Ordinary Differential Equations (noc 24 ma 78)

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Week-04

Lecture-17 : Fundamental theorem of linear Systems

Hello students, in this video we are going to talk about fundamental theorem of linear systems. Sorry, let me write it properly. Fundamental theorem for linear systems. Systems. So, essentially, you see, we talked about the linear system, right, variable coefficient.

So, we know that if we are looking at this problem, x prime equals to At times x, okay, x at the point 0 is x 0, then the problem has a unique solution, problem admits an unique solution, admits an unique solution. unique solution and how is the, what is the solution here? It is given by x of t is the fundamental matrix phi of t times a constant which is a n cross 1 constant, okay, sorry, this initial data is given, right, so it is nothing but phi inverse of 0 times x 0, right, phi is the fundamental matrix, the fundamental matrix. fundamental matrix okay now the point is this this is okay but the thing is for real life calculations it is not a very feasible option so what we do is this and this is why we talked about matrix exponential in the first place in the last video okay so what we do is this see let us say so as a motivation let us look at some motivation motivation.

So, let us look at this problem. See, x prime equals to Ax, let us say, yeah, and x0 equals to x0, yeah. You can easily check that xt equals to x0 times e power at, okay, is a solution, is the only solution, is the only solution, right, only solution. A is in R, right, A is in R given, Okay.

Xt is the only solution. Now the thing is this from here we can actually motivate ourselves that what should be. So now the question is can one find similar results for x prime equals to x x at the point 0 is x 0 okay i am writing 0 you can take it to be t 0 no problem okay now here you see please remember remark what i what i am going to do now okay whatever follows whatever follows in this video in this video is only valid for a constant coefficient system.

Constant coefficient system. Clear? Okay. See, this thing, this actually holds for fundamental matrix solution. That actually holds for variable coefficient.

No issue there. But the thing is, we can actually, we expect that, let us say, the variable coefficient becomes a constant coefficient. We can actually say more, right? Okay. We can see much more details.

So, that is what we are trying to do here. So, let us say you have a constant coefficient problem, x prime equals to x, x0 equals to x0. Let us call that as star. We want to find the solution. Okay.

From the earlier thing, motivation, we can say that this is the thing M. You see, you can actually see, I mean, you can think of it like this, that e power a t times x 0 should solve star. You see, here, e power At times x 0 is a solution of this problem, right? Now, if you are making it a way an n cross n system, then that for that system e power At times x 0 has to be the solution. You have to write it like this because e power At is a n cross n matrix and this is a n cross 1 vector, right?

So, then make sense. You cannot write x 0 times e power At. Please remember that. You cannot write x 0 times e power AT, yeah? Okay.

So, please remember this part. clear now you see the thing is xt equals to e power At times x 0 should be the solution right that's what we can expect from this particular thing okay so we are going to do that part okay right so this is the main aim however okay so let's look at a lemma first lemma let a a Be a n cross n matrix. Be a n cross n matrix. Then we want to talk about the derivative of e power at.

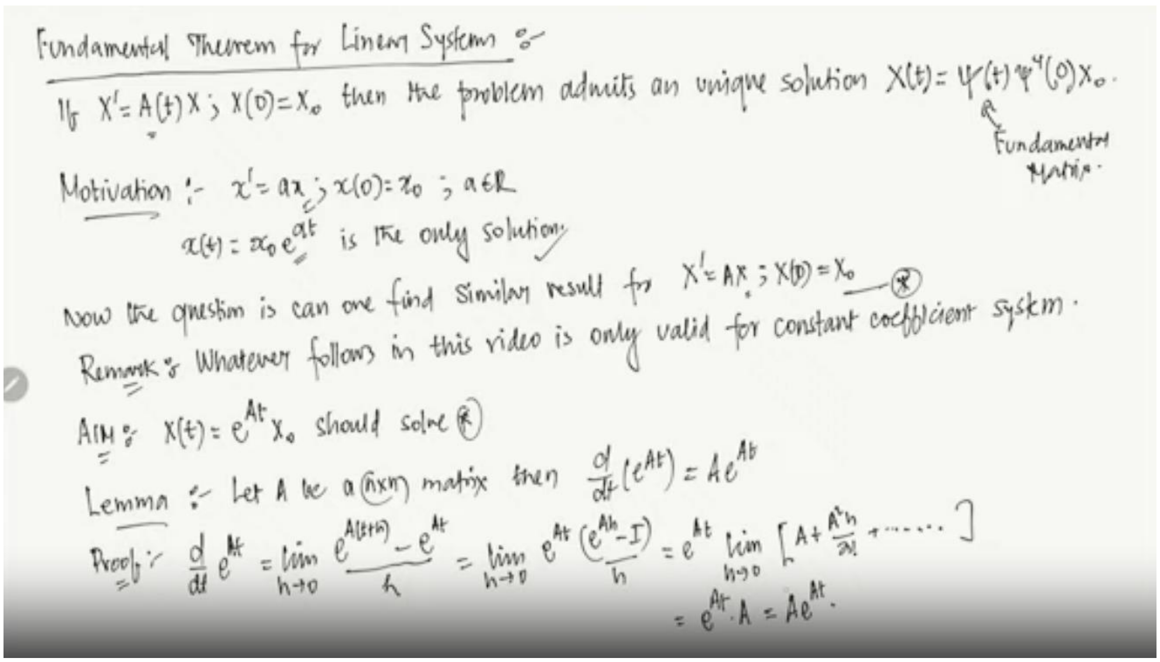
Then d dt of e power at. What happens here? Okay, now from one variable you do realize it should be, this should look like this. E power A, E power AT. It should look like this.

Yes, and actually it does. Okay, so you do not have to worry about it. Okay, but the thing is we of course need a proof of that. So let us look at a proof. Proof.

So, d dt of e power a t is nothing but this is our usual definition limit h tends to 0 e power a t plus h minus e power a t by h. So, this is nothing but limit h tends to 0 e power a t okay e power a h minus identity you can write it like this right by h clear so this is nothing but e power a t limit h tends to 0 i can take the e power a t outside because it does not depends on h so limit h tends to 0 e power a h what is e power a h it is nothing but minus identity it is nothing but 1 by h okay a plus a square h by 2 factorial plus so 1 by h it will get cancelled

I am now moving this thing okay okay okay now this is nothing but what happens this is nothing but e power a t times a right, okay, which is again, I mean, this, this actually you can see that this actually commutes, so you can write it as A e power At, clear. So, we have our, so, you know, result, okay.

See, the last equality which we are doing, so basically e power AT converges uniformly, okay, so that is why you can change the, you know, the two limits essentially, okay. fine um i did miss something no i have missed something limit h tensors below what is the k okay fine yeah this okay so right.



So, now we talk about the fundamental theorem. Okay. So, theorem. So, let A be a n cross n matrix. n cross n matrix. Okay. Then given a x 0in Rn. okay. The initial value problem, the initial value problem x prime equals to x, x0 equals to x0. So, please remember this is a constant coefficient, okay. Admits an unique solution, unique solution. What is the unique solution? Given by

given by x of t is e power a t times x1, okay? This is what you see. We were expecting this, right? So, this theorem actually says that you have a unique solution. Unique solution is guaranteed by Picard's existence and uniqueness.

And you actually say that there is a unique solution which is given like this, yeah? Okay. So, how do you prove something like this? See proof. if x t is nothing but e power a t times x 0, okay, that will imply x prime of t is d dt of e power a t times x 0, okay.

See, x 0 does not, it does not independent of t, right. So, we can, and so d dt of e power a t, we know that it is a e power a t times x 0. So, you see and e power a t times x 0 is nothing but x t. So, this is a times x of t. So, x prime t equals to a times x t. So, x t satisfies the equation for all t in R. So, x t given by e power a t times x 0 satisfies the equation for all t in R. Also, x of 0 is nothing but identity times x 0 e to the power a 0 which is identity times x 0 which is nothing but x naught right okay so therefore okay see the thing is we have to show

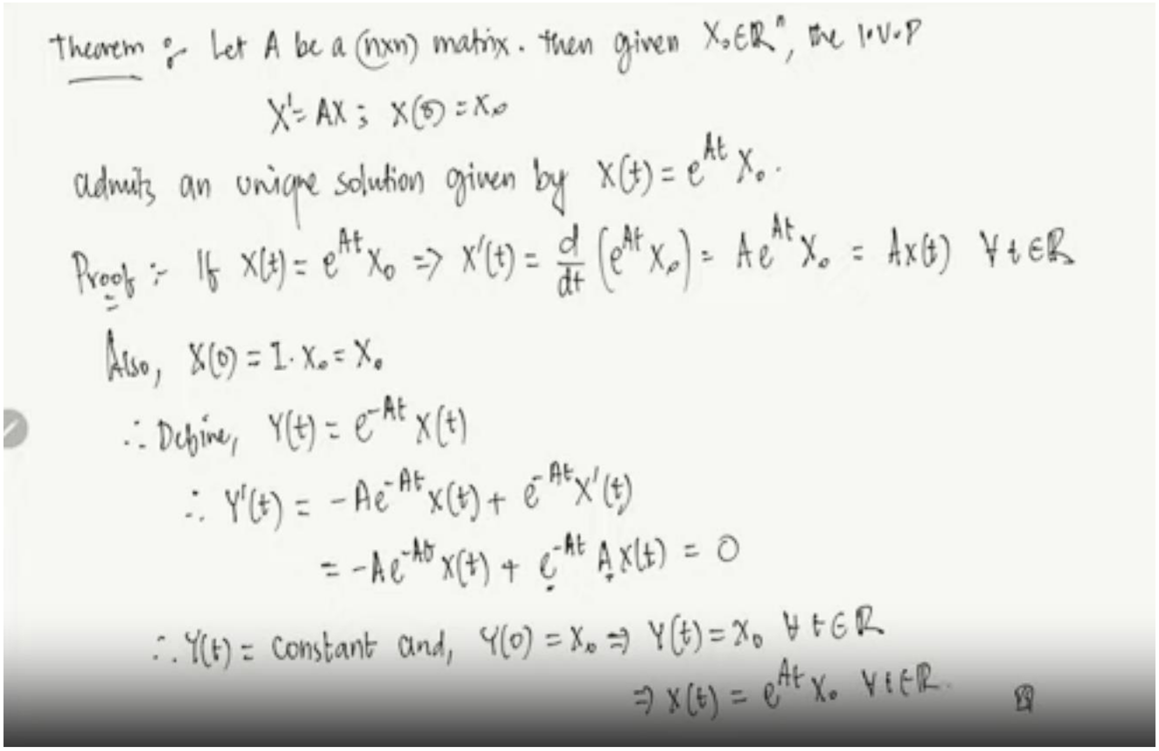
that see what we showed is x t equals to e to power a t x 0 satisfy the equation yes now the thing is this how do you show this is the only solution of course you can use directly picker's theorem and you can say that this is has to be the only solution there is no other way without picker's theorem also you can prove let's just look at that proof okay so what you do is you define Define y of t to be e to the power minus a t times x t. You see similar sort of thing we did it in the last video also if you remember. y of t equals to e to the power a t minus a t times x t. Then what is y prime of t? y prime of t is nothing but minus a times e power minus a t x of t plus e to the power minus a t x prime of t. So, this is nothing but minus a e to the power minus a t x t and what is x prime of t?

It is nothing but a of x. So, e to the power minus a t a of x of t. Now, a and e power minus a t, they actually commutes. Okay. So, these two commutes, this one and this one commutes with each other. So, basically, we can write this is essentially 0.

Right. That will imply y of t is nothing but constant. Okay. y t is constant. And what is y at the point 0?

This is nothing but x of 0, which is x 0. Okay. So, if this is x 0, then what happens is that will imply y of t is always x 0, right? x naught for all t in R. And then we can actually say what is y of t? It is x of t then is nothing but e to the power a t times x 0.

So this is proved. So this holds for all t in R. Now so what we understood here is this. See if you have a constant coefficient equation at least homogeneous constant coefficient equation what we have is you have a unique solution given by e to the power At times x 0 and that solution is valid for all of R. All of R.



See the thing is here you may ask Why don't we use Picard? You can use Picard for the uniqueness thing. But the thing is that is not guaranteeing that you can have a solution for all r. It only guarantees in a small neighborhood of 0. But here you can actually using this proof you can actually say this works for all r. So let us look at a simple example. So let us say that We have this problem, x prime equals to minus 2, minus 1, 1, minus 2x, okay? And x at the point 0 is a vector given by 1 and 0.

Okay, now I want to find out what the solution is. So, the solution x of t, what do we know? We know this is a t times x 0. Okay, so you just have to calculate e power a t and you are done. Okay, and what is e power a t?

You see this is in this form. If you remember, we did one problem, right? This problem, you see a minus b, b, a. Yes, in this form, e power a we calculated, which is looks like e power a cosine b minus sine b, sine b cosine b. Okay, so we can use this property here. So, since e power a t, what is e power a t? Let us just write it down.

I am just writing it directly. I am not calculating anything, writing it directly. Cosine t minus sine t, sine t and cosine of Clear? Okay.

So, you just put this vector here and acting at 1, 0. So, the answer which you are going to get, I am not calculating again. You can calculate it yourself. So, you have to check it yourself. It is cosine t and sine t. Okay.

So, please check it yourself. Clear? Okay. So, this is how you calculate this. Very, very simple.

Okay. Now, the thing is this. If I, the question is, which I want to pose is this question. Okay. You see, x prime equals to x, x0 equals to x0 has a unique solution in terms of fundamental matrix, right?

Solution in terms of fundamental matrix given by, given by x of t, which is phi of t, fundamental matrix phi of t, phi inverse of 0 times x 0, right? That we know, okay? So, phi is a fundamental matrix, fundamental matrix, okay? And you can show that this holds for all t in R,

Okay. Now, the thing is, but again, this theorem says that x prime equals to ax, x0 equals to x0 has, you know, this unique solution e power At times x0. Okay. So, is there a relation? So, is there a relation of the fundamental matrix, fundamental matrix

matrix phi of A with exponential of A. What do you think? See, the thing is, you have to understand that in both the situations, it is a unique solution, exactly same equation and it passes through the same vector x 0 in Rn, right? So, therefore, one can say that without, you know, putting much emphasis here, phi of t phi inverse of 0 has to be e power At, no? It has to be e power At.

There is no other option here, okay? So, you see, if there is, so this is another way of, I mean, calculating e power a if you want, okay? So, if one can calculate, okay, We will explore all these things in the future videos. But for now, this is just for information.

If one can calculate fundamental matrix or if it is given, let us say, the fundamental matrix, then calculating exponential is easy. Calculating exponential is easy. Exponential of a matrix is easy. is easy. Now, you have to understand, please remember this thing.

This actually holds only when it is constant coefficient. Variable coefficient equation, you can of course write it in terms of fundamental matrix, the solution, but you cannot have this thing. So, phi t phi inverse of 0 is nothing but e power a times t. So, with this, I am going to end this video.

Ex: $X' = \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix} X$; $X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$X(t) = e^{At} X_0 = e^{-2t} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ (Check it yourself).

$\left[\because e^{At} = e^{-2t} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \right]$

Question :- $X' = AX$; $X(0) = X_0$ has a unique solution given by $X(t) = \psi(t) \psi^{-1}(0) X_0$. $\forall t \in \mathbb{R}$.

$X_0 \in \mathbb{R}^n$ — Fundamental Matrix

so, Is there a relation of the Fundamental Matrix ψ of A with e^A.

$\therefore \underline{\psi(t) \psi^{-1}(0) = e^{At}}$.

so, if one can calculate F.M then calculating exponential of a matrix is easy.