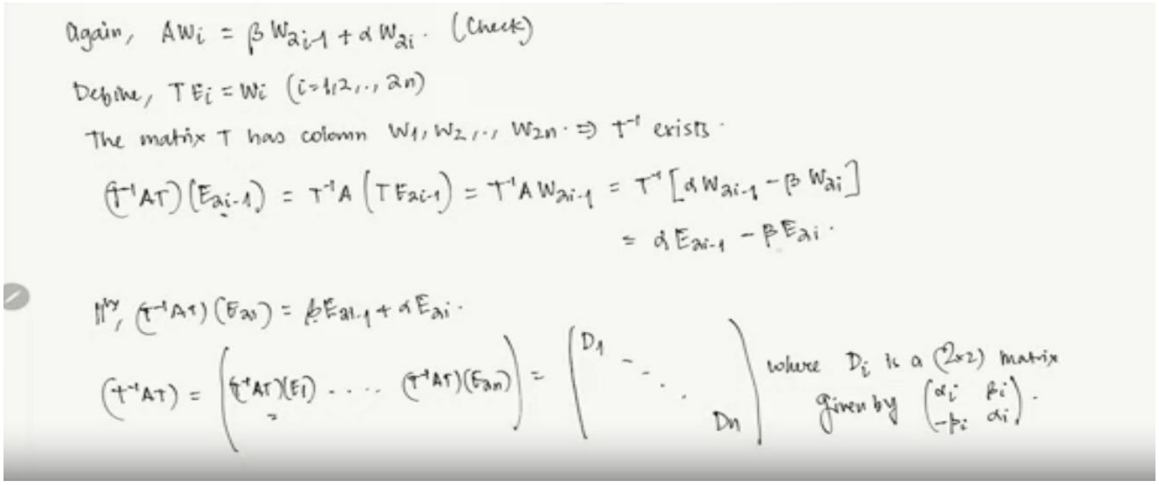
So, I can take everything inside. So, this becomes alpha times T inverse W 2i is nothing but E i, right? E 2i minus 1 minus beta E 2i. Clear? Now, similarly, you can check this part that T inverse At acting at E T U I is nothing but beta times E 2 I minus 1 plus alpha times E 2 I. Yes.

So, now, let us just write down what is T inverse At. What is T inverse At? T inverse 80, this matrix is nothing but T inverse At acting acting at E1. Clear?

Okay. And it will go on like this and then T inverse A T acting at E 1 to n. Right. So, let us write it down if we is possible. So, you see it is nothing but this matrix D1 Dn.

Okay. Where d i is a 2 cross 2 matrix, 2 matrix given by, given by, you see, here alpha minus beta, beta alpha, alpha i, beta i, okay, and then minus beta i, alpha i. If you write it down, you see, I mean, T inverse of, let us say here, E1. What is E1?

It is alpha, so T inverse A T of E1. It is like alpha E1 minus beta, okay? e 2 okay so you write put everything together and this is what you are going to get okay now you again t inverse a t times e 2 if you want to calculate then you just put alpha f so basically you put i equals to 1 i equals to 2 and then you just write it down then you are going to get a matrix which looks like this okay.



So the theorem which we are going to get is this theorem Let A has n cross n, has distinct, distinct, so A is a n cross n matrix, n cross n matrix, has distinct eigenvalues, eigenvalues, okay. Then we can say there exists a linear map, linear map such that such that T inverse A T. How does it look like? T inverse A T, it will look like this. See, lambda 1, lambda K. These are corresponding to the real, this thing, the real eigenvalues, okay? So basically, you see, distinct eigenvalues, it can be real or complex, right? Of course, if they are complex, there is nothing to distinct.

So basically, the real part, lambda 1, lambda 2, lambda K, and for the complex part, what you have is this, D1, D2, Dn. So, you have D1, D, let us just write it as K, P, where D, P will look like this, alpha P, beta P, minus beta P. alpha is this clear so c If they are distinct eigenvalues, I am not saying they are real or complex. Let us say there is a bunch of real eigenvalues.

So, corresponding to real eigenvalues, we know that it will be, you know, the lambda 1, lambda 2, lambda k, right? Now, what happens to the complex eigenvalues? For the complex eigenvalues, we know that it will look like d1, d2, dn. So, that is what we write, d1, d2, dn, okay? So, that is your t inverse a t, okay?

So, if a has distinct eigenvalues, this is what it happens, that you have a linear map t. such that this happens. So, with this I am going to end this video. In the next video, we are going to see what happens when they have repeated eigenvalues. That is a complicated case.